A Tribute to Warren Ambrose

I. M. Singer and H. Wu

I have not forgotten my first day at MIT. In 1950 Moore instructors had to teach summer school. On a sunny afternoon early in July, I crossed the bridge in search of Building 2. Math headquarters was on the second floor; it still is. I introduced myself to Ruth Goodwin, who handled all secretarial services, and I asked to see the chairman. When I gave Ruth my name, a chap sitting across from her, head buried in the *Boston Globe*, lowered his paper and said: “Singer, I’m Ambrose. There is a seminar in Lie groups in five minutes. You can see Martin later. Come.”

I did, and met John Moore, Barrett O’Neill, and George Whitehead, who became lifelong friends. After getting my teaching assignment from Ted Martin, Ambrose told me the seminar met at midnight in the Hayes-Bickford coffee shop. He would pick me up at 11:45. Kay Whitehead joined us for these evening sessions; the coffee was deadly, the conversation lively. Ambrose gave me a tour of Boston that first night and by the time he dropped me off, we were close friends.

One day Ambrose said, “Singer, you listened to Chern’s lectures. What did he say?” At Chicago, I had passively taken notes of S. S. Chern’s course, while writing a dissertation in another subject. What with interpreting my notes, reading Chern’s papers, pouring over Elie Cartan, and Ambrose insisting on absolute clarity in every detail, we learned differential geometry together.

Ambrose designed the Geometry of Manifolds course, and we taught it in alternate years. It is pretty much the same today as it was then: standard manifold theory the first term and instructor’s choice of topics the second. Our students wrote some well-known graduate texts based on this course: Bishop-Crittenden, Hicks, and Warner.

With customary zeal, Ambrose changed the undergraduate program in pure mathematics. Whereas in 1948 Andre Weil explained differential forms to the faculty at the University of Chicago, less than a decade later we were using them in undergraduate differential geometry. Ambrose taught the Lebesgue integral in the analysis course for juniors and seniors “because it’s simpler than the Riemann integral.” For almost twenty years Ambrose was the guiding spirit of pure mathematics at MIT. His efforts were key in making it a great department.

Ambrose and I regularly drove around Boston late at night talking about mathematics and life. We knew every street; and to this day, with the inevitable traffic jam, I’ll remember an alternate route and, often enough, a special moment: yes, here is where we finally understood holonomy.

I. M. Singer is professor of mathematics at the Massachusetts Institute of Technology in Cambridge, MA.

H. Wu is professor of mathematics at the University of California at Berkeley. His e-mail address is `hwu@math.berkeley.edu`.

Warren Ambrose

Ambrose taught me jazz. I had been an enthusiast of Dixieland and swing. But by 1950 bop was dominant. Charlie Parker was king, and everyone else his student. We heard all the great jazz musicians at all the jazz joints. There was a casualness and directness then that makes it difficult for me to hear jazz live today. It's too rigid and formal now. Imagine watching Bird take on a long line of young sax players, listening to each intently and then playing their variation as it should be played. Teaching at its best, I felt. Imagine having coffee between sets with Billie and thanking her for her early records that meant so much to many of us. Dick Kadison and I used to hear Ella night after night at Birdland when we took a break from work. Ambrose explained jazz to me and talked about it in the fifties the way Wynton Marsalis does now.

Though mathematics was wonderful, I was heavily burdened. My oldest son was blinded at birth and, as I learned later, brain damaged. I could not have survived as a mathematician without Ambrose's steady support and the steady support of my very good friend, Dick Kadison. Those who knew Ambrose know that expressing gratitude was forbidden. He walked rapidly away the one time I tried. Occasionally, I can provide special help to a young mathematician. I think of Ambrose and feel that by the time I have helped a hundred, I'll have begun to pay my debt to him.

I loved Ambrose for his absolute honesty, his generosity, his wit, his energy, and above all for his tenderness, which he tried so hard to hide. I am sad that so few mathematicians knew what a great man he was. But I was happy for him when he found his wife, Jeannette, with whom he could be himself for twenty years.

I. M. Singer

Warren Ambrose, a pioneer in differential geometry, passed away on December 4, 1995, in Paris. He was eight-one.

Differential geometry has not always been the popular subject that it is today. In the 1950s Ambrose (together with I. M. Singer) made MIT into the only center in geometry in the United States outside of the University of Chicago.

Ambrose was born on October 25, 1914. He obtained his Ph.D. in probability with J. L. Doob at the University of Illinois at Urbana (now Urbana-Champaign) in 1939, but his interest soon switched to functional analysis. His most notable work in the latter area is probably his structure theory of what he called the $H*$-algebras, a generalization of the L^2 group algebra of a compact group (Trans. Amer. Math. Soc. **57** (1945), 364–386). By the early 1950s his interest had changed yet again, this time to differential geometry. In his later years, he was to explain this change by saying that he wanted to be in a field where "the theorems come less easily".

