Model Theory and Exponentiation

David Marker

Model theory is a branch of mathematical logic in which one studies mathematical structures by considering the first-order sentences true of those structures and the sets definable in those structures by first-order formulas.

The fields of real and complex numbers have long served as motivating examples for model theorists. Many model theoretic concepts arose by abstracting classical algebraic phenomena to a more general setting (see, for example, [29]). In the past five years significant progress has been made in the other direction. Model theoretic methods have been used to develop new insights into real analytic geometry. In particular, this has led to a generalization of semialgebraic and subanalytic geometry to a setting in which one studies global exponentiation on the reals.

Tarski's Theorem

The logical study of the field of real numbers began with the work of Tarski. These results were announced by Tarski in [33], but publication was interrupted by the war, and the proofs did not appear until [34]. Tarski was primarily interested in showing that there is an algorithm for deciding first-order sentences about the real numbers and hence showing the decidability of elementary Euclidean geometry.

Tarski's decidability of the real numbers stands in stark contrast to Gödel's undecidability of the natural numbers. The great freedom we have to solve polynomial equations in the real numbers insures that the real solution sets of systems of polynomial equations are topologically well behaved. In the natural numbers such solution sets can behave very badly. This wild behavior allows considerable latitude for coding complicated combinatorial phenomena, while, as we will see below, in the reals we cannot even define the integers.

The natural language for studying the real numbers is the language of ordered rings $\mathcal{L} = \{+, -, \cdot, <, 0, 1\}$. In this language the *basic $\mathcal{L}$-formulas* are of the form $p(v_1, \ldots, v_n) = q(v_1, \ldots, v_n)$ and $p(v_1, \ldots, v_n) < q(v_1, \ldots, v_n)$ where $p, q \in \mathbf{Z}[X_1, \ldots, X_n]$ for some $n \in \mathbf{N}$. The set of all $\mathcal{L}$-formulas is obtained by taking the basic $\mathcal{L}$-formulas and closing under the Boolean connectives $\wedge, \vee, \neg$ and quantifiers $\exists$ and $\forall$. For example,

$$\exists y \; x = y^2 - y \tag{1}$$

$$\forall x \exists y \; (x = y^2 \vee x + y^2 = 0) \tag{2}$$

$$\exists w \; (x - w^2 + xy = 0 \wedge w < 3y - 4) \tag{3}$$

are legitimate $\mathcal{L}$-formulas. We say that an occurrence of a variable in a formula is *free* if it is not inside the scope of a quantifier; otherwise,

---

David Marker is professor of mathematics at the University of Illinois at Chicago. His e-mail address is `marker@math.uic.edu`.

This paper is based on an invited lecture presented in October 1994 at the AMS Central regional meeting in Stillwater, Oklahoma.

Partially supported by NSF grants DMS-9306159 and INT-9224546, and an AMS Centennial Fellowship.

we say it is *bound*. For example, in (1) x is a free variable and y is bound, while in (2) both x and y are bound. If a formula has free variables, we often denote the formula in a way that explicitly shows which variables are free. For example, we might denote (3) by $\phi(x, y)$. We think of a formula $\phi(x_1, \dots, x_n)$ as expressing a relation that holds among the variables $x_1, \dots, x_n$. For example,

$$\exists w \; xw^2 + yw + z = 0 \tag{4}$$

expresses the relationship "x, y, z are coefficients of a solvable quadratic polynomial", while

$$\forall x \forall y \; ux^2 + vxy + zy^2 \geq 0 \tag{5}$$

expresses that u, v, z are the coefficients of a positive semidefinite binary quadratic form.

A formula is a *sentence* if it has no free variables. If $\mathcal{A}$ is an ordered ring, then any $\mathcal{L}$-sentence ϕ will either be true or false for $\mathcal{A}$. For example, (2) is true for the field of real numbers $\mathbf{R}$ but false for the field of rational numbers $\mathbf{Q}$. Formulas that are not sentences may hold for some assignments of the free variables but not for others. For example, in $\mathbf{Q}$ (4) holds for (1,5,6) but fails for (1,0,-2). Of course, (4) holds for (1,0,-2) in $\mathbf{R}$. The first-order theory of $\mathbf{R}$, $Th(\mathbf{R})$, is the set of all $\mathcal{L}$-sentences that are true about $\mathbf{R}$. Many important properties of $\mathbf{R}$ can be expressed as $\mathcal{L}$-sentences. For example, (2) expresses that for any x either x or $-x$ is a square. For each n the sentence

$$\forall a, b \forall w_0, \dots, w_n \, [(a < b \wedge (\sum_{i=0}^{n} w_i a^i)(\sum_{i=0}^{n} w_i b^i) < 0) \rightarrow \exists x(a < x < b \wedge \sum_{i=0}^{n} w_i x^i = 0)]$$

expresses that the sign change property holds for polynomials of degree n. One sees that all of the axioms for real closed ordered fields (see [21]) are contained in $Th(\mathbf{R})$. On the other hand, there is no first-order sentence that expresses the completeness of the ordering of $\mathbf{R}$. Indeed, we cannot hope to characterize the real numbers up to isomorphism by $Th(\mathbf{R})$, as the Löwenheim-Skolem theorem (see [3]) insures there will be models of $Th(\mathbf{R})$ of every infinite cardinality.

