

Optional Mathematics Is Not Optional

John F. Price

The mathematical practice and theory of option pricing has many surprises. Who would have thought that to price an option on a stock you can completely ignore the expected value of the stock price and only consider its standard deviation? Or that options are redundant, meaning that their outcomes can be completely replicated by buying and selling bonds and the underlying stock? Another unexpected fact is that there is an exact formula (or algorithm) for the price of an option; if the quoted price differs from the calculated price, then you can always make a profit, called an arbitrage profit, no matter what the market does. Of course, all these results depend upon your chosen model of the marketplace, but extensive empirical tests show that pricing options in the usual setting based on geometric Wiener processes and the assumption of no arbitrage possibilities is reasonably accurate.

Buying and Selling Risk Using Options

It is unusual for companies to do business without fire insurance. By purchasing such insurance, they are shifting the financial consequences of a major fire to the insurance company. Now increasing numbers of companies are realizing that they need to do something similar for other areas such as energy prices or foreign exchange costs.

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Suppose you are the financial controller of a major airline and you expect that over the next two years your company will be purchasing specified quantities of aviation fuel every month. Market indicators suggest that prices will not exceed the current price by more than 5 percent. To prepare a budget for the next two years, you would like a guarantee that you will never have to pay above this 5 percent ceiling. On the other hand, you would like to be able to take advantage of any lower prices over the planning period. What financial strategies are available to achieve this goal—and how much would they cost?

Or imagine that because you are able to borrow money at 2 percent above the U.S. three-month Treasury rate, your real concern is that fuel costs and interest rates are both going to go up. In addition, you really only want protection for the average cost of fuel and interest payments over the next two years. This means that your main concern is with the average of some type of weighted sum of fuel costs and the Treasury rate. You want a strategy that reimburses you for the amount this average is above a particular level but that does not penalize you if the difference is negative.

The required protection in both these cases can be obtained by using options. In the first case, the company could purchase a string of call options expiring at the end of each month with strikes equal to 5 percent over the current price of aviation fuel. These could be purchased (at least for gasoline) on the New York Mercantile Exchange. The option needed in the second case

is a type of average basket option that could be purchased from many large financial institutions.

In both these examples, the corporation already has a risk and they are willing to pay someone to take over some or all of it. But from whom

would it buy these options? Often the individual or corporation on the other side has no risk, but they are willing to take it on for a fee. Or perhaps they are aviation fuel producers whose risk is opposite to that of the airline company. In the jargon of the area, all these transactions are reshaping their risk profiles.

These are simplified versions of typical problems that are being tackled every day throughout the world using powerful mathematical tools principally from probability, stochastic calculus, differential and integral equations, statistics, and numerical methods. Related option-based problems occur in asset allocation and fund management. Hence, at least for the finance industry, optional mathematics is not optional.

European and American Options

Starting with options on the purchase and sale of tulip bulbs in Holland during the seventeenth century, a bewildering range of types of options (averaging, barrier, compound, digital, and other "exotic" species) are today offered on almost every conceivable asset and financial indicator, including stocks, stock indices, agricultural products (corn, soybeans, etc.), livestock, oil and gas, metals (copper, gold, silver, etc.), currencies (Japanese yen, British pounds, etc.), mutual funds, and bonds. Even options on "catastrophes" have recently been introduced for the (re)insurance market by an option exchange.

The following discussion will be confined to foreign exchange options, since most of the interesting aspects of the theory can be found there.

Consider options on Australian dollars (AUD) in terms of U.S. dollars. Purchasing a *European call option* on AUD with expiration time T and strike K gives the purchaser the right to buy one Australian dollar at time T for a price of K dollars. Let S_T denote the price of one AUD (that is, the exchange rate) at time T ; if $S_T \leq K$, the expiration value of the option is zero; if $S_T > K$, the expiration value is $S_T - K$, since the option holder can purchase one AUD for K and immediately sell it for S_T . In the first case, the option is said to expire out-of-the-money, in the second in-the-money. Note that unless there has been some special agreement, the holder of an option

