# Finsler Geometry Is Just Riemannian Geometry without the Quadratic Restriction

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In his *Habilitationsvortrag* of 1854, Riemann introduced a metric structure in a general space based on the element of arc

$$ds = F(x^1, \dots, x^n;\ dx^1, \dots, dx^n)\ . \tag{1}$$

Here, $F(x, y)$ is a positive (when $y \neq 0$) function on the tangent bundle $TM$ and is homogeneous of degree one in $y$. An important special case is when

$$F^2 = g_{ij}(x)\, dx^i dx^j\ . \tag{2}$$

Historical developments have conferred the name Riemannian geometry to this case while the general case, Riemannian geometry without the quadratic restriction (2), has been known as Finsler geometry.

The name "Finsler geometry" came from Finsler's thesis of 1918. It is actually the geometry of a simple integral and is as old as the calculus of variations. Hilbert attached great importance to the field, and in his famous Paris address of 1900 devoted Problem 23 to the variational calculus of $\int ds$ and its geometrical overtones. A rewrite (using Euler's theorem for homogeneous functions) of the integrand here—the Hilbert form—will be discussed in the following section.

Finsler geometry is *not* a generalization of Riemannian geometry. It is better described as Riemannian geometry without the quadratic restriction (2). A special case in point is the interesting paper [11]. They studied the Kobayashi metric of the domain bounded by an ellipsoid in $\mathbb{C}^2$, and their calculations showed that the Kobayashi metric on such a domain exhibits many of the nice properties of a Riemannian metric but is plainly not quadratic.

There are developments in Finsler geometry in recent years which deserve attention. It has been shown that modern differential geometry provides the concepts and tools to effect a treatment of Riemannian geometry, without the quadratic restriction, in a direct and elegant way so that all results, local and global, are included. This not only gives a better understanding of the geometry but opens a vista comparable to the developments of algebraic geometry from quadrics to general algebraic varieties.

Our report will be divided into the following sections:

1. Connections and the equivalence problem
2. The second variation of arc length and comparison theorems
3. Harmonic theory
4. Complex Finsler geometry
5. The Gauss-Bonnet formula

These topics do not cover all the important recent developments in Finsler geometry, nor do they highlight the pivotal role played by the local theory. For a more comprehensive overview of the subject, see the proceedings volume [8] (especially the prefaces) and references therein.

## Connections and the Equivalence Problem

The fundamental problem in local Finsler geometry is the equivalence problem: To find a complete system of invariants or to decide when

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two Finsler metrics differ by a coordinate transformation. In the Riemannian case this was the form problem, solved in 1870 by E. B. Christoffel and R. Lipschitz. In his solution Christoffel introduced the requisite covariant differentiation, which Ricci developed into his tensor analysis, making it a fundamental tool in classical differential geometry.

The key idea in Finsler geometry is to consider the projectivized tangent bundle $PTM$ (i.e., the bundle of line elements) of the manifold $M$. The main reason is that all geometric quantities constructed from $F$ are homogeneous of degree zero in $y$ and thus naturally live on $PTM$, even though $F$ itself does not.

To describe one such quantity, let $x^i$, $1 \leqq i \leqq n$, be local coordinates on $M$. Express tangent vectors as $y = y^i \frac{\partial}{\partial x^i}$, so that $(x^i, y^i)$ can be used as local coordinates of $TM$ and, with $y^i$ homogeneous, as local coordinates of $PTM$. The function $F$ in (1) gives rise to a function $F(x^1, \dots, x^n;\ y^1, \dots, y^n)$, linearly homogeneous in the $y$'s. The fundamental tensor $g_{ij}$ is defined as the $y$-Hessian $(\frac{1}{2} F^2)_{y^i y^j}$. Each $g_{ij}$ is homogeneous of degree zero in $y$, hence lives on $PTM$. In case $F^2$ is quadratic, these $g_{ij}$'s simply reduce to the usual $g_{ij}(x)$'s of (2), which then also live on $M$.