Tarski's Theorem. The first-order theory of the field of real numbers is decidable. That is, there is an algorithm that will decide on input ϕ whether or not ϕ is a sentence true of $\mathbf{R}$.

The proof of Tarski's theorem breaks into two steps.

Step 1: (Elimination of Quantifiers) If $\phi(v_1, \dots, v_n)$ is an $\mathcal{L}$-formula, then there is an equivalent quantifier-free $\mathcal{L}$-formula $\psi(v_1, \dots, v_n)$. (Here by equivalent we mean that ϕ holds of $\overline{a} = (a_1, \dots, a_n)$ if and only if ψ holds of $\overline{a}$ for all $\overline{a} \in \mathbf{R}^n$.) Moreover, there is an algorithm that transforms ϕ to ψ.

Note that quantifier-free formulas are just finite Boolean combinations of $p(x_1, \dots, x_n) = q(x_1, \dots, x_n)$ and $p(x_1, \dots, x_n) > q(x_1, \dots, x_n)$ where $p, q \in \mathbf{Z}[X_1, \dots, X_n]$ for some $n \in \mathbf{N}$.

We are familiar with some natural examples of quantifier elimination. For example, (4) is equivalent to the quantifier-free formula

$$y^2 - 4xz \geq 0.$$

Also, while the formula in free variables $\{v_{i,j} : 1 \leq i, j \leq n\}$ that expresses that the matrix $(v_{i,j})$ has an inverse involves existential quantifiers, it is equivalent to the quantifier-free formula that asserts that the determinant of $(v_{i,j})$ is nonzero.

Tarski's original proof relied on techniques from elimination theory and Sturm's algorithm. Other proofs are based on model-theoretic ideas (see, for example, [24]) or cylindric decomposition (see, for example, [2]).

The second step is just an easy observation.

Step 2: Quantifier-free sentences are finite Boolean combinations of ones of the form $m = n$ and $m > n$ where m and n are integers. These can be decided easily.

Thus, to decide if a sentence ϕ is true in $\mathbf{R}$, we first transform it into a quantifier-free sentence and then test to see if the quantifier-free sentence is true.

While Tarski's result proves the decidability of the theory of the real numbers, the algorithm obtained is far from feasible. The best-known algorithm for eliminating quantifiers is doubly exponential in the number of quantifier blocks [17], and it is known that even deciding first-order sentences for the ordered group $(\mathbf{R}, +, <)$ is exponentially hard [14].

It is also worth noting that for any formula $\phi(v_1, \dots, v_n)$, the formula $\psi(v_1, \dots, v_n)$ obtained by quantifier elimination is actually equivalent to $\phi(v_1, \dots, v_n)$ in any other real closed field as well. Since the decision made in Step 2 depends only on $\mathbf{Z}$, it follows that a sentence ϕ is true in $\mathbf{R}$ if and only if it is true in every real closed field if and only if it is true in some real closed field. This means that the axioms for real closed fields completely axiomatize $Th(\mathbf{R})$.

In [35] Tarski asked if his results could be extended to include the exponential function. Namely, let $\mathcal{L}_{\exp} = \mathcal{L} \cup \{e^x\}$. Is the first-order

$\mathcal{L}_{\exp}$-theory of $\mathbf{R}$ decidable?[1] We will give a partial answer to Tarski's problem in Section 4.

One might hope to generalize the arguments used for the ordered field, but serious difficulties arise. First, quantifier elimination fails. In [5], van den Dries showed that the formula

$$y > 0 \wedge \exists w \, (wy = x \wedge z = ye^w)$$

is not equivalent to a quantifier-free $\mathcal{L}_{\exp}$-formula.

There are rather silly ways to extend the language so that we do have elimination of quantifiers, but the trick is to extend the language so that we have quantifier elimination and the quantifier-free formulas are still natural and meaningful.

Even if we could eliminate quantifiers, there is no obvious uniform algorithm for deciding quantifier-free sentences like

$$e^{e^2-2} - e^5 = e^{3+e^{-3}},$$

since, as far as we know, there may be some perverse exponential-algebraic relations that hold in $\mathbf{Z}$. One expects that there are no surprising exponential algebraic relations holding between integers, but the only known proof of this relies on a conjecture in transcendental number theory.