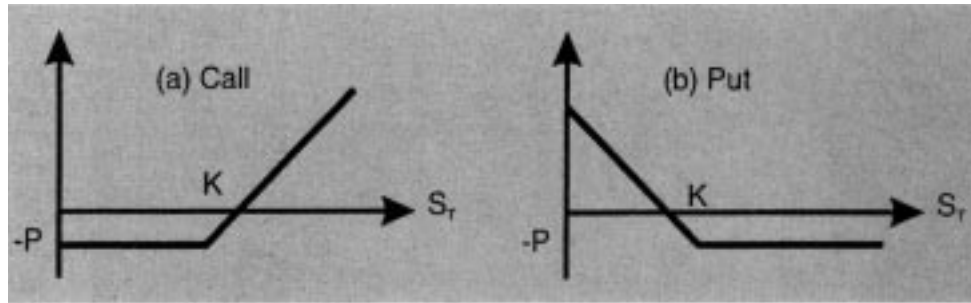


Figure 1. Profit and loss at expiration.

that is expiring in-the-money does not actually have to buy the Australian dollars (or 5,000 bushels of soybeans or 40,000 pounds of live cattle!), but just receives the difference $S_T - K$ in cash. If P is the price paid for the option, then the final profit and loss profile is $(S_T - K)^+ - P$ as shown in Figure 1(a).

A *European put option* with expiration T and strike K gives the right to sell one AUD at time T for K dollars. Figure 1(b) shows its payoff profile $(K - S_T)^+ - P$.

Most options traded on stock and futures markets around the world are, however, American options. These are similar to their European cousins except that they can be exercised at any time on or before the expiration date. As we shall see, this introduces some intriguing mathematical questions.

Binomial Model for Option Pricing

The paper that showed that European option pricing could be put on a rational mathematical basis was Black and Scholes [1] published in 1973. It was so revolutionary that the authors had to submit it to a number of journals before it was accepted. Although there are now numerous approaches to the result, they mostly require specialized methods, including Itô calculus and partial differential equations, and perhaps Girsanov theory and Feynman-Kac methods.

But it is the binomial method due initially to Sharpe [13] and substantially extended by Cox, Ross, and Rubinstein [3] that made the theory of option pricing accessible to everyone with limited mathematical background. Even though it requires only routine algebraic manipulations, the method is still able to elucidate many of the ideas behind the full theory. Furthermore, all the surprising results mentioned in the opening can be located in this approach. For these reasons it is usually the first method presented in textbooks and finance courses; we shall follow this trend and step through it. The binomial method is, however, much more than a pedagogical breakthrough, since it allows for the development of numerical approximation methods for a wide range of options for which there are no known analytic solutions.

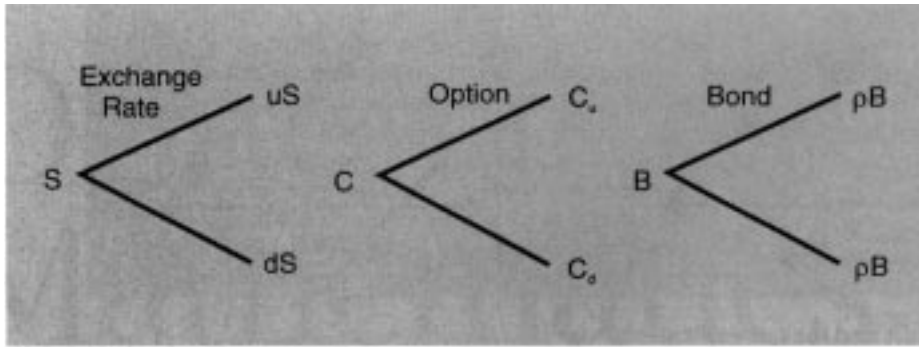


Figure 2. Movements for the binomial model.

The general “first approximation” assumptions for the mathematical theory of option pricing are that there are no transaction costs and taxes, there is a constant riskless interest rate and dividend rate, and markets are competitive so that there are no riskless arbitrage possibilities. This means that if two financial portfolios have the same values at some time in the future under all market scenarios, then their initial values must be equal.

The defining additional assumption for the binomial model is that in one unit Δt of time the market can go to just one of two states. For instance, the current price S of the currency AUD can move to either uS or dS , where $d < u$. Also suppose that \$1 invested in a riskless (or default-free) U.S. bond returns ρ (face value plus interest) to the investor after time Δt . Similarly, suppose that such an investment of one AUD in Australia returns η AUD at the end of the period. This means that if the AUD moves to uS , one AUD would generate wealth $u\eta S$. Similarly for a d movement.