Ambrose entered geometry at a time when the dawn of a new era was just around the corner. The works of J. L. Synge, H. Hopf, and S. Cohn-Vossen in the decade after 1925 and those of S. B. Myers and S. S. Chern in the forties made the shift of focus in geometry from the local to the global all but inevitable. Then in 1950 C. Ehresmann published a paper that cleaned up the foundations; availing himself of the latest works in Lie theory and topology, he gave the first rigorous definition of a connection on a fiber bundle. The stage was thus set for a nontrivial theorem that would embrace this new spirit and new machinery. It can be argued that the Ambrose-Singer holonomy theorem (Trans. Amer. Math. Soc. **75** (1953), 428–443) was exactly that theorem. This theorem concerns a principal G-bundle P with connection C over a connected manifold M, where G is a Lie group. The connection C associates to each curve γ on M joining two points x and y an isomorphism γ_* from the fiber P_x over x to the fiber P_y, called the *parallel translation* from x to y. (If P is the bundle of bases in the tangent spaces of M so that G is the general linear group $Gl(n, \mathbf{R})$, then γ_* is the parallel translation in the classical sense, i.e., it maps the bases in M_x (tangent space at x) to those in M_y.) Now fix an x in M and consider all the loops that begin and end at x. γ_* then becomes an automorphism of P_x which turns out to coincide with the action of an element $\tilde{\gamma}$ of G on P_x. The set of all such $\tilde{\gamma}$ as γ runs through all the loops at x is easily seen to be a subgroup of G. The identity component of this subgroup, to be denoted by H, is a Lie group. Up to conjugation in G, H is independent of the choice of x. H is the (restricted) *holonomy group* of the connection C. The Ambrose-Singer holonomy theorem asserts that the Lie algebra $\mathcal{H}$ of H is exactly the linear span of the values of the curvature form Ω of C. (Technical aside: Ω should be restricted to the holonomy subbundle of P.) This is therefore a vast generalization of the fact that if the Levi-Civita connection of a Riemannian metric is flat (zero curvature), then its parallel translation is trivial (no holonomy).

A similar (but slightly weaker) result was obtained independently by A. Nijenhuis around the same time, but it was the proof of Ambrose and Singer that captured the imagination of the geometers. Although the geometric idea of the proof is clearly in the forefront, the whole argument is carried out with total precision and in an entirely abstract setting. There was never anything vaguely resembling this in the classical literature. This proof is still the one that is reproduced essentially verbatim in the standard texts of today.

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The Ambrose-Singer paper has been rightfully recognized as one of the most influential in the recent history of the subject, but the full realization of its import came only in the last ten years. In greater detail, suppose M is a Riemannian manifold with metric g, P is the bundle of orthonormal bases in M with respect to g, G is the orthogonal group $O(n)$, and C is the Levi-Civita connection of g. Then the holonomy algebra $\mathcal{H}$ is now the linear span of a set of skew-symmetric matrices (Lie algebra of $O(n)$) which are the values of the curvature form Ω of g, and Ω must further satisfy a collection of identities, the Bianchi identities. It is then clear that $\mathcal{H}$ is far from arbitrary. This was the idea that motivated Marcel Berger in 1955 to use the holonomy theorem to prove that the number of Lie subgroups of $O(n)$ that can be holonomy groups of a Riemannian manifold is drastically small. In particular, if H acts irreducibly on a tangent space M_x but is not transitive on the unit sphere of M_x, then M must be *locally isometric to a symmetric space of rank* ≥ 2. (In 1962 Jim Simons gave a direct proof of this fact.) This startling fact lay dormant for almost twenty years before geometers realized that it provides a powerful tool for singling out the symmetric space of rank ≥ 2 among Riemannian manifolds. Among its striking applications, one can cite the characterization up to isometry of compact Hermitian symmetric spaces among compact Kähler manifolds in terms of bisectional curvature (N. Mok, 1988) and the characterization up to isometry of symmetric spaces of noncompact type of rank ≥ 2 among complete Riemannian manifolds of nonpositive sectional curvature in terms of geometric rank (P. Eberlein, W. Ballman, M. Gromov, and others, 1990). These results would be impossible without the work of Berger, and ultimately of Ambrose-Singer.

In 1955 Ambrose published the isometry theorem that now bears his name (Ann. Math. **64** (1956), 337–363). Equally influential was his work in the foundations of Riemannian geometry, which, after much delay, eventually made it to print in J. Indian Math. Soc. **24** (1960), 23–76. The fact that the basic theorems of Riemannian geometry should be developed using only properties of the Levi-Civita connection without the intervention of the calculus of variations is now taken for granted, but it was not so until Ambrose persuasively argued his case.

After 1960 Ambrose's interest turned to partial differential equations, and his publications soon reflected this shift. He had ambitious plans but did not live to see their fruition.

Those who knew Ambrose in the fifties remember his superb leadership and organizing ability that did so much to make the MIT department an exciting one. As a teacher Ambrose was noted for the clarity of his presentations. Since clarity in geometric writing was one quality that was in short supply back then, it comes as no surprise that Ambrose's lectures in geometry soon acquired an international audience through the writing of his friends and students.

Ambrose was a man of absolute integrity. His was not the temperament that would gladly suffer pretentiousness or dishonesty. Yet those close to him were also privileged to experience his self-deprecating humor and great kindness.

Ambrose's whole career was essentially spent at MIT. He retired in 1985 and moved to Paris in 1990. He is survived by his wife, Jeannette, and his children, Ellen and Adam, from a previous marriage.

H. Wu