Schanuel's Conjecture. Let $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ be linearly independent over $\mathbf{Q}$. Then $\mathbf{Q}(\lambda_1, \ldots, \lambda_n, e^{\lambda_1}, \ldots, e^{\lambda_n})$ has transcendence degree at least n over $\mathbf{Q}$.

Schanuel's conjecture generalizes the Lindemann-Weierstrass theorem, viz., if $\alpha_1, \ldots, \alpha_n$ are algebraic numbers that are linearly independent over $\mathbf{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbf{Q}$. Schanuel's conjecture has many nice consequences, such as the algebraic independence of e and π. In [22] Macintyre proved that if Schanuel's conjecture holds, one does not obtain any unexpected exponential-algebraic relations on $\mathbf{Z}$. This leads to a decision procedure for the quantifier-free $\mathcal{L}_{\exp}$-sentences.

Tarski also considered the theory of the field of complex numbers. Here too one can prove quantifier elimination and decidability. The theory of the complex field is just the theory of algebraically closed fields of characteristic zero. On the other hand, the theory of $(\mathbf{C}, +, -, \cdot, e^x)$ will behave very badly, since one can use the formula

$$\forall z, w \, (z^2 = -1 \wedge e^{wz} = 1 \rightarrow e^{xwz} = 1)$$

to define the integers. Since we can define $\mathbf{Z}$, Gödel's results show that this theory is far from decidable.

Definable Sets and o-Minimality

We say that $X \subseteq \mathbf{R}^n$ is *definable* using $\mathcal{L}$ if there is an $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $b_1, \ldots, b_m \in \mathbf{R}$ such that

$$X = \{(a_1, \ldots, a_n) \in \mathbf{R}^n : \phi(a_1, \ldots, a_n, b_1, \ldots, b_m) \text{ holds}\}.$$

For example, let $\phi(x_1, x_2, y)$ be the formula

$$\exists u \, (x_1 - u)^2 + (x_2 - u^2)^2 < y^2.$$

Substituting the real number $\frac{\pi}{2}$ for the variable y, we get the definable set

$$\{(x_1, x_2) : \text{the distance from } (x_1, x_2) \text{ to the parabola } y = x^2 \text{ is less than } \frac{\pi}{2}\}.$$

We say that a function $f : \mathbf{R}^n \to \mathbf{R}^m$ is definable if its graph $\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m : f(x) = y\}$ is definable.

By quantifier elimination every definable subset can be defined by a quantifier-free formula. This simple idea was given an important geometric reformulation by Seidenberg [31]. We say that a subset of $\mathbf{R}^n$ is *semialgebraic* if it is a finite Boolean combination of sets of the form $\{\overline{x} = (x_1, \ldots, x_n) : f(\overline{x}) > 0\}$ and $\{\overline{x} : g(\overline{x}) = 0\}$ where $f, g \in \mathbf{R}[X_1, \ldots, X_n]$. It is easy to see that the semialgebraic sets are exactly the sets defined by quantifier-free formulas. Thus the semialgebraic sets are exactly the definable sets. If X is a semialgebraic subset of $\mathbf{R}^{n+1}$, then the projection of X to $\mathbf{R}^n$ is $\{\overline{x} \in \mathbf{R}^n : \exists y \, (\overline{x}, y) \in X\}$. Since this is definable, it is also semialgebraic[2]

These are very useful ideas. For example, let $A \subseteq \mathbf{R}^n$ be definable. Then the closure of A is

$$\{\overline{x} \in \mathbf{R}^n : \forall \epsilon > 0 \; \exists \overline{y} \, \overline{y} \in A \wedge \sum_{i=1}^{n} (x_i - y_i)^2 < \epsilon\},$$

a definable set. Thus the closure of a semialgebraic set is semialgebraic.

Semialgebraic subsets of $\mathbf{R}$ are particularly simple. For any polynomial $f(X) \in \mathbf{R}[X]$, $\{x \in \mathbf{R} : f(x) > 0\}$ is a finite union of open intervals. Thus any semialgebraic subset of $\mathbf{R}$ is a finite union of points and intervals. This simple fact is the starting point of the modern model-theoretic approach to $\mathbf{R}$. Let $\mathcal{L}^*$ be a language extending $\mathcal{L}$, and let $\mathbf{R}^*$ denote the reals as an

[1] *Of course, even if this theory is decidable, decision computations will, in general, be unfeasible.*

[2] *Similarly for the field of complex numbers the definable sets are exactly the constructible sets of algebraic geometry; quantifier elimination is equivalent to Chevalley's theorem that the projection of a constructible set is constructible.*

$\mathcal{L}^*$-structure. For example, $\mathcal{L} = \mathcal{L}_{\exp}$ and $\mathbf{R}^* = \mathbf{R}_{\exp} = (\mathbf{R}, +, -, \cdot, e^x, <, 0, 1)$. We say that $\mathbf{R}^*$ is *o-minimal* if every subset of $\mathbf{R}$ definable using $\mathcal{L}^*$-formulas is a finite union of points and intervals. Pillay and Steinhorn [27] introduced o-minimality, generalizing an earlier idea of van den Dries [5]. Remarkably, o-minimality has strong consequences for definable subsets of $\mathbf{R}^n$ for $n > 1$.