Consider a European call option on Australian dollars with expiration time T and strike K . Suppose that by some means we know that the value of such an option is C_u if the exchange rate moves to uS and C_d otherwise. These possible movements are shown in Figure 2 where C , the current value of the option, is what we wish to determine.

To avoid arbitrage we must have $d\eta < \rho < u\eta$. Suppose that we form a portfolio consisting of Δ units of AUD and an investment of B in (riskless) U.S. bonds for a total value of $\Delta S + B$. If the market goes to uS , this portfolio is worth $\Delta u\eta S + \rho B$; and if it goes to dS , it is worth $\Delta d\eta S + \rho B$. The question now is, does there exist Δ and B so that this portfolio matches the value of the option after the movement of the market? Clearly the answer is yes; the required values of Δ and B are the solution of the set of equations

$$(1) \quad \Delta u\eta S + \rho B = C_u, \quad \Delta d\eta S + \rho B = C_d,$$

namely,

$$(2) \quad \Delta = \frac{C_u - C_d}{\eta S(u - d)}, \quad B = \frac{uC_d - dC_u}{\rho(u - d)}.$$

With these values, in the absence of arbitrage, the value $\Delta S + B$ of the original portfolio must be equal to the initial option price C and so

$$(3) \quad C = \frac{1}{\rho} (pC_u + (1 - p)C_d), \quad \text{where } p = \frac{\rho/\eta - d}{u - d}.$$

Hence we have obtained an explicit value for the option price C , given that we know the option values C_u and C_d .

Since we know the value of a European call option at expiration is $(S_T - K)^+$, it is routine to partition the interval $[0, T]$ into n equal parts to build an n -step binomial tree for calculating the initial price of a European call (or put) option. As before, we just assume that at each partition point the market can go to one of two states. Then we work backwards step-by-step from the known values at time $t = T$ by applying equation (3) to calculate the option price at each node. This results in the foreign-exchange version of the binomial option pricing formula. Figure 3 shows the n -step tree with the payoffs and their weights. The binomial theorem is attained by discounting the expected payoff back to the present time.

Theorem 1. The binomial model price C of a European call option with expiration time T and strike K is

$$C = S\eta^{-n}\Phi[a; n, p'] - K\rho^{-n}\Phi[a; n, p]$$

where

$$\Phi[b; m, q] = \sum_{j=b}^m \binom{m}{j} q^j (1 - q)^{m-j},$$

$$p = \frac{\rho/\eta - d}{u - d}, \quad p' = \frac{u\eta}{\rho} p,$$

$$a = \text{smallest } j \text{ satisfying } u^j d^{n-j} S - K \geq 0.$$

The steps of the proof show how to construct what is called a self-financing trading strategy for currency and bonds that replicates the behavior of an option. The strategy starts at time $t = 0$ with a mixture or portfolio of currency AUD and bonds such that it is equal in value to the initial price of the option. Then, at each subsequent partition time, by solving equations of the form (1), it describes the quan-

tivity of currency to sell and the quantity of bonds to buy with the proceeds, or vice versa, so that, at all the partition times, the value of the portfolio (as a function of the exchange rate) exactly matches the value of the option. In particular, the portfolio and the option are equal at the expiration time T .

The binomial method is also the most common approach of pricing American options. Since these are options that can be exercised at any time before their expiration dates, at each node the maximum of the exercise value and the backward-induction value as computed using equation (3) replaces this latter value. This means, however, that there is no closed form summation formula, with the result that the method requires a substantial increase in computational time.

Equivalent Martingale Measures

Even though we have obtained our goal for pricing a European call option (put options can be handled similarly) in the binomial setting, there is still much to learn and a few puzzles to resolve. For example, we have obtained the value of C without any mention of probabilities. Let's analyze this more carefully.

Start with the one-step case. Since $d < \rho/\eta < u$, p defined by equation (3) satisfies $0 < p < 1$. Now assign probabilities p and $1 - p$ to the uS and dS events respectively. (Notice that this has nothing to do with the true or observed probabilities of these events.) Then

$$C = \tilde{E}\left(\frac{1}{\rho}C_{\Delta t}\right),$$

where $C_{\Delta t}$ denotes the random variable of the option price at time $t = \Delta t$ and \tilde{E} the expected value with respect to the assigned probability.