The tangent and cotangent bundles, $TM$ and $T^*M$ respectively, over $M$, when pulled back to $PTM$, have distinguished properties. The object $g_{ij}\, dx^i \otimes dx^j$ defines a scalar product on each fibre of these bundles. With respect to this scalar product, the pulled-back $TM$ admits a globally defined section $\ell := \frac{y^i}{F} \frac{\partial}{\partial x^i}$ of unit length. Its natural dual is, modulo a rewrite, the integrand of Hilbert's invariant integral in the calculus of variations, namely, the one-form

$$\omega = \frac{\partial F}{\partial y^i}\ dx^i \ . \qquad (3)$$

We will call it the Hilbert form. This is a global section of the pulled-back $T^*M$, and it has unit length. One can also view it as a globally defined linear differential form on $PTM$. By Euler's theorem, one can rewrite the length integral $\int_a^b ds$ on $M$ as $\int_a^b \omega$.

The Hilbert form is a powerful piece of data. Hilbert devoted almost all his discussion of the 23rd problem to it. As an example of its utility, we mention that a typical nondegeneracy hypothesis (cf. [13, 14]) on $F$, when expressed in terms of $\omega$, reads

$$\omega \wedge (d\omega)^{n-1} \neq 0 \ .$$

That is, $\omega$ defines a contact structure on $PTM$.

As a second example, consider the differential ideal $\{\omega, d\omega\}$, $\{\omega \wedge d\omega, (d\omega)^2\}$, $\dots$, $\{\omega \wedge (d\omega)^{n-1}\}$ generated by the Hilbert form. Now, $PTM$ is a Riemannian manifold with a natural metric induced by the $g_{ij}$'s. In [10] it has been shown that, with respect to the Laplace-Beltrami operator on $PTM$, the first pair of forms has eigenvalue $(n-1)$, the second pair $2(n-2)$, and so on. Also, the last item, namely, the contact form itself, is harmonic.

It turns out that by simply taking the exterior derivative of $\omega$ one can define a connection in the pulled-back bundles over $PTM$. This connection is characterized by being torsion-free, together with the property that the scalar product remains invariant when the canonical section $\ell$ is parallel displaced. In the Riemannian case this construct reduces to the Christoffel-Levi-Civita connection. For a detailed treatment using moving frames, see [14]. An explicit formula of this connection in natural coordinates can be found in [18] or [10].

The above connection gives connection forms in the relevant principal bundles and leads to a solution of the equivalence problem in a standard way.

Exterior differentiation of the connection forms gives the notion of curvature, as quadratic exterior differential forms. Their components consist of two curvature tensors, $R_j{}^i{}_{kl}$ and $P_j{}^i{}_{kl}$. To get numerical invariants, one considers a flag based at $x \in M$, with flagpole $0 \neq y \in T_xM$ and a unit length transverse edge $V$ which is perpendicular to the flagpole. Here, the length of $V$ and its orthogonality to $y$ are measured by $g_{ij}(x, y)\, dx^i \otimes dx^j$. The flag curvature is then $K(x, y, V) := V^i\, (\ell^j\, R_{jikl}\, \ell^l)\, V^k$. It takes the place of the sectional curvature in the Riemannian case. It is also independent of whichever of the standard connections (among several !) we choose and can even be obtained from a dynamical systems point of view (cf. Foulon [15]).

To close this section, let us mention a result of Akbar-Zadeh's [3]: A compact boundaryless Finsler manifold is locally Minkowskian if and only if it has zero flag curvature.

## The Second Variation of Arc Length and Comparison Theorems

A systematic study of global Riemannian geometry began with Heinz Hopf. His 1932 report in the *Jahresberichte der deutschen Mathematiker Vereinigung* was a landmark. Hopf and Rinow introduced the notion of completeness, and in his thesis Hopf gave a description of all the Clifford-Klein space forms, i.e., the Riemannian manifolds of constant sectional curvature.

