Budding Mathematician Wins Westinghouse Competition

John H. Conway and Allyn Jackson

On the Work of Jacob Lurie

It is a particular pleasure for me to congratulate Jacob Lurie on his prize-winning entry for the Westinghouse Science Talent Search, not only because it is a magnificent piece of work, but also because it is about the surreal numbers, specifically, about computable surreal numbers.

However, it would be hard for me to describe the contents of Lurie's essay in any detail in an article addressed to lay mathematicians, even to professional ones, so I shall not even try. Instead, I shall merely set the stage for this particular play and then describe some of the actors and a few of the things that went on in the first few scenes.

I must first describe the surreal numbers, which are best thought of as the most natural collection of numbers that includes both the usual real numbers (which I shall suppose you know) and the infinite ordinal numbers discovered by Georg Cantor. Cantor's ordinal numbers

$$0, 1, 2, 3, \dots, \omega, \omega+1, \omega+2, \dots,$$
$$\omega+\omega, \omega+\omega+1, \dots \quad \dots \quad \dots$$

were used by him to count the members of arbitrary sets, and their defining principle can be roughly summarized by saying that no matter how many of them you've already seen, there will always be a next one later than all of those. If we write $\{\alpha, \beta, \gamma, \dots |\}$ for the ordinal number that comes first after all of $\alpha, \beta, \gamma, \dots$, then the above informal principle can be made slightly more precise by saying that the class of all ordinals is the closure of this operator.

We can regard this as giving us a sort of construction procedure for the ordinals: if $\alpha, \beta, \gamma, \dots$ are all the ordinals you've found so far, then you may construct from them the new ordinal $\{\alpha, \beta, \gamma, \dots |\}$. The first few steps of this procedure yield 0 and the other natural numbers:

$$\{|\} = 0, \quad \{0|\} = 1, \quad \{0,1|\} = 2, \quad \{0,1,2|\} = 3,$$

and so on. After all these comes

$$\{0, 1, 2, 3, \dots |\},$$

which Cantor called ω, and then

$$\{0, 1, 2, 3, \dots, \omega|\} = \omega + 1,$$

and so on.

There is no reason why the numbers before the vertical bar should be *all* the ordinals up to some point, but they might as well be. For example, the next number after both of 3 and 5 is 6, so that

$$\{3, 5|\} = 6 = \{0, 1, 2, 3, 4, 5|\},$$

and in general

$$\{\lambda, \mu, \nu, \dots |\} = \{\alpha, \beta, \gamma, \delta, \dots |\},$$

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where the numbers before the bar on the right-hand side are all those that are less than or equal to at least one of those before the bar on the left. Using this principle, we can see that the same ordinal can be defined or constructed in many different ways, for example

$$\omega = \{0, 1, 2, 3, \ldots |\} = \{0, 1, 4, 9, 16, \ldots |\} = \{1, 3, 9, 27, 81, \ldots |\}.$$

The surreals are obtained by generalizing this construction method. We now allow some numbers to appear to the right of the vertical bar, provided that they exceed all those that appear on its left, and interpret

$$\{\alpha, \beta, \gamma, \ldots | \delta, \epsilon, \ldots\}$$

as the simplest number ν that lies in between:

$$\alpha, \beta, \gamma, \ldots < \nu < \delta, \epsilon, \ldots .$$

Thus

$$\{0|1\} = \frac{1}{2} = \{0|1, 3\}, \quad \{|0\} = -1, \quad \{-1|5, 7\} = 0.$$

Once again the same number may have many different constructions. This procedure enables us to construct all the rational numbers, and indeed all the reals; for example,

$$\sqrt{2} = \{ \text{rational} \quad \alpha \quad \text{with} \quad \alpha^2 < 2 \quad | \quad \text{rational} \quad \beta > 0 \quad \text{with} \quad \beta^2 > 2\}.$$

However, it also finds some new numbers scattered in between the reals, such as

$$\{0|1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = \frac{1}{\omega},$$

which lies between 0 and all the positive real numbers, and of course it also finds all the ordinal numbers and some new numbers such as

$$\{0, 1, 2, 3, \ldots |\omega\} = \omega - 1$$

and

$$\{0, 1, 2, 3, \ldots |\omega, \omega - 1, \omega - 2, \omega - 3, \ldots\} = \frac{\omega}{2}.$$

That lamentably brief introduction to the surreals will have to suffice. I discovered them in 1969 and described their algebra and arithmetic in my book *On Numbers and Games*, published in 1976. Some of their applications are described in *Winning Ways*, which I wrote jointly with Elwyn Berlekamp and Richard Guy, and in *The Book of Numbers*, written jointly with Guy, which will appear this summer. I refer the reader who wants to know more to these books and to the references at the end of this article. Let us press on!