But what is this new probability? Turning to the price of Australian dollars, the \tilde{E} -expected value of an investment in a single AUD after the time interval Δt is $\eta\tilde{E}(S_{\Delta t})$, where $S_{\Delta t}$ is the random variable of the AUD price at that time. However,

$$\begin{aligned} \eta\tilde{E}(S_{\Delta t}) &= \eta puS + \eta(1 - p)dS \\ &= \eta\left(\frac{\rho/\eta - d}{u - d}u + \frac{u - \rho/\eta}{u - d}d\right)S = \rho S. \end{aligned}$$

In other words, under this assigned probability, the expected value of an AUD investment after one unit of time equals the value if the same

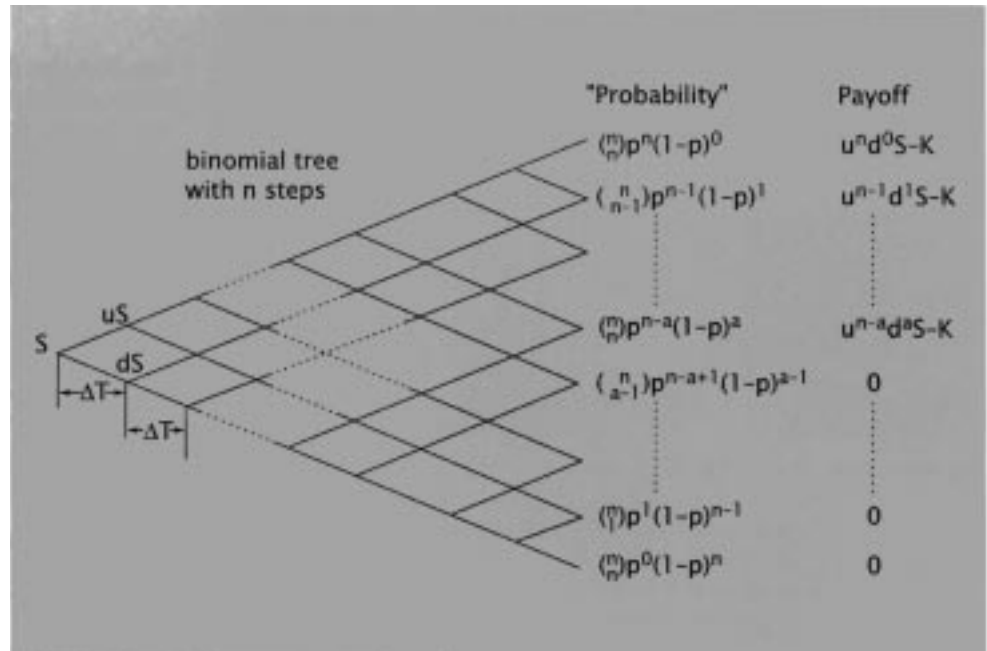


Figure 3. Binomial tree.

amount of money had been invested in local bonds. This is referred to as *risk-neutrality*.

Let $S_0 = S$ and extend the discussion to an n -step binomial tree as a binomial random process with probabilities p and $1 - p$ at each step. Given that we are at the k -th step, the expected value at the $k + 1$ step multiplied by η/ρ will be precisely the value at the k -th step. Formally,

$$\tilde{E}\left(\frac{\eta}{\rho}S_{(k+1)\Delta t} \mid S_0, S_{\Delta t}, \dots, S_{k\Delta t}\right) = S_{k\Delta t}.$$

This is precisely the condition for the process $(\eta/\rho)S_t$, $t = 0, \Delta t, 2\Delta t, \dots$, to be a martingale, a type of stochastic process used to describe "fair" games. For this reason, the assigned probability structure is called an *equivalent martingale measure*.

Theorem 1 can now be rewritten to state that the price of a call option is the expected value of the payoff random variable $(S_T - K)^+$ (calculated with respect to the equivalent martingale measure) discounted to present time.

Theorem 2. Under the conditions for Theorem 1, the price of a call option is

$$C = \tilde{E}(\rho^{-n}(S_T - K)^+)$$

We shall see that this general form reemerges in a continuous format on the way to the Black-Scholes theorem.