A natural question in this area is the relation between curvature and topology. Of particular interest is the study of Riemannian manifolds whose sectional curvature keeps a constant sign. A fundamental tool for this study is a formula for the second variation of arc length. Here the remarkable fact is that the same formula holds, with the sectional curvature in the Riemannian case replaced by the flag curvature in the general Finsler case. As a consequence all the classical theorems, such as the Hadamard-Cartan theorem on manifolds of nonpositive curvature, the Bonnet-Myers theorem, the Synge theorem, the first comparison theorem of Rauch as well as the Bishop-Gromov volume comparison theorem, extend to the Finsler setting. For details, see [5, 6, 13, 17].

There has been a great development on Riemannian manifolds of positive curvature. A parallel theory of Finsler manifolds of positive flag curvature deserves to be worked out. As a start, the case of constant positive flag curvature is gradually being understood; see Shen [18].

Since $PTM$ plays a central role in Finsler geometry, the most fundamental invariant should be the Ricci scalar $Ric$. It is a scalar function on $PTM$ and is defined as $g^{ik}(\ell^j R_{jikl}\,\ell^l)$. Its companion (cf. [4]) is the Ricci tensor $Ric_{jk} := (\frac{1}{2} F^2 Ric)_{y^j y^k}$.

A natural question is: Can every manifold be given a Finsler metric with constant Ricci scalar or perhaps one whose Ricci scalar does not depend on $y$? It is well known that, in dimension $\geq 3$, such a Riemannian metric does not always exist. For some interesting (albeit preliminary) considerations in the Finsler case, see the article [4] by Akbar-Zadeh.

The study of the deformation theory of Finsler structures is an equally worthwhile endeavor and should also be pursued.

## Harmonic Theory

A fundamental work in global Riemannian geometry is Hodge's theory of harmonic forms. Briefly it says that on a compact Riemannian manifold without boundary every cohomology class has a unique harmonic representative. The main ingredient is the definition of the Laplace operator $\Delta$ (on $M$, *not* on $PTM$), since a form is harmonic when it is annihilated by the Laplacian.

D. Bao and B. Lackey succeeded in formulating the definition of a Laplacian for Finsler manifolds which reduces to the standard one in the Riemannian case. Their work makes use of an unusual scalar product on the $p$-forms on the manifold $M$. If $d^*$ denotes the adjoint of the exterior differentiation operator relative to this scalar product, then the Laplacian is defined by the usual formula

$$\Delta = d \circ d^* + d^* \circ d \ ,$$

where $d$ is ordinary exterior differentiation. Their definition of the scalar product comes from a careful study of the Hodge star operator on $PTM$ and involves integration over the fibers of $PTM$. For details cf. [9].

With this Laplacian, which is an elliptic operator, we have Hodge's theorem:

***Theorem.*** On a compact oriented Finsler manifold without boundary, every cohomology class has a unique harmonic representative. The dimension of the space of all harmonic forms of degree $p$ is the $p$-th Betti number of the manifold.

The introduction of the Laplacian opens up a whole host of problems, for example Bochner-type vanishing theorems and eigenvalue estimates.

## Complex Finsler Geometry

It is possible that Finsler geometry will be most useful in the complex domain, because every complex manifold, with or without boundary, has a Caratheodory pseudo-metric and a Kobayashi pseudo-metric. Under favorable (though somewhat stringent) conditions these are $C^2$ metrics and, most importantly, they are *naturally* Finslerian. The analysis on the manifold is thus intimately tied to the geometry. For an explicit example of the Kobayashi metric, see [11].

Complex Finsler geometry is extremely beautiful. Again the bundle of line elements $PTM$ plays the important role. The scalar product on the pulled-back $TM$ gives rise to a hermitian structure on the complexification of the latter. Here the geometrical properties mix well with the complex structure; connection forms are of type (1,0) and curvature forms are of type (1,1). A real-valued holomorphic curvature, as a function on $PTM$, can be introduced.

From this viewpoint an important class of complex manifolds consists of those whose Kobayashi metric has constant holomorphic curvature. When it is a negative constant or zero, they have been studied by Abate and Patrizio, cf. [2]. The case of positive constant holomorphic curvature deserves investigation.

