The Beacon of Kac-Moody Symmetry for Physics

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Although the importance of developing a mathematics which transcends practical use was already understood by the Greeks over 2000 years ago, it is heartening even today when mathematical ideas created for their abstract interest are found to be useful in formulating descriptions of nature. Historically, the idea of symmetry has its scientific origin in the Greeks' discovery of the five regular solids, which are remarkably symmetrical. In the nineteenth century, this property was codified in the mathematical concept of a *group* invented by Galois and then that of a *continuous group* by Sophus Lie.

In 1967, Victor Kac (MIT), then working in Moscow [26], and Bob Moody (Alberta) [38] independently enlarged the paradigm of classical Lie algebras, resulting in new algebras which are infinite-dimensional. The representation theory of a subclass of the algebras, the affine Kac-Moody algebras, has developed into a mature mathematics.

By the 1980s, these algebras had been taken up by physicists working in the areas of elementary particle theory, gravity, and two-dimensional phase transitions as an obvious framework from which to consider descriptions of nonperturbative solutions of gauge theory, vertex emission operators in string theory on compactified space, integrability in two-dimensional quantum field theory, and conformal field theory. Recently Kac-Moody algebras have been shown to serve as duality symmetries of nonperturbative strings appearing to relate all superstrings to a single theory. The infinite-dimensional Lie algebras and groups have been suggested as candidates for a unified symmetry of superstring theory.

In addition to this wide application to physical theories, the Kac-Moody algebras are relevant to number theory and modular forms. They occur in the relation between the sporadic simple Monster group and the symmetries of codes, lattices, and conformal field theories [22, 6]. Much in the same way that certain finite groups were understood to be the symmetry groups of the regular solids, the Monster and certain affine Lie algebras are seen to be automorphisms of conformal field theories [14, 9]. Kac's generalization of Weyl's character formula for the affine algebras leads to a deeper understanding of Macdonald combinatoric identities relating infinite products and sums [35, 27]. Topological properties of the groups corresponding to the affine algebras have been analyzed [17, 42], as well as varieties related to the singularity theory of Kac-Moody algebras [28, 39, 48]. Several reviews emphasize the physical applications of the algebras [8, 19, 25].

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Note: Please see ***Mathematics People*** *in this issue for announcement of Wigner Prize winners Victor Kac and Robert Moody.*

Familiar Symmetry

In physics, the symmetries of a theory provide significant information about the general solution of the system. For example, a general solution to Laplace's equation $\nabla^2 U(x, y, z) = 0$ can be compounded from particular solutions in a coordinate system suggested by the symmetry of the problem. For a spherical conductor, limited to the case where U is $U(r, \theta)$, the solution can be expressed in terms of specific functions of r and the Legendre polynomials $P_n(\theta)$. In quantum field theories, useful applications of symmetry include the invariance imposed on the regularization of an approximation scheme, symmetry transformations among different specific solutions, and Noether constants of the motion used to label them. As evidenced in the works of Weyl [53], van der Waerden [51], and Wigner [55], finite groups and then continuous Lie groups with their associated algebras were central to the understanding of atomic spectra and the formulation of quantum mechanics in the 1930s [54].

In addition to (this) wide application to physical theories, the Kac-Moody algebras are relevant to number theory and modular forms.

Familiar symmetries reflect not only spacetime invariance such as the Poincaré group or the general linear group of gravity, but also internal gauge invariance. For example, gauge invariance given by the group $U(1)$ results in the conservation of electric charge in Maxwell's equations. The notions of connection and curvature in modern differential geometry describe the gravity, electromagnetic, and Yang-Mills gauge theories of the fundamental interactions of physics. Already in the 1920s Kaluza and Klein [30] suggested a unification of the classical theories of gravity and electromagnetism by observing that the Maxwell field (later identified as a connection on a principal U(1) fiber bundle) could be understood as extra components of gravity defined in five spacetime dimensions, with the fifth dimension made compact. This related the symmetries of gravity and electromagnetism in four dimensions and provided one of Einstein's models for a unified field theory. The Kaluza-Klein mechanism for curling up dimensions is eventually seen to accommodate even higher-dimensional models such as string theory.

