The Mathematical Millennium

David Gale

It seems that most of the letters to the editor and some of the other communications in these *Notices* over the last few years have been on the subject of mathematics education. By way of contrast, I would like to express a view on a subject related to what is presumably the Society's primary mission, namely, that of promoting and facilitating mathematical research.

As the century draws to a close, some sort of stocktaking would seem appropriate. It is natural to ask the obvious question, What has been achieved during these years? Well, to me it is overwhelmingly clear that the second half of this century has been a golden age for mathematics, far and away more productive than any similar period in history, and I am moved to write about this because I have a strong sense that very few people aside from the comparatively small community of research mathematicians are aware of what has been going on. By way of illustration, most people who keep up with science would probably know about the great revolution in physics that occurred during the first half of the century, but these same people would probably be surprised to learn that what the first fifty years were for physics the second fifty have been for mathematics. (The other subject that has made giant strides during this same period is, of course, biology.) Indeed, even in the mathematical community itself, I get the feeling that many people do not sufficiently appreciate what has been happening around them.

To be sure, there is a trivial sense in which mathematical "productivity" gets bigger every year, since the number of research mathematicians keeps growing with each new crop of mathematics Ph.D.s. Just looking at the content of current issues of *Mathematical Reviews*, I would estimate that something like ten thousand new mathematical results are added to the literature every month, but this purely quantitative phenomenon obviously does not in itself constitute "progress".

The notion of progress is, of course, rather subjective and is to some extent in the eye of the beholder. Nevertheless, I do expect many mathematicians would agree that the past fifty years have been quite exceptional. No doubt future historians of the subject will be able to pinpoint the development quite specifically. I would speculate, though, that the mathematical explosion began right at mid-century with Serre's thesis, followed some years later by Milnor's discovery of exotic differential structures. Each of us has our own ideas of which were the most significant landmarks, but there is no question that they were many. Indeed, in some ways these years have had an almost storybook quality, culminating in true fairytale fashion in a climactic moment—the spectacular achievement of Andrew Wiles in solving the Fermat problem. As I said, I expect many, perhaps most, research mathematicians are aware of all this, but I have never seen it stated explicitly in print, and it seems to me very worthwhile to say it loud and

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clear, especially since apparently even among mathematicians I am not sure there is a full awareness of what has been happening. In fact, sometimes when I express these views to other mathematicians, there are challenges. If this has been such a great period, they will say, who are its giants? Where are the Hilberts, Lebesgues, Minkowskis, Poincarés, Godels, Kolmagorovs, and Cartans? Perhaps we are too close to the era to see who our own giants are, but another possibility is that, to continue the fairytale metaphor, the age of giants may have passed. The subject of mathematics has become so large and has spread in so many directions that there simply cannot be dominating figures like those of the past. They, the forefathers, were the ones who laid the foundations. Their discoveries and insights are what we now use daily and perhaps even take for granted, but it was those insights which bore fruit in the extraordinary takeoff that occurred with the coming to maturity of the first postwar generation of mathematicians. (Some people have said that they believe the great surge may have peaked and things are now starting to slacken off. In any case, even if this is true, it was quite something while it lasted.)

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Everything that has been said so far applies to the international mathematical community, but there is also a special sense in which the achievements are relevant for the AMS. Note that there are no Americans in my list above (though perhaps one could have included Birkhoff and Wiener). In the postwar mathematics boom the situation has been quite different. It is perhaps better to let the reader fill in the names at this point, but any history of the progress of mathematics over the past fifty years will have to include, for example, the already-mentioned results in differential topology, the proofs of the Poincaré conjecture in dimensions four and greater, the solution of a celebrated Hilbert Problem, the independence of the continuum hypotheses, the classification of the finite simple groups. In all of these areas Americans played the leading role. Whether the mathematicians of the future will consider them giants or merely geniuses does not really make much difference.

Concluding Remarks: We of the mathematical community are often told that we do a poor job of communicating to the public what it is we do and why what we do is important. When I have tried to tell nonmathematicians this golden age story, they are always surprised and even skeptical. If this is so, they wonder, why haven't we heard about it? Unfortunately, it is difficult for us to make headlines. The trouble is we have no double helices or charmed quarks to expound upon, and it is hard to make the front page with solitons or gauge fields or strange attractors or Mandelbrot sets, even though these things have been among the many breakthroughs of the past decades. I suppose we could all go around shouting from the rooftops about the glories and achievements of our discipline, but it is not clear that anyone would listen. It is also unsatisfactory to tell people that this is all great stuff, but you'll just have to take our word for it. Perhaps the only answer is to face the daunting task of providing expository materials which will make some of these things as comprehensible as possible to the general public without getting into the technicalities that are necessary for full understanding. Well, it seems I have ended up talking about education after all.

Addendum: I am grateful to several colleagues for opinions on the foregoing remarks; the most frequently heard comment was, how can you write about the state of contemporary mathematics without even mentioning the impact of computers? Good point. My excuse is that I am limiting my observations to what might be called mainstream mathematics or, more appropriately, the mathematics we inherited (from the masters of the past). Equally impressive, however, are the brand-new branches of mathematics that have been created during this same period, and of course this is where computers come in; e.g., they have been the inspiration for complexity theory, automata theory, etc. Indeed, "I'm glad you asked that question," because it only reinforces my main point by providing yet more examples of the remarkable originality and creativity that have characterized the mathematics of the past fifty years.