We inductively define the notion of *cells* as follows

- $X \subseteq \mathbf{R}$ is a cell if and only if it is either a point or an interval.
- If $X \subseteq \mathbf{R}^n$ is a cell and $f : X \to \mathbf{R}$ is a continuous definable function, then the graph of f is a cell.
- If $X \subseteq \mathbf{R}^n$ is a cell and $f, g : X \to \mathbf{R}$ are continuous definable functions and $f(x) > g(x)$ for all $x \in X$, then $\{(x, y) : x \in X$ and $f(x) > y > g(x)\}$ is a cell, as are $\{(x, y) : x \in X$ and $f(x) > y\}$ and $\{(x, y) : x \in X$ and $y > f(x)\}$.

Theorem. (van den Dries [5]/Knight-Pillay-Steinhorn[20])

i) If $\mathbf{R}^*$ is o-minimal, then every definable set X can be partitioned into finitely many disjoint cells.

ii) If $f : X \to \mathbf{R}$ is a definable function, then there is a partition of X into finitely many cells such that f is continuous on each cell.

This is just the beginning. In any o-minimal structure, definable sets have many of the good topological and geometric properties of the semialgebraic sets (see, for example, [6] and [11]). For example:

- Any definable set has finitely many connected components.
- Definable bounded sets can be definably triangulated.
- Suppose $X \subseteq \mathbf{R}^{n+m}$ is definable. For $a \in \mathbf{R}^m$ let

  $$X_a = \{\overline{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n : (\overline{x}, a) \in X\}.$$

  There are only finitely many definable homeomorphism types for the sets X_a.
- (Curve selection) If $X \subseteq \mathbf{R}^n$ is definable and a is in the closure of X, then there is a continuous definable $f : (0, 1) \to X$ such that

  $$\lim_{x \to 1} f(x) = a.$$
- (Pillay [26]) If G is a definable group, then G is definably isomorphic to a Lie group.
- If we assume in addition that all definable functions are majorized by polynomials, then many of the metric properties of semialgebraic sets and asymptotic properties of semialgebraic functions also generalize.

One is led to the question: Is the real field with exponentiation o-minimal? We will see Wilkie's positive answer to this question in the section "Wilkie's Theorem".

$\mathbf{R}_{\mathrm{an}}$ and Subanalytic Sets

Most of the results on o-minimal structures mentioned above were proved before we knew of any interesting o-minimal structures other than the real field. In [4] van den Dries gave the first new example of an o-minimal theory.

Let $\mathcal{L}_{\mathrm{an}} = \mathcal{L} \cup \{\hat{f}$: for some open $U \supset [0, 1]^n$, $f : U \to \mathbf{R}$ is analytic $\}$.

We define $\hat{f} : \mathbf{R}^n \to \mathbf{R}$ by

$$\hat{f}(x) = \begin{cases} f(x) & x \in [0, 1]^n \\ 0 & \text{otherwise.} \end{cases}$$

We let $\mathbf{R}_{\mathrm{an}}$ be the resulting $\mathcal{L}_{\mathrm{an}}$-structure. Denef and van den Dries [7] proved that $\mathbf{R}_{\mathrm{an}}$ is o-minimal and that $\mathbf{R}_{\mathrm{an}}$ has quantifier elimination if we add $x \mapsto \frac{1}{x}$ to the language. Quantifier elimination is proven by using the Weierstrass preparation theorem to replace arbitrary analytic functions of several variables by analytic functions that are polynomial in one of the variables. Tarski's elimination procedure is then used to eliminate this variable.

Denef and van den Dries also showed that if $f : \mathbf{R} \to \mathbf{R}$ is definable in $\mathbf{R}_{\mathrm{an}}$, then f is asymptotic to ax^q for some $a \in \mathbf{R}$ and $q \in \mathbf{Q}$. In particular, although we can define the restriction of the exponential function to bounded intervals, we cannot define the exponential function globally. It is also impossible to define the sine function globally, for its zero set would violate o-minimality.

Although $\mathbf{R}_{\mathrm{an}}$ may seem unnatural, the definable sets form an interesting class.