In the 1970s, high-energy physicists pursued Lie algebra theory as a valuable tool to characterize all the gauge interactions. These are now understood to be $SU(3)$ for the strong force (which describes the interactions between quarks, which are the constituents of hadrons such as the proton) and $SU(2) \times U(1)$ for both the weak and electromagnetic interactions of quarks and leptons (such as the electron). This is an important feature of the standard model of particle physics [57, 52, 43, 18]. *Grand unification* was an effort to combine these symmetries as subgroups of a unifying group such as $SU(5)$. *Superstring unification* provides an alternative mechanism to combine symmetries.

The interpretation of fundamental force laws in terms of group theory is now commonly understood in terms of E. Noether's theorem which identifies the elements of the Lie algebra with the charges conserved in the interactions [40]. The symmetry is used to label the physical states: the eigenvalues of the Cartan subalgebra are the quantum numbers of the elementary particles. The quantum numbers are known as charge, spin, hypercharge, isospin, etc., depending on which group is being considered. Modern high-energy theorists effectively think of the elementary particles of the strong, weak, and electromagnetic forces as irreducible representations of the direct product of the Poincaré group and $SU(3) \times SU(2) \times U(1)$.

Selection rules derived from the conserved charges limit allowed transitions of quantum numbers. Explicit solutions of four-dimensional quantum field theory transition amplitudes, however, are known only in a perturbative expansion, i.e. they are not known exactly. This weak coupling perturbative approximation is extremely useful in electro-weak theory, but the strong coupling problem of why the quarks are confined inside the hadrons remains more elusive.

In two spacetime dimensions the situation is different. Both discrete systems (for e.g. the Ising model) and continuous theories (for e.g. the sine-Gordon equation and the principal chiral models) can be solved exactly; i.e. they are integrable due to the occurrence of an infinite number of conservation laws. Essentially each theory has as many constants of motion as it has degrees of freedom. For example, the affine algebra has been used to construct a general class of solutions for the Korteweg-deVries equation and to linearize the periodic Toda lattice. In the case of a finite parameter algebra, the method of orbits of Lie groups has led to the quantization of the integrable Toda chains [31]. These stunning results of mathematical physics suggested that infinite-dimensional algebras as well as the finite-parameter symmetry algebras might be important for physical theories.

Kac-Moody Symmetry and Conformal Field Theory

The theory for the finite-dimensional semisimple Lie algebras was worked out by E. Cartan and Killing about a hundred years ago. They associated with each algebra a finite integer matrix with positive-definite conditions. Generalizing the Cartan matrix by relaxing the positivity conditions, one can obtain the Kac-Moody algebras, of which a subclass constitutes the conventional Lie algebras. Another subclass of Kac-Moody algebras referred to as *affine* replaces positive-definite with positive-semidefinite; and the corresponding structure and representation theory has a precise analogy to that of the semisimple finite-dimensional algebras. The commutation relations for the affine algebra are:

$$[T_n^a, T_m^b] = if_{abc}T_{n+m}^c + kn\delta_{n,-m}\delta_{ab}$$

where $n, m \in \mathrm{Z}$, $T_0^a \in g$ for any semisimple Lie algebra g with structure constants f_{abc}, so $1 < a, b, c < \dim g$. The central extension k is proportional to the identity and is related to the level of the affine algebra. Since the generators are moments of currents, i.e. densities $T^a(z) = \sum_n T_n^a z^{-n-1}$, their commutators can be expressed equivalently in terms of an operator product expansion for $|z| > |\zeta|$ by

$$T^a(z)T^b(\zeta) = (z-\zeta)^{-1} if_{abc}T^c(\zeta) + (z-\zeta)^{-2}k\delta_{ab} + \text{regular terms.}$$

This expansion is reminiscent of Gell-Mann's current algebra approach for local particle symmetries. This emphasized that not only do the *charge* operators Q^a form an algebra, such as $SU(3)$ in the eightfold way used to classify elementary particles, but also the local *currents* $J^a(x)$ satisfy algebraic commutation relations [1, 49].

Infinite-dimensional Kac-Moody algebras which are not affine are called *hyperbolic*, and recent connections with superstring theory have led to a better understanding of their representations [16] as well.