Black-Scholes Formula for Option Pricing

The mathematical theory of option pricing started with the 1900 dissertation of Louis Bachelier, who used continuous-time stochastic

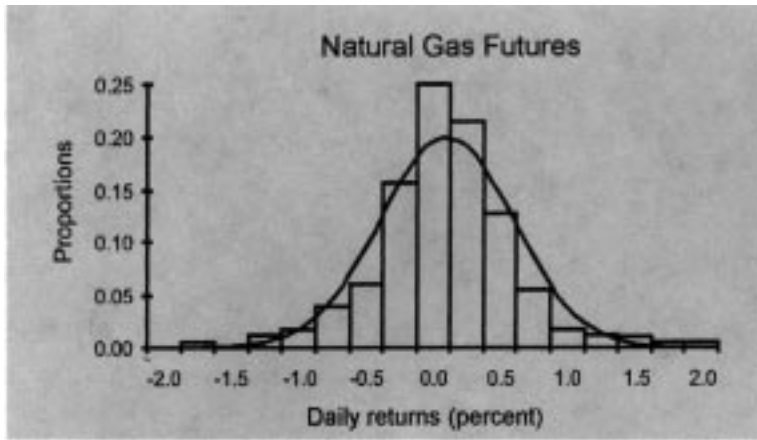


Figure 4. Two years of daily natural gas futures data.

processes to model and price options and derivatives. His work was unknown for over half a century, and it wasn't until the work of such people as Arrow, Debreu, Lintner, Markowitz, Merton, Miller, Modigliani, Samuelson, Sharpe, and Tobin that current mathematical techniques were again applied to finance theory.

It was, however, the seminal paper by Black and Scholes [1] that showed just how effective stochastic methods could be in modeling a vital part of applied finance, namely, the pricing of options. The primary assumption is that the price ($S_t : t \in \mathbb{R}^+$) of the underlying asset or exchange rate follows a geometric Wiener process:

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = S$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and ($W_t : t \in \mathbb{R}^+$) is a standard Wiener process or Brownian motion defined on a probability space (Ω, \mathcal{F}, P) . In addition, suppose that this space is equipped with the augmented filtration $F = (\mathcal{F}_t : t \in \mathbb{R}^+)$ generated by W . All subsequent processes will be defined on $(\Omega, \mathcal{F}, F, P)$, a filtered probability space.

Perhaps a reason for the ubiquity of this process in finance theory is that markets are driven by small upward and downward nudges through purchases and sales. The size of these nudges are roughly proportional to the price of the asset; now assume everything is independent, let the size of the nudges go to zero, and apply the central limit theorem to achieve a geometric Wiener process.

Figure 4 compares the histogram of the daily returns (that is, the logarithms of ratios $S_{t+\Delta t}/S_t$, where S_t and $S_{t+\Delta t}$ represent prices on consecutive trading days) of natural gas futures prices to a normal probability density function with the same mean and standard deviation. The natural gas data, in common with most other markets, is heteroskedastic, meaning that it is more peaked or has fatter tails than a normal distribution.

Our aim in the remainder of this section is to indicate the main ideas for a proof of the Black-Scholes theorem in the setting of the exchange rate between the U.S. and Australia. An assump-

tion for the theorem is that in these countries there are risk-free bonds paying constant continuously compounded interest rates; denote the rates by r and y respectively.

The midpoint of most of the proofs of the Black-Scholes theorem is the establishment of a partial differential equation relating the option price to the input variables. This can be achieved through constructing a self-financing trading strategy, the continuous analog of the trading strategy described above in the binomial setting, that precisely replicates the effect of owning an option.

A *trading strategy* is a rule for buying and selling an asset. More specifically, it is a predictable process ($a_t : t \in \mathbb{R}^+$) defined on $(\Omega, \mathcal{F}, F, P)$. (Predictability or nonanticipation is a technical requirement that ensures that there can be no "clairvoyance" involved in the strategy.) If a_t units are purchased at time t and sold at time $t + \Delta t$, the profit is

$$\begin{aligned} a_t(S_{t+\Delta t} - S_t) + a_t S_t(\exp(y\Delta t) - 1) \\ \approx a_t \Delta S_t + a_t S_t y \Delta t, \end{aligned}$$

the first term representing the profit from a change in the price S_t of the asset and the second from interest paid on the asset. Taking the limit in the usual sense of stochastic calculus suggests defining the profit resulting from the trading strategy as the sum

$$\int_0^t a_u dS_u + y \int_0^t a_u S_u dt.$$

Now suppose that we have a trading strategy (a, b) with a the strategy for AUD and b the strategy for U.S. bonds where the differential equation for bond prices is $d\beta_t = r\beta_t dt$. Such a strategy is called *self-financing* if the value of the resulting portfolio at any time equals the initial value plus the intermediate profits or losses from following the strategy; no extra funds are added or withdrawn after the initial investment. The condition for this strategy to be self-financing is that for each t

$$(4) \quad V_t - V_0 = \int_0^t a_u dS_u + y \int_0^t a_u S_u du + \int_0^t b_u d\beta_u$$

where $V_s = a_s S_s + b_s \beta_s$.