We say that $X \subseteq \mathbf{R}^n$ is *semi-analytic* if for all x in $\mathbf{R}^n$ there is an open neighborhood U of x such that $X \cap U$ is a finite Boolean combination of sets $\{\overline{x} \in U : f(\overline{x}) = 0\}$ and $\{\overline{x} \in U : g(\overline{x}) > 0\}$ where $f, g : U \to \mathbf{R}$ are analytic. We say that $X \subseteq \mathbf{R}^n$ is *subanalytic* if for all x in $\mathbf{R}^n$ there is an open U and $Y \subset \mathbf{R}^{n+m}$ a bounded semianalytic set such that $X \cap U$ is the projection of Y into U. It is well known (see [1] for an excellent survey) that subanalytic sets share many of the nice properties of semialgebraic sets.

If $X \subset \mathbf{R}^n$ is bounded, then X is definable in $\mathbf{R}_{\mathrm{an}}$ if and only if X is subanalytic. Indeed $Y \subseteq \mathbf{R}^n$ is definable in $\mathbf{R}_{\mathrm{an}}$ if and only if it is the image of a bounded subanalytic set under a semialgebraic map. Most of the known properties of subanalytic sets generalize to sets defined in any polynomial bounded o-minimal theory.

Wilkie's Theorem

The big breakthrough in the subject came in 1991. While quantifier elimination for $\mathbf{R}_{\text{exp}}$ is impossible, Wilkie [37] proved the next best thing.

Wilkie's Theorem. Let $\phi(x_1,\ldots,x_m)$ be an $\mathcal{L}_{\text{exp}}$ formula. Then there is $n \geq m$ and $f_1,\ldots,f_s \in \mathbf{Z}[x_1,\ldots,x_n,e^{x_1},\ldots,e^{x_n}]$ such that $\phi(x_1,\ldots,x_n)$ is equivalent to

$$\exists x_{m+1}\ldots\exists x_n\, f_1(x_1,\ldots,x_n,e^{x_1},\ldots,e^{x_n}) = \ldots = f_s(x_1,\ldots,x_n,e^{x_1},\ldots,e^{x_n}) = 0.$$

Thus every formula is equivalent to an existential formula (this property is called *model completeness*), and every definable set is the projection of an exponential variety.

Wilkie's proof depends heavily on the following special case of a theorem of Khovanski [19]. Before Wilkie's theorem, Khovanski's result was the best evidence that $\mathbf{R}_{\text{exp}}$ is o-minimal; indeed Khovanski's theorem is also the crucial tool needed to deduce o-minimality from model completeness.

Khovanski's Theorem. If $f_1,\ldots,f_m : \mathbf{R}^n \to \mathbf{R}$ are exponential polynomials, then $\{x \in \mathbf{R}^n : f_1(x) = \ldots f_n(x) = 0\}$ has finitely many connected components.

If $X \subseteq \mathbf{R}$ is definable in $\mathbf{R}_{\text{exp}}$, then by Wilkie's theorem there is an exponential variety $V \subseteq \mathbf{R}^n$ such that X is the projection of V. By Khovanski's theorem, V has finitely many connected components and X is a finite union of points and intervals. Thus $\mathbf{R}_{\text{exp}}$ is o-minimal.

Using the o-minimality of $\mathbf{R}_{\text{exp}}$, one can improve some of Khovanski's results on "fewnomials". From algebraic geometry we know that we can bound the number of connected components of a hypersurface in $\mathbf{R}^n$ uniformly in the degree of the defining polynomial. Khovanski [19] showed that it is also possible to bound the number of connected components uniformly in the number of monomials in the defining polynomial. We will sketch the simplest case of this. Let $\mathcal{F}_{n,m}$ be the collection of polynomials in $\mathbf{R}[X_1,\ldots,X_n]$ with at most m monomials. For $p \in \mathcal{F}_{n,m}$ let

$$V^+(p) = \{\overline{x} = (x_1,\ldots,x_n) \in \mathbf{R}^n : \bigwedge_{i=1}^{n} x_i \geq 0 \wedge p(\overline{x}) = 0\}.$$

We claim that there are only finitely many homeomorphism types of $V^+(p)$ for $p \in \mathcal{F}_{n,m}$. Let $\Phi_{m,n}(x_1,\ldots,x_n,r_{1,1},\ldots,r_{1,n},\ldots,r_{m,1},\ldots,r_{m,n},a_1,\ldots,a_m)$ be the formula

$$\exists w_1,\ldots,w_m\, \Big(\bigwedge_{i=1}^{m} e^{w_i} = x_i\Big) \wedge \sum_{i=1}^{m} a_i \prod_{j=1}^{n} e^{w_i r_{i,j}} = 0).$$

We see that Φ expresses

$$\sum_{i=1}^{m} a_i \prod_{j=1}^{n} x_j^{r_{i,j}} = 0.$$

Let $X_{\overline{r},\overline{a}}$ denote the set of $\overline{x} \in \mathbf{R}^n$ such that $\Phi(\overline{x},\overline{r},\overline{a})$ holds. By o-minimality, $\{X_{\overline{r},\overline{a}} : \overline{r} \in \mathbf{R}^{mn}, \overline{a} \in \mathbf{R}^m\}$ represents only finitely many homeomorphism types.