For a Kac-Moody algebra, the Cartan matrix, and thus the Cartan subalgebra, remains finite-dimensional. Each affine algebra $\hat{g}$ has a finite-dimensional semisimple Lie subalgebra g whose rank, i.e. number of simple roots, is one less than that of the whole affine algebra. Therefore, theories with the larger symmetry have one additional quantum number called the level. We can also extend the affine algebra to include an element called the derivation operator $-L_0$ where $[L_0, T_n^a] = -nT_n^a$, whose eigenvalue can be thought of as the affine charge, and measures the mass level in physical theories. The derivation is the zero mode of another infinite-dimensional algebra, the Virasoro algebra. These generators can be derived via the Sugawara construction and are bilinear in the affine generators $L(z) \equiv 1/(2k + c_\psi) \sum_a {}^\times_\times T^a(z)T^a(z)^\times_\times$, so that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$
$$[L_n, T_m^a] = -mT_{n+m}^a$$

where the central charge c is related to the level of the affine algebra and $f_{abc}f_{abe} = c_\psi \delta_{ce}$. The Virasoro algebra corresponds to the infinite-dimensional conformal symmetry of two-dimensional conformal field theories. These field theories are relevant to two-dimensional spin systems at a phase transition and to the two-dimensional world sheet traced out by a string as it moves through time [4]. In statistical models, representations of the conformal symmetry lead to an identification of the highest weights with critical exponents [15]. It is when physicists began to ask detailed questions about the representation theory of the infinite-dimensional algebras that progress took off in those physical theories.

The quantum number associated with the affine charge is the conformal weight. An irreducible representation of an affine algebra $\hat{g}$, with nonzero level, is labelled by the level and the highest weight and is comprised of an infinite tower of irreducible representations of g. The tower is a stack of different "grades", with a grade being distinguished by its conformal weight. The pattern is worked out for the first few grades in terms of $su(2)$ representations in Figure 1. The g content of a given representation can be expressed by multiplicity formulas [29]. For example, the multiplicity $\phi_\kappa(x)$ for the level one highest weight singlet representation of $\hat{s}u(2)$ is

$$\phi_\kappa(x) = \sum_{s=0}^{\infty} n_\kappa(s)\, x^s = x^{\kappa^2}(1 - x^{2\kappa+1}) \prod_{\ell=1}^{\infty}(1 - x^\ell)^{-1}\,; \quad \kappa = 0, 1, 2, \ldots,$$

where $n_\kappa(s)$ is equal to the number of $su(2)$ multiplets with highest weight κ in grade s. These algebras and their representations are applicable to physical theories with an infinite number of states, such as string theory or bound states (strong coupling limits) of gauge theories. See Figure 1.

The "value added" in moving from Lie groups to Kac-Moody algebras in physical theories is that larger symmetry groups give more information about the solution. In the case of conformally invariant systems for example, there are primary fields which transform as

$$[L_n, \phi(z)] = z^{n+1}\frac{d\phi(z)}{dz} + h(n+1)z^n\phi(z),$$

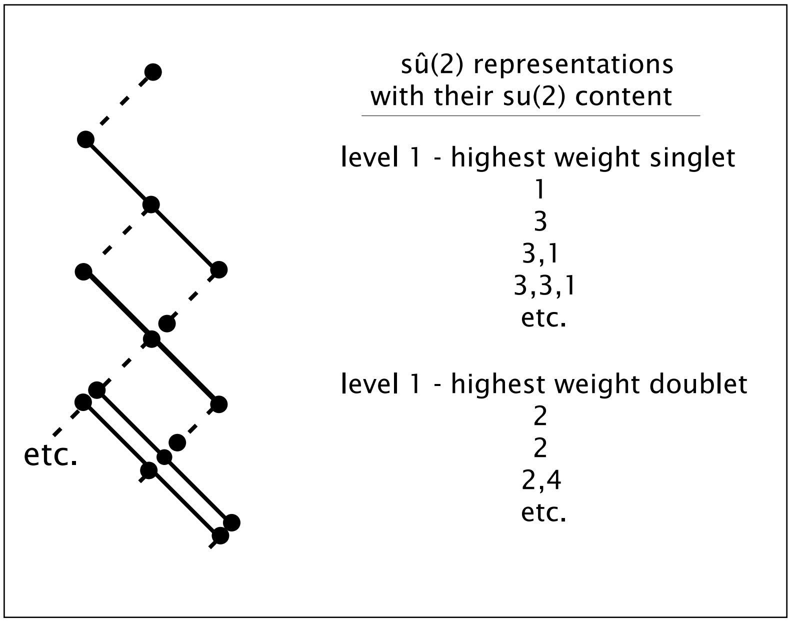

Figure 1. Pictorial view of su (2) level 1 representations. The highest weight singlet tower follows the solid lines. The doublet representation follows the dotted lines.