Take as our goal to price a European option that at time T pays $g(S_T)$ for some measurable function g with polynomial growth. For example, $g(x) = (x - K)^+$ leads to a European call option. We want to find a self-financing strategy (a, b) such that

$$a_T S_T + b_T \beta_T = g(S_T).$$

In this case, $V_t = a_t S_t + b_t \beta_t$ is referred to as the *arbitrage price* at time t of the option that pays $g(S_T)$ at time T .

If the option was traded at time t for any price C_t other than this, then an arbitrage profit can be

made. For example, if $d = C_t - (a_t S_t + b_t \beta_t) > 0$, sale of the option together with implementation of the trading strategy locks in a profit of d . As a result, the trading strategy (a, b) describes a portfolio of assets and bonds with value $V_t = a_t S_t + b_t \beta_t$ equal, at each time t , to the value of the option.

Suppose that the value process V_t is of the form $V_t = f(S_t, \tau)$ for $0 \leq \tau \equiv T - t \leq T$, where $f \in C^{2,1}((0, \infty) \times [0, T])$. From Itô's lemma:

$$(5) \quad \begin{aligned} V_t - V_0 &= \int_0^t f_x(S_u, T - u) dS_u \\ &- \int_0^t f_\tau(S_u, T - u) du \\ &+ \frac{1}{2} \sigma^2 \int_0^t S_u^2 f_{xx}(S_u, T - u) du. \end{aligned}$$

Comparison of equations (4) and (5) shows that the process (V_t) comes from a self-financing strategy $V_t = a_t S_t + b_t \beta_t$ with the required value when $t = T$ provided f satisfies the parabolic partial differential equation

$$(6) \quad \frac{1}{2} \sigma^2 x^2 f_{xx} + (r - y) x f_x - r f - f_\tau = 0$$

with boundary values

$$(7) \quad \begin{aligned} f(x, 0) &= g(x), \quad f(0, \tau) = \exp(-r\tau)g(0) \\ &\text{for } x \geq 0, \quad 0 \leq \tau \leq T. \end{aligned}$$

Originally Black and Scholes solved equation (6) by transforming it to the standard heat equation, at least when $g(x) = (x - K)^+$ or $(K - x)^+$. Duffie [4] shows how it can be done using Feynman-Kac theory. The method that currently has the widest ramifications for financial theory is due to Harrison and Kreps [6] and Harrison and Pliska [7]. It uses Girsanov's theorem to convert the underlying measure P so that the discounted price process $\exp(-(r - y)(T - t))S(t)$ becomes a martingale. In terms of expected values conditional on knowing the exchange rate at time t , the solution to equation (6) is:

Theorem 3. The arbitrage price at time t of an option paying $g(S_T)$ at time T is

$$\begin{aligned} V_t &= f(S_t, \tau) = \tilde{E}(e^{-r\tau} g(S_T) | S_t) \\ &= E(e^{-r\tau} g(\tilde{S}_T) | S_t) \end{aligned}$$

where $\tau = T - t$ and (\tilde{S}_t) is the process described by

$$d\tilde{S}_t = \tilde{S}_t((r - y)dt + \sigma dW_t), \quad \tilde{S}_0 = S.$$

Notice the similarity with Theorem 2 and that the parameter μ does not appear. The expected values in the theorem can be evaluated as

$$V_t = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(S_t e^{(r-y-\frac{1}{2}\sigma^2)\tau + \sqrt{\tau}\sigma u}) e^{-u^2/2} du.$$