In addition to answering the question of o-minimality, some headway has been made on the problem of decidability. Making heavy use of Wilkie's methods and Khovanski's theorem, Macintyre and Wilkie [23] have shown that if Schanuel's conjecture is true, then the first-order theory of $\mathbf{R}_{\text{exp}}$ is decidable. One would expect that the computational complexity of this theory would be horrific, but the only known lower bounds are those imposed by the theory of the additive ordered group. Macintyre and Wilkie also gave an axiomatization of $Th(\mathbf{R}_{\text{exp}})$ (assuming Schanuel's conjecture); unfortunately the axioms are quite complicated and ugly.

Miller [25] provided an interesting counterpoint to Wilkie's theorem. Using ideas of Rosenlicht [30], he showed that if $\mathbf{R}^*$ is any o-minimal expansion of the real field that contains a function that is not majorized by a polynomial, then exponentiation is definable in $\mathbf{R}^*$.

$\mathbf{R}_{\text{an, exp}}$

Let $\mathcal{L}_{\text{an,exp}}$ be $\mathcal{L}_{\text{an}} \cup \{e^x\}$, and let $\mathbf{R}_{\text{an, exp}}$ be the real numbers with both exponentiation and restricted analytic functions. In [8] van den Dries, Macintyre, and I examined this structure. Using the Denef-van den Dries quantifier elimination for $\mathbf{R}_{\text{an}}$ and a mixture of model-theoretic and valuation theoretic ideas, we were able to show that $\mathbf{R}_{\text{an, exp}}$ has quantifier elimination if we add log to the language. Using quantifier elimination and Hardy field style arguments (but avoiding the geometric type of arguments used by Khovanski), we were able to show that $\mathbf{R}_{\text{an, exp}}$ is o-minimal. Using arguments similar to Wilkie's, o-minimality was also proved in [10].

Since the language $\mathcal{L}_{\text{an,exp}}$ has size $2^{\aleph_0}$, one would not expect to give a simple axiomatization of the first-order theory of $\mathbf{R}_{\text{an, exp}}$. Ressayre noticed that the model-theoretic analysis of $\mathbf{R}_{\text{an, exp}}$ uses very little global information about exponentiation. This observation leads to a "relative" axiomatization. The theory $Th(\mathbf{R}_{\text{an, exp}})$ is axiomatized by the theory of $\mathbf{R}_{\text{an}}$ and by axioms asserting that exponentiation is an increasing homomorphism from the additive group onto the multiplicative group of positive elements that majorizes every polynomial.

Using this axiomatization and quantifier elimination, one can show that any definable function is piecewise given by a composition of polynomials, exp, log, and restricted analytic functions on $[0,1]^n$. For example, the definable function $f(x) = e^{e^x} - e^{x^2} - 3x$ is eventually increasing and unbounded. Thus for some large enough $r \in \mathbf{R}$ there is a function $g : (r, +\infty) \to \mathbf{R}$ such that $f(g(x)) = x$ for $x > r$. The graph of g is the definable set $\{(x,y) : x > r$ and $e^{e^y} - e^{y^2} - 3y = x\}$. Thus g is a definable function, and there is some way to express g explicitly as a composition of rational functions, exp, log, and restricted analytic functions. In most cases it is in no way clear how to get these explicit representations of an implicitly defined function. One important corollary is that every definable function is majorized by an iterated exponential.

Logarithmic-Exponential Series

The axiomatization of the theory of $\mathbf{R}_{\text{an, exp}}$ leads to a relatively natural algebraic construction of nonstandard models of $\mathbf{R}_{\text{an, exp}}$. These models can be used to study asymptotic properties of definable functions. For example, we use them to answer a problem of Hardy.

We first show how to build many nonstandard models of $\mathbf{R}_{\text{an}}$.

Suppose $(\Gamma, +)$ is an abelian group and $<$ is an ordering of Γ compatible with addition. For formal sums

$$f = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma$$

we define the *support* of f to be the set $\{\gamma \in \Gamma : a_\gamma \neq 0\}$ and let $\mathbf{R}((t^\Gamma))$ be the set of formal sums $f = \sum a_\gamma t^\gamma$ such that the support of f is well ordered. We can make $\mathbf{R}((t^\Gamma))$ an ordered field by defining addition and multiplication in the usual way and $f = \sum a_\gamma t^\gamma > 0$ iff $a_{\gamma_0} > 0$ where γ_0 is least such that $a_{\gamma_0} \neq 0$. If $r \in \mathbf{R}$ and $r > 0$, then $t < r = rt^0$. Thus t is an infinitesimal element. If Γ is also a $\mathbf{Q}$-vector space, then $\mathbf{R}((t^\Gamma))$ is a real closed field and hence a nonstandard model of the theory of the real field.