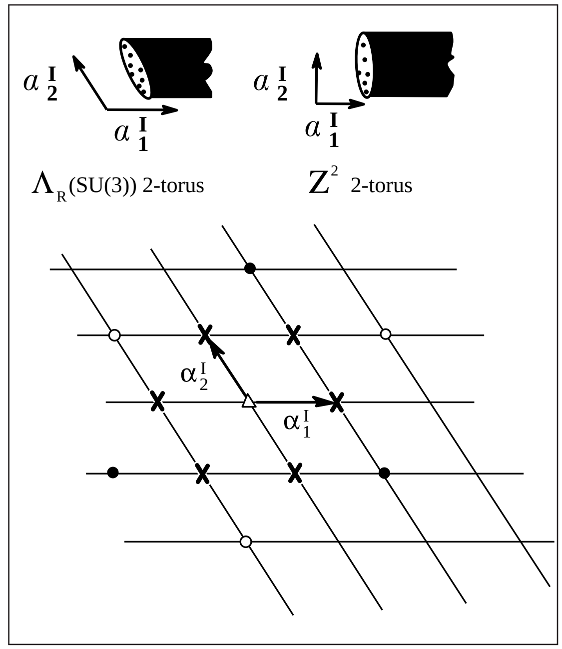

Figure 2. Compact Dimensions

and amplitudes can be computed exactly as follows. Since the vacuum state $|0\rangle$ satisfies $L_n|0\rangle = 0$, $n \geq -1$, then $L_0\phi(0)|0\rangle = h\phi(0)|0\rangle$, $L_n\phi(0)|0\rangle = 0$, $n > 0$, and the two-point function is determined up to a multiplicative constant,

$$\langle 0|\phi(z_1)\phi(z_2)|0\rangle = r^{-h}e^{-i\theta h}\langle 0|\phi(1)\phi(0)|0\rangle$$

where $z_1 - z_2 = re^{i\theta}$.

Integrable Systems and Gauge Theories

The first hint that strong interaction gauge theory might contain a hidden symmetry came from an observation by Polyakov [41] that a functional formulation of the nonabelian theory was similar to the local equations of the two-dimensional integrable chiral models. In this approach, the fundamental role of the gauge field $A^a_\mu(x)$ is replaced by the path-dependent field

$$\psi[\xi] = P\exp(\oint Ad\xi)$$
$$= P\exp\{\oint ds\dot{\xi}_\mu(x)A_\mu(\xi(s))\},$$

an element of the holonomy group, i.e. a path-dependent element of the gauge group. Here $A_\mu \equiv A^a_\mu T^a$ where $T^a$are the elements of $su(N)$. A functional differential equation for $\psi[\xi]$ can be derived:

$$\frac{\delta}{\delta\xi_\mu(s)}(\psi^{-1}\frac{\delta\psi}{\delta\xi_\mu(s)}) = 0$$

when A_μ satisfies the Yang-Mills equations of motion $D_\mu F_{\mu\nu} = 0$. These loop equations look like the chiral model equations:

$$\partial_\alpha(g^{-1}\partial_\alpha g) = 0$$

which have an infinite number of symmetries [34], and the algebra of a class of these symmetries was shown to be given by that of the positive modes of an affine Kac-Moody algebra [8].

It was then natural to conjecture that the same symmetry algebra should be responsible for such transformations in both theories and that representations of Kac-Moody algebras will give information about the gauge theory bound state spectrum, which is comprised of confined sets of gluons. This is consistent with the belief that bound states of gluons lie on linear trajectories, dictating an infinite number of them with a linear relation between the mass and the spin of the states, which is a generic feature of a string theory. Certainly knowledge of the complete symmetry group of the theory is important information.

Properties of instantons in two dimensions and in the four-dimensional self-dual Yang-Mills equations demonstrate similarity in the symmetry of the models as well [2]. Affine symmetry has also been identified as potentially useful in describing the physical particle content of extended supergravity models whose scalar fields parameterize a symmetric space G/H, where G is a noncompact global symmetry group with H as its maximal compact subgroup [12].