In the mathematics of this century there has been a tendency, when once a mathematical theory has been reasonably well understood, to develop a theory of that theory, which discusses the ways in which the objects of the original theory can be named, constructed, or computed. I shall use the the word "metatheory" for this new kind of theory but warn the reader that this usage differs from the standard one, although it has much of the same spirit.

The theory of recursive functions (those functions of integers that can be computed by a purely mechanical method) that was independently developed by a number of mathematicians in the 1930s may be regarded as a metatheory of the integers in this sense. The corresponding metatheory for the reals was set up in A. M. Turing's famous paper on computable real numbers.

These metatheories produced a lot of important mathematics; for instance, Turing's famous theorem that the halting problem for a computer is undecidable. Indeed, they led to what some people consider to be the most important mathematical theorem of all: Kurt Gödel proved his famous Incompleteness Theorem by showing how to arithmetize the metatheory of any formal logical system.

A number of metatheories of ordinal numbers have also been constructed, notably the Church-Kleene theory of notations for ordinals and various theories of computable ordinals.

I can (at last!) come to the subject of Jacob Lurie's prize-winning essay, namely, the "metatheory" in this sense of computable surreal numbers, as initiated by Leon Harkleroad. The difficulties are that there are many "obvious" ways to define notions of "computable surreal numbers" that are not equivalent, or at least not obviously equivalent, and Lurie's paper studies the various implications between these notions.

His writing is already very much like that of a professional mathematician, and any professional mathematician would be justly proud if he could produce arguments as subtle and deep as Lurie's. Of course, this makes it difficult to follow all of his work in complete detail, and I must confess that I have not yet done so. But the part that I did follow reads like the work of an established mathematician, and from a high school student it is astonishing.

It is a pleasure to congratulate Jacob Lurie on this magnificent piece of work and to welcome him to the fold!

—John H. Conway

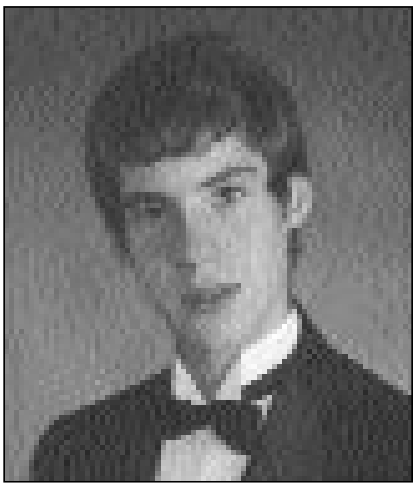
References

[1] John H. Conway, *On numbers and games*, Academic Press, 1976.

[2] DONALD KNUTH, *Surreal numbers*, Addison-Wesley, 1974.
[3] NORMAN L. ALLING, *Foundations of analysis over surreal number fields*, North-Holland, Amsterdam, 1978.
[4] ELWYN BERLEKAMP, JOHN H. CONWAY, and RICHARD K. GUY, *Winning ways*, Academic Press, 1982.
[5] JOHN H. CONWAY and RICHARD K. GUY, *The book of numbers*, to be published in 1996 by Springer-Verlag.
[6] HARRY GONSHOR, *An introduction to the theory of surreal numbers*, Cambridge University Press, 1986.

Young Mathematician Impresses Many

The press release about the 1996 Westinghouse Science Talent Search described the winner, Jacob Lurie, as "an 18-year-old mathematician from Bethesda, Maryland." Never mind that 18-year-olds are usually not thought of as having a vocation, much less a profession requiring years of study. The designation seems to be appropriate.



Jacob Lurie

Lurie took the first-place prize with a mathematics paper that brought together three branches of the field to examine the computable sets in the surreal numbers. Other winners submitted projects such as a sensor to detect ice accumulation on aircraft wings, or a plasma generator. In choosing the top entry, Science Service, which administers the competition, clearly did not aim at a project that has an immediate application or was easily explainable to the general public. "It's not really flashy, it's not an experiment, it's not chemicals exploding or things like that," says University of Maryland computer scientist William Gasarch, who is Lurie's mentor and attended the competition with him. "It was a total surprise to us that it won."