It is possible to view all of the restricted functions as acting on $\mathbf{R}((t^\Gamma))$. For example, if $x \in [0,1]$, then we can write $x = r + \epsilon$ where $r \in \mathbf{R} \cap [0,1]$ and ϵ is infinitesimal. Then we define e^x as $e^r \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}$. Using the Denef-van den Dries quantifier elimination, one shows that $\mathbf{R}((t^\Gamma))$ is a model of the theory of $\mathbf{R}_{\text{an}}$.

To build a nonstandard model of exponentiation, we start with $\Gamma_0 = \mathbf{R}$. We cannot define a total exponential on $\mathbf{R}((t^{\mathbf{R}}))$ because t^{-1} is infinite, so we want $\exp(t^{-1}) > t^r$ for all $r \in \mathbf{R}$. To remedy this, we extend Γ_0 to $\Gamma_1 = \{\sum a_r t^r : a_r = 0 \text{ for all } r > 0\} \oplus \mathbf{R}$ which we order lexiographically. We then define an exponential $\exp : \mathbf{R}((t^{\Gamma_0})) \to \mathbf{R}((t^{\Gamma_1}))$ by

$$\exp(\alpha + r + \epsilon) = t^{-\alpha} e^r \sum \frac{\epsilon^n}{n!}$$

where $r \in \mathbf{R}$, ϵ is infinitesimal and all elements of the support of α are negative. We will still not be able to define an exponential of $\exp(t^{-1})$, so we must repeat this process infinitely many times. The resulting field will have an exponentia, but not every positive element will have a logarithm. So a further infinite construction needs to be done to add logs (for details see [9]).

The resulting model, which we call $\mathbf{R}((t))^{\text{LE}}$, is very useful for examining the asymptotic behavior of definable functions, as the formal series representing the function reflects its asymptotic expansion. Consider the function $f(x) = (\log x)(\log\log x)$, and let g be a compositional inverse to f defined on $(r, +\infty)$ for some $r \in \mathbf{R}$. In [16] Hardy conjectured that g is not asymptotic to a composition of exp, log and semialgebraic functions. Shackell [32] came close to verifying this conjecture by proving that the inverse to $(\log\log x)(\log\log\log x)$ is not asymptotic to such a composition.[3] Using the series expansion for g in $\mathbf{R}((t))^{\text{LE}}$, we are able to show that Hardy's conjecture is correct (see [9]).

We also use asymptotic expansion in $\mathbf{R}((t))^{\text{LE}}$ to show, for example, that $\Gamma|(0,+\infty)$, $\int_0^x e^{t^2} dt$ and the bold Riemann zeta function restricted to $(1,+\infty)$ are not definable in $\mathbf{R}_{\text{an, exp}}$.

And Beyond

The most pressing current problem in the subject is to find o-minimal expansions of $\mathbf{R}$ allowing richer collections of functions. Recently van den Dries and Speissegger [12], building on work of Tougeron [36] and Gabrielov [15], have shown how to prove model completeness and o-minimality for expansions containing functions with divergent Taylor series coming from certain quasi-analytic classes. For example, they have shown that one can add a function with domain $(0,\epsilon)$ and asymptotic expansion

$$\sum_{n=1}^{\infty} x^{\log n}.$$

There is some hope that in a sufficiently rich o-minimal expansion of $\mathbf{R}$ one would have all of the Poincaré return maps for polynomial vector fields in $\mathbf{R}^2$. If this is true, it is likely that one could give a new proof that such a vector field

[3] *On the other hand, the inverse to* $x \log x$ *is asymptotic to* $x/\log x$.

has only a finite number of limit cycles.[4] Moreover, one expects that such a proof would yield a bound on the number of limit cycles uniform in the degrees of the polynomials defining the vector field. We are far from answering this, and undoubtedly this problem will fuel the subject for some time to come.

References

[1] E. BIERSTONE and P. MILMAN, *Semialgebraic and subanalytic sets*, IHES Publ. Math. **67** (1988), 5–42.

[2] J. BOCHNAK, M. COSTE, and M. F. ROY, *Géométrie algébrique réelle*, Springer-Verlag, 1986.

[3] C. C. CHANG and H. J. KEISLER, *Model theory*, North-Holland, 1977.

[4] J. DENEF and L. VAN DEN DRIES, *p-Adic and real subanalytic sets*, Ann. Math. **128** (1988), 79–138.

[5] L. VAN DEN DRIES, *Remarks on Tarski's problem concerning* $(\mathbf{R}, +, \cdot, \exp)$, Logic Colloquium '82 (G. Longi, G. Longo, and A. Marcja, eds.), North-Holland, 1984.

[6] ____, *Tame topology and o-minimal structures*, monograph in preparation.

[7] ____, *A generalization of the Tarski-Seidenberg theorem, and some nondefinability results*, Bull. Amer. Math. Soc. (N. S.) **15** (1986), 189–193.