Recently, an approach using discrete duality symmetries in supersymmetric gauge theories has led to some analytic control over a mechanism for describing quark-gluon confinement [46]. These theories, when viewed as the low-energy limits of string theories, lead the way to discovering the larger string dualities and the fundamental symmetry group of the string.

String Theory

In 1968, around the time Kac and Moody formulated their algebras, Veneziano [50] wrote down particle-scattering amplitudes with certain analyticity properties. The amplitudes are functions of the momenta of the particles and can be derived as the scattering of modes of a massless relativistic string, a one-dimensional object whose length is characterized by the nature of the coupling. For closed strings, which contain gravity as well as the quarks and leptons, the scale is the Planck length, 10^{-33} centimeters; and the different modes of oscillation describe the different particles, much in the same way that different modes on a violin string result in different notes. Later as the interacting string picture, which set up the unitarity and general consistency of quantum string theory, was being developed [36], mathematicians constructed irreducible representations of the affine algebras. They used string theory vertex operators whose momenta were restricted to take on discrete values, such as the points on the root lattice of g. The discretization of momenta implies a compactified, closed, periodic condition on the conjugate position space. See Figure 2. The number of compactified dimensions is equal to the rank of g.

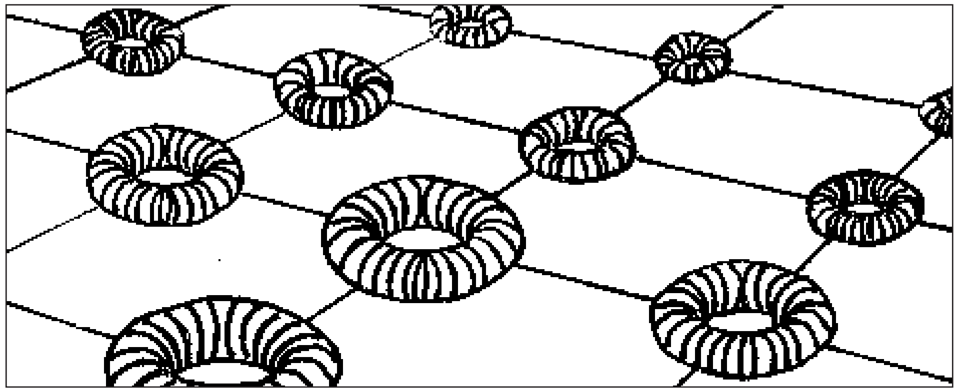

Figure 3. HIDDEN DIMENSIONS of the universe, proposed in a theory that seeks to unify the forces of nature, can be viewed as a small, compact structure such as a 6-torus that is associated with every point in 3-space and time.

In the early 1980s it was understood by physicists that since the superstring was quantizable only in ten dimensions, discrete momenta corresponding to the curling up of six of these was the obvious Kaluza-Klein answer to achieve a string theory relevant for physics in four spacetime dimensions. See Figure 3.

The level one representations were constructed from the Veneziano open bosonic string Fock space oscillators, with the compactification size R fixed to be the string scale α' [13, 45]. See Figure 4. Earlier, representations had been found for twisted affine algebras with use of the Corrigan-Fairlie [7] oscillators corresponding to strings with different boundary conditions, i.e. one end of the string is held at a fixed position [33]. Higher-level quark model representations were given in terms of fermionic oscillators [3].