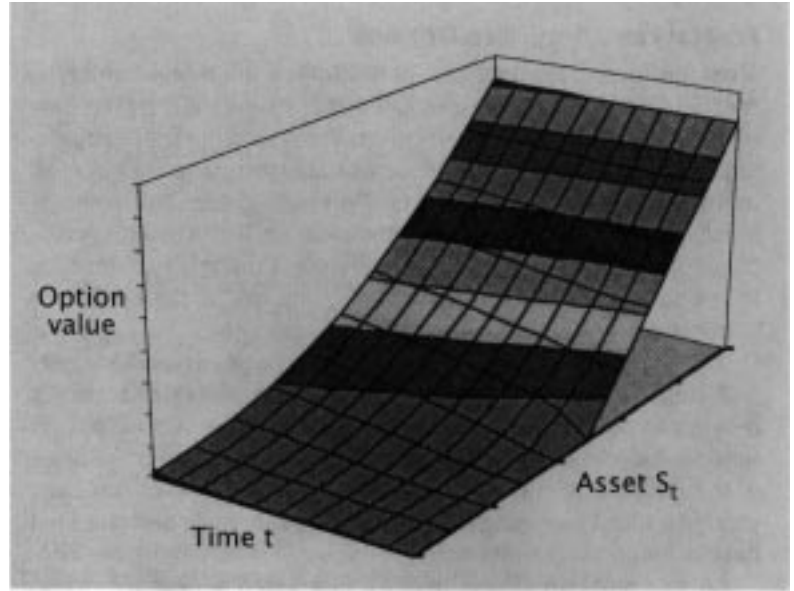


Figure 5. European call option values as a function of t and S_t .

When $g(x) = (x - K)^+$, this collapses to the Garman-Kohlhagen [5] formula:

$$(8) \quad V_t = e^{-y\tau} S_t N(d_1) - e^{-r\tau} K N(d_2)$$

where N is the cumulative normal distribution and

$$d_1, d_2 = \frac{\log(S_t/K) + (r - y \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Figure 5 displays the value of a call option as function of time t and the exchange rate S_t . When $y = 0$, (8) is the standard expression for the Black-Scholes formula for a European call option on an asset that does not pay dividends.

Finally we mention that it is a subtle question to determine the financial implications of the assumptions in this (and other) approaches to the Black-Scholes equation in the continuous setting. Since we have to deal with such mathematical concepts as predictable processes and Itô integrals, things are clearly more complicated than the binomial method discussed above where the financial assumptions were straightforward.

It is possible, however, to deduce the partial differential equation (6) as a certain type of limit of the binomial model by letting the number of steps n tend to infinity [8]. Specifically, let $\rho = \exp(rT/n)$ and $\eta = \exp(yT/n)$. Choosing $u = \exp(\sigma\sqrt{\Delta t})$, $d = \exp(-\sigma\sqrt{\Delta t})$ and $p = 1/2 + (r - y - \sigma^2/2)\sqrt{\Delta t}/2\sigma$, where $\Delta t = T/n$, the mean and standard deviation of the discrete distribution of the binomial method converges to those of the risk-neutral probability distribution of S_T , that is of \tilde{S}_T , as $n \rightarrow \infty$. At the same time, the corresponding option value defined by Theorem 1 will converge via the central limit theorem to the value given by equation (8) (see [3]).

Fractals and American Options

Most options traded in the marketplace are American-style, meaning that they can be exercised at any time before their date of expiration. These have no closed-form solution for their pricing and analysis and so the industry usually resorts to the binomial method described in the article. Typically 150 to 200 steps or more are used to ensure adequate accuracy with resulting computational expense. (Computational time is roughly proportional to the square of the number of steps.) But not all data needs so many steps.

The green areas in the images on the cover show where only 1–3 steps are necessary to get two decimal places of accuracy moving to over 150 steps in the red areas. (The vertical axis is time ranging from zero to six months, the horizontal is the price of the underlying commodity ranging from 90 to 110. Different volatilities and interest rates give different images and the final data is mapped to a disk.)

An examination of the images suggested a new approach, called the PB method since it “prunes” and “bends” the binomial tree [10]. It speeds up the existing binomial method by a factor of over 60 by making use of the scattering of the regions with high accuracy—the green regions. Based on any initial data, the method mathematically transforms the green stripes so that the data will lie on one of them with the result that only a few steps are needed to achieve the desired accuracy.

For example, in the top disk there is a blue dot in a red region indicating that to get reasonable accuracy for data corresponding to this point would require approximately 150 steps. But after applying the PB transformation, in the bottom disk the same data lies on a green stripe so only a few steps are needed to get the same accuracy.