[8] L. VAN DEN DRIES, A. MACINTYRE, and D. MARKER, *The elementary theory of restricted analytic fields with exponentiation*, Ann. of Math. **140** (1994), 183–205.

[9] ____, *Logarithmic-exponential power series*, J. London Math. Soc. (to appear).

[10] L. VAN DEN DRIES and C. MILLER, *On the real exponential field with restricted analytic functions*, Israel J. Math. **85** (1994), 19–56.

[11] ____, *Geometric categories and o-minimal structure*, Duke Math. J. (to appear).

[12] L. VAN DEN DRIES and P. SPEISSEGGER, *The real field with convergent generalized power series is model complete and o-minimal*, preprint.

[13] J. ECALLE, *Finitude des cycles-limites et accéléro-sommation de l'application de retour*, Bifurcations of Planar Vector Fields (J. P. Françoise and R. Roussarie, eds.), Lecture Notes in Math., vol. 1455, Springer-Verlag, 1990, pp. 74–159.

[14] M. J. FISCHER and M. O. RABIN, *Super-exponential complexity of Pressburger arithmetic*, Complexity of Computation (R. Karp, ed.), Proc. SIAM-AMS Sympos. Appl. Math., Amer. Math. Soc., 1974.

[15] A. GABRIELOV, *Existential formulas for analytic functions*, preprint.

[16] G.H. HARDY, *Properties of logarithmico-exponential functions*, Proc. London Math. Soc. **10** (1912), 54–90.

[17] J. HEINTZ, M-F. ROY, and P. SOLERNO, *Complexité du principe de Tarski-Seidenberg*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), 825–830.

[18] Y. IL IASHENKO, *Finiteness theorems for limit cycles*, Transl. Math. Monographs, vol. 94, Amer. Math. Soc., 1991.

[19] A. HOVANSKII, *On a class of systems of transcendental equations*, Soviet Math. Dokl. **22** (1980), 762–765.

[20] J. KNIGHT, A. PILLAY, and C. STEINHORN, *Definable sets in ordered structures II*, Trans. Amer. Math. Soc. **295** (1986), 593–605.

[21] S. LANG, *Algebra*, Addison-Wesley, 1971.

[22] A. MACINTYRE, *Schanuel's conjecture and free exponential rings*, Ann. Pure Appl. Logic **51** (1991), 241–246.

[23] A. MACINTYRE and A. WILKIE, *On the decidability of the real exponential field* (to appear).

[24] D. MARKER, *Introduction to the model theory of fields*, Model Theory of Fields (D. Marker, M. Messmer, and A. Pillay, eds.), Lecture Notes in Logic, vol. 5, Springer-Verlag, 1996.

[25] C. MILLER, *Exponentiation is hard to avoid*, Proc. Amer. Math. Soc. **122** (1994), 257–259.

[26] A. PILLAY, *Groups and fields definable in o-minimal structures*, J. Pure Appl. Algebra **53** (1988).

[27] A. PILLAY and C. STEINHORN, *Definable sets in ordered structures I*, Trans. Amer. Math. Soc. **295** (1986), 565–592.

[28] J. P. RESSAYRE, *Integer parts of real closed exponential fields*, Arithmetic, Proof Theory, and Computational Complexity (P. Clote and J. Krajicek, eds.), Oxford Univ. Press, 1993, pp. 278–288.

[29] A. ROBINSON, *Model theory as a framework for algebra*, Studies in Model Theory (M. Morley, ed.), Math. Assoc. Amer., 1973.

[30] M. ROSENLICHT, *The rank of a Hardy field*, Trans. Amer. Math. Soc. **280** (1983), 659–671.

[31] A. SEIDENBERG, *A new decision method for elementary algebra*, Ann. of Math. **60** (1954), 365–374.

[32] J. SHACKELL, *Inverses of Hardy's L-functions*, Bull. London Math. Soc. **25** (1993), 150–156.

[33] A. TARSKI, *Sur les ensembles définissables de nombres réels*, Fund. Math. **17** (1931), 210–239.

[34] ____, *A decision method for elementary algebra and geometry* (prepared for publication by J. C.C. McKinsey), RAND Corp. monograph, 1948.

[35] ____, *The completeness of elementary algebra and geometry*, Institut Blaise Pascal, Paris, 1967 (a reprint of page proofs of a 1940 work).

[36] J.-CL. TOUGERON, *Sur les ensembles semi-analytiques avec conditions Gevrey au bord*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 173–208.

[37] A. J. WILKIE, *Some model completeness results for expansions of the ordered field of reals by Pfaffian functions and exponentiation*, J. Amer. Math. Soc. (to appear).

---

[4] *This is part of Hilbert's 16th problem that was solved by Ecalle [13] and Il'iashenko [18].*