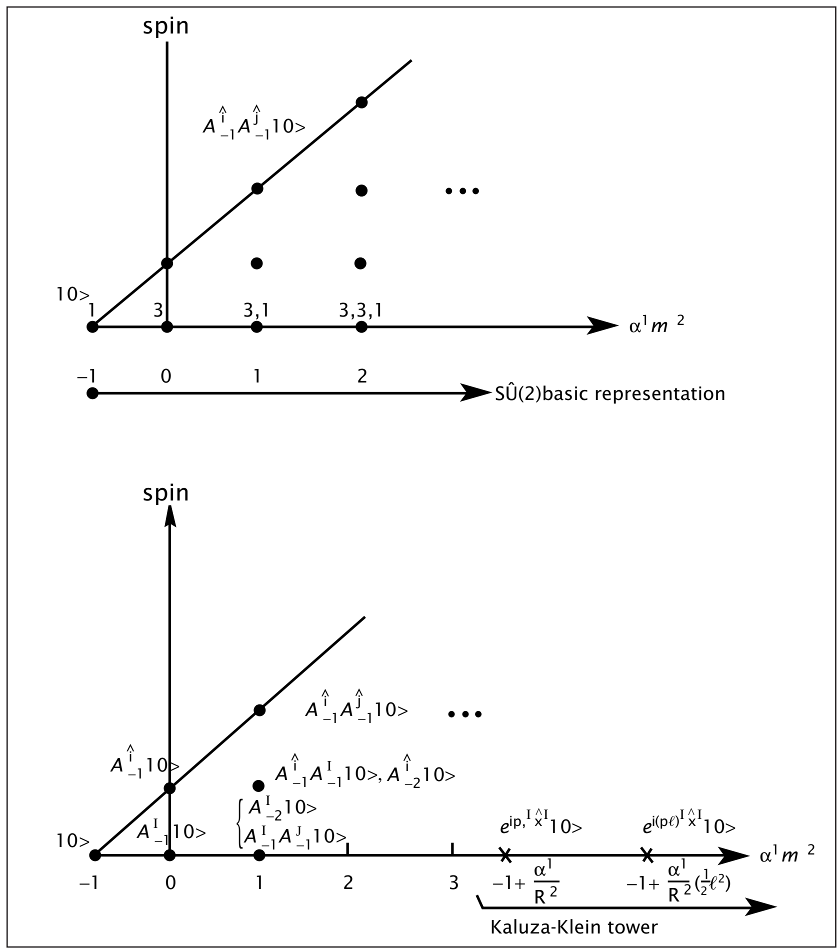

Figure 4. Bosonic String Affine Representations

These realizations were then incorporated into realistic string models, which appear to have a chance of describing standard model physics. The heterotic string [23] makes use of level one affine representations, while compactifications of the Green Schwarz superstring [20] employ representations with level equal to the dual Coxeter number of g [5]. In both models, the zero-mode generators which are elements of g correspond to the Yang-Mills gauge symmetry of the particle spectrum.

At the time of the rebirth of string theory in 1985 [21], physicists had been working hard on solving the problem of quark confinement in four-dimensional nonabelian gauge theory. It was believed that the added structure in string theory, whose low-energy limit was gauge theory, might prove sufficient to formulate a nonperturbative solution. In the last year or so, developments concerning duality symmetries have occurred which carry this program further.

Duality symmetry that maps between strong and weak coupling in gauge theories and in string theory has been tested [47]. The simplest notion of this kind of symmetry surfaced originally in the study of integrable models, where solvability and strong-weak maps were seen to go hand in hand. In fact, the Kramers-Wannier self-duality of spin models [32], such as the Ising model and the X-Z model, can be used to construct an infinite commuting algebra of conserved charges [10]. A kind of Kramers-Wannier duality was incorporated by Montonen and Olive [37] to formulate a conjecture for electric-magnetic duality in gauge theories, since the Dirac quantization condition for monopoles fixes the electric and magnetic couplings to be inversely proportional. This duality, which occurs naturally in a supersymmetric gauge theory, has been extended to the strong and weak coupling limits of string theory. The duality transformations can be used to relate different superstring models to one another [11, 24, 56], so that in fact there may be just one structure whose quantum ground state describing elementary particles is unique.

It now seems feasible [44, 16] that the infinite discrete set of duality symmetries of string theory may be related to both E_9, i.e. the affine E_8 algebra, and to a hyperbolic Kac-Moody Lie algebra E_{10}. The idea of using Kac-Moody symmetry to get at nonperturbative information in theories of particle physics remains viable.

Physics and Mathematics

Many of the properties of the Kac-Moody algebras were rediscovered in physical theories. They supply the precise Fock space states and field operators which provide a practical way to compute structure constants and construct representations and realizations. The occurrence of these symmetries in physics has been useful to the development of the mathematical theory. Nonetheless, the sophisticated mathematical structure which has evolved serves as a beacon to physicists in that it signals that the theories under investigation probably have further properties which are not yet obvious but may be instrumental in understanding the consistency of nature.

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