Further work is proceeding to develop new methods of computer visualization to process and analyze the massive amounts of data coming from financial markets and their models rather than just using the basic two-dimensional and three-dimensional charts.

Using these values of the parameters, for each n we can obtain a discrete version of the partial differential equation (6). From there we can easily move to discrete versions of the results in this section. For instance, a consequence of equations (4) and (5) is that $a_t = f_x(S_t, T - t)$ which has expression (2) for Δ as its discrete analog. By adopting this perspective, we do not assume, for example, that options can be replicated by trading a portfolio of assets and bonds but that this fact follows from the no-arbitrage assumption on the underlying tree.

American Options

Since American options allow the possibility of early exercise, boundary conditions similar to those described by (7) will no longer suffice. Consider the case when $g(x) = (x - K)^+$. It turns out that the domain of the equation splits into two parts: a *continuation region*, where it is ad-

vantageous to keep the option, and a *stopping region*, where the correct strategy is to exercise the option. The problem now falls under the theory of partial differential equations with free boundaries.

There is a continuous curve $S_t = b(t)$ separating these two regions that is called the *optimal boundary* or the *early exercise boundary*. In this setting, the price of the American option is $V_t = f(S_t, T - t)$, where f is the solution of equation (6) with the boundary conditions

$$f(x, 0) = (x - K)^+, \quad f(b(t), \tau) = b(t) - K, \\ f_x(b(t), \tau) = 1,$$

where $\tau = T - t$. The nature of this boundary and its relationship with the option price is an active area of research, as is the study of how well the marketplace recognizes the location of the boundary and the consequences of prices crossing it.

As mentioned above, another approach to pricing American options is to use the binomial model; for n around 150 to 200, the previous values for u , d , and p give values for American options that lie within the tolerances of the marketplace.

Other assignments for the parameters u , d , p are possible, but Omberg [10] shows that the convergence is never monotonic. The chaotic nature of this convergence is evident from the image on the cover. Price [11] makes use of the scattering of the regions where convergence is fastest to develop a new algorithm for pricing American options that is around 60 times faster than the standard binomial method with minimal loss of accuracy. (See the box at left.)

Of Shoes—and Ships—and Sealing Wax

Of course, all the assumptions described above have been relaxed, both individually and in various groupings, and a huge amount of work continues in these directions. Interest rates and volatility can be stochastic processes, stocks can pay discrete random dividends, trading costs can be incorporated.

Most importantly, geometric Wiener processes can be replaced by processes with leptokurtic returns and random jumps. Restraints on short selling and borrowing can be incorporated as restrictions on the quantities of the assets and bonds used to form self-financing trading strategies as described above. As well, approaches that require risk to be measured (and then minimized) globally are being proposed as replacements for the differential methods outlined above in which the risk of owning an option can be instantaneously matched at each point in time by selling the corresponding self-financing replicating portfolio [2].

Another emerging direction is the implementation in financial mathematics of group theoretic methods that formalize the existence of various symmetries in the market [9]. Even though still in an embryonic stage, these methods demonstrate their power in a full range of applications, starting with the purely theoretical, where these methods restrict possible classes of stochastic processes and provide via a product integral of noncommutative nonlinear operators a unified framework for analyzing various exotic options, and finishing with such applied areas as the enhancement of computational efficiency.

Despite such advances, the Black-Scholes framework is tenacious with its simple formula and intuitive parameters.

A further active area of research in financial mathematics is to price options on various types of bonds. Here the situation is more complex, since we need to model the dynamics of the yield curve, the curve that describes interest rates as a function of their term or period of investment. Major contributions to the area have been made by Black-Derman-Toy, Brace-Gatarek-Musiela, Heath-Jarrow-Morton, Ho-Lee, Hull-White, Jamshidian, Li-Ritchken-Sankarasubramanian, Miltersen-Sandmann-Sonderman and Vasicek. Rogers [12] provides a useful overview of the different approaches. Even though there have been a number of developments since then, it is safe to say that no model for yield curves has the same dominance as that of Black-Scholes for assets with constant interest rates.

The Black-Scholes model has provided the risk-management and investment industries with a substantial first course, but it has only sharpened their appetites for more. It is a challenge to the mathematics community to provide it for them.

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