## The Gauss-Bonnet Formula

Among the relationships between curvature and topology is the Gauss-Bonnet formula. In his thesis published in 1925 Hopf proved this formula for an orientable closed hypersurface of even dimension in a Euclidean space, expressing its Euler characteristic as an integral of a curvature function. As a result of using tubes, C. B. Allendoerfer and W. Fenchel extended it to an arbitrary submanifold in Euclidean space (1940). In 1943 Allendoerfer and Weil extended the formula to a Riemannian polyhedron; their study of the boundary solid angles is masterful.

It turns out that the key idea lies in the consideration of the bundle $SM$ of the unit tangent vectors of $M$, as Chern showed in his proof of the Gauss-Bonnet formula

$$\int_M -\Omega \ = \ \chi(M)$$

by transgression. The extension of the Gauss-Bonnet formula to Finsler geometry has been considered in [7], in which the relevant curvature forms were introduced and studied. Several factors occur in this generalization. The most notable one is the role played by the total volume of $S_xM$ at each point $x \in M$. A Gauss-Bonnet formula is obtained whenever this volume function is constant (on $M$), though its value is not necessarily equal to that of the Riemannian case. Thus Finsler geometry has features to make the study of this problem interesting.

The relation of other characteristic classes, particularly the Pontryagin classes, to curvature remains to be studied.

## Conclusion

I believe I have shown that almost all the results of Riemannian geometry can be developed in the Finsler setting. It is remarkable that one needs only some conceptual adjustment, no essential new ideas being necessary. This not only implies more general results but also gives a better geometrical understanding.

At this point it might be interesting to quote Riemann:

> In space, if one expresses the location of a point by rectilinear coordinates, then $ds = \sqrt{\Sigma(dx^i)^2}$. Space is therefore included in this simplest case. The next simplest case would perhaps include the manifolds in which the line element can be expressed as the fourth root of a differential expression of the fourth degree. Investigation of this more general class would actually require no essential different principles, but it would be rather time-consuming [*zeitraubend*, in the original German] and throw relatively little new light on the study of Space, especially since the results cannot be expressed geometrically.

As is well known, Riemannian geometry can be handled, elegantly and efficiently, by tensor analysis on $M$. Its handicap with Finsler geometry arises from the fact that the latter needs more than one space, for instance $PTM$ in addition to $M$, on which tensor analysis does not fit well. However, this problem can be remedied by working on $TM$ and making sure that all constructions are invariant under rescaling in $y$.

Riemann's emphasis on Riemannian geometry could be based on the Pythagorian nature of the metric. His allusion to general Finsler geometry was a remarkable insight. After more than a century of mathematical development, his vision was justified.

In addition to what is yet to be done on the subject in order for some obvious questions to be answered, I am inclined to think that future developments lie in further generalizations. The geometry of a metric space is always an attractive subject. Finsler geometry has been studied from this vantage point by A. D. Alexandrov [1], H. Busemann [12], and M. Gromov [16]. A combination of geometric and analytic methods remains a challenging open field.

This brief report has addressed the raison d'être of Finsler metrics. For example,

a. In the function theory of several complex variables, the Kobayashi and Caratheodory metrics are naturally Finslerian and user friendly; they also render holomorphic mappings distance decreasing.
b. For comparison theorems except Toponogov's, the effortless replacement of sectional curvature by flag curvature extends their validity to the Finsler category. Admittedly, despite Akbar-Zadeh's seminal results [3] the concept of space forms is more complicated than that in Riemannian geometry, but workers in the field seem to embrace this as a refreshing challenge.
c. Recent work on Hodge theory has brought forth some natural but unorthodox elliptic operators, together with some clues as to which geometrical data are more important than others.

Finslerian constructs also assert themselves in applications, most notably in control theory, mathematical biology/ecology, and optics. Nevertheless, in spite of the above arguments about the relevance and timeliness of the Finslerian viewpoint, Riemannian geometry will remain a most important chapter of Finsler geometry.

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