In another way, it was no surprise: Gasarch believes that Lurie's paper "could easily be half a Ph.D. thesis." Carol Wood, a mathematician at Wesleyan University, was a judge in the competition, which involves not only an assessment of the projects submitted but also interviews with the students participating. With Lurie, "his project was impressive, but he is more impressive than his project," says Wood. "It was lovely for me, a mathematician, to meet a kid asking the right questions, with enormous brain power and, in Jacob's case, a large dose of judgment and taste."

As it turns out, Gasarch's best friend and next-door-office neighbor at Maryland is Clyde Kruskal, son of mathematician Martin Kruskal of Rutgers University. The elder Kruskal has done research on surreal numbers with the inventor of the concept, John H. Conway of Princeton University. Kruskal knew when he met Lurie that he had won a gold medal at the 1994 International Mathematical Olympiad, at which the U.S. team made history with a perfect score for each team member. "So, of course, he was very brilliant," says Kruskal. But, with his paper on surreal numbers, "what he's done now is to go much beyond that."

"They are a very small percentage, but there are still quite a few students who are extremely quick, facile, and gifted mathematically" such as those on the Olympiad teams each year, Kruskal notes. "But to go on and study a subject in depth on their own—that's very unusual already. And then to go and write a paper on it—that's astounding." Kruskal says he cannot fully understand Lurie's paper, because it uses some parts of mathematics that he doesn't know well. "I can't judge it entirely," he says, "but it seems to be marvelous."

Gasarch, who notes that the paper is entirely Lurie's own work and that he himself doesn't understand all of it, says Lurie breaks the stereotype of a young, brilliant mathematician. "Stereotypically, I think of a younger person as someone who is quick and clever" and relies on those qualities to do research, he says. "I think of an older person as relying on a database of knowledge to do work. And that's what's so remarkable and unusual: this is an old man's paper. Of course, it's clever in parts, but it draws on a vast amount of knowledge, and therefore it's just not what you would expect from a high school student." In addition, he says, Lurie seems to have none of the usual "splits" one finds in mathematicians—he can think both geometrically and algebraically, he is a good problem creator and a good problem solver, he is a hard worker and also brilliant. "In Jacob's case, no matter what split you give him, he's good on both sides," says Gasarch.

What does Jacob Lurie think of all the accolades? It doesn't seem that winning this prestigious award, which includes a $40,000 scholarship that he intends to use at Harvard next year, has turned his head. Asked how he felt about having won the award, he replies, "Just like I did the week before." The clearest characteristics he shows in a short conversation are modesty and seriousness. He is also quiet and rather reserved but warms up to the topic of mathematics. When asked what draws him to mathematics, he explains that “mathematical structures are natural, more natural than in many other fields of knowledge.” While physics also has that quality, he feels that physics research is often diverted to trying to understand specialized phenomena rather than general principles. This is why he has a special interest in logic: “Questions in logic are the most fundamental.”

After he won the Westinghouse award, Lurie was interviewed by many newspapers, appeared on the *CBS Morning* program, and was featured on National Public Radio. What was the most fun? “I was contacted by someone at Indiana University,” Lurie says. “I mentioned his book on NPR, and he invited me to work on some stuff with him over the summer, which is stuff that I’m very interested in.” That someone is logician Jon Barwise, whose book, *Admissible Sets and Structures*, has probably never gotten this kind of press before. But that is not what really mattered to Barwise. “I must say, nothing in my professional life has pleased me more than the thought of Jacob off studying my book, written before he was born, being inspired by it, and doing something original with it”, Barwise said in an e-mail message to a friend.

What is in the future for Jacob Lurie? First Harvard, then graduate school in mathematics, then research. He intends to pursue his interest in logic and its connections to other areas of mathematics. “I also want to develop some analysis for the surreal numbers,” he says, “and I think that some issues related to logic are really strongly tied up in that.” He was asked on *CBS Morning* if he thought he would win the Nobel Prize in mathematics. He had the perfect answer, which revealed neither conceit nor lack of confidence: “There is no Nobel Prize in mathematics.” With his modesty, seriousness, and intelligence, this 18-year-old mathematician is doing some good mathematics—and spreading a lot of joy among the grownups.

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