Forum

The Mathematician and the Mathematics Education Reform

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E. H. Moore's Retiring Address as AMS president took place in 1902. Moore had a firm commitment to both teaching and research, but the theme of his address centered on mathematics education. On the role of the AMS in education he said: "Do you not feel with me that the AMS, as the organic representative of the highest interests of mathematics in this country, should be directly related with the movement of [education] reform?" ([7], p. 671) With the current mathematics education reform movement in place for almost a decade, Moore's words of a century ago become all the more relevant now.

The "reform" referred to in this article will cover both the *K–12 mathematics education reform* and the *calculus reform*, since these two reforms share an almost identical outlook and ideology. (See, for example, [29].) Unbeknownst to most mathematicians, the AMS has already taken part in this reform: p. vi of the *NCTM Standards* [18][1] carries the following statement:

ENDORSERS

> The following mathematical science organizations join with the National Council of Teachers of Mathematics in promoting the vision of school mathematics described in *Curriculum and Evaluation Standards for School Mathematics*: Amer. Math. Assoc. of Two-Year Colleges, AMS, Amer. Stat. Assoc., AWM, Assoc. of State Supervisors of Math., CBMS, Council of Presidential Awardees in Math., Council of Scientific Society Presidents, Inst. of Management Sc., MAA, MSEB, National Council of Supervisors of Math., Operations Research Soc. of Amer., School Sc. and Math. Assoc., SIAM.

There are valid reasons why we as members of the AMS should, as Moore said, be "directly related with the reform." To the outside world the AMS has spoken for all of us by endorsing NCTM's "vision", and the reform is settling in, becoming codified in law in some states and being mandated on local and national levels. It is now incumbent on us to consider, even if a trifle too late, whether this vision is indeed the one that we could—or should—endorse personally. The purpose of this article is to present some facts to help the mathematical community make up its collective mind.

The dictionary definition of *reform* is "the improvement or amendment of what is wrong, corrupt, unsatisfactory". Does this reform then *improve* on what is unsatisfactory in the so-called

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[1] *This document is available on WWW; please see the references.*

traditional curriculum? The answer is both yes and no: the reform presents not an unalloyed improvement but a set of uneasy tradeoffs. The succeeding paragraphs give some of the details. My comments on the reform are based on the following documents and texts: [1, 3, 4, 5, 6, 10, 11, 12–14, 15, 16, 22, 18–21, 24, 26, 27, 28, and 30]. (It may be added that most of these documents are either basic to the reform, such as [18–20] and [10]; or highly praised, such as [12] and [22]; or widely used either nationally or within a state, such as [6] and [4].) A discussion of the impact of the reform on school and university mathematics education together with specific suggestions of how mathematicians can help the cause of education will be given in a separate article [31].

First, a word of caution: I have referred to **the** reform and **the** traditional curriculum as if they were monolithic entities, but of course they are not. Insofar as they are in the social domain, general statements in this article must be understood to have some exceptions.

In the following, the *traditional curriculum* will refer to the generic school mathematics curriculum of the 80s. By the time the idea of the latest reform took hold in 1986—the year NCTM convened its first meeting to draft the *NCTM Standards* [18]—the concept of a "proof" in the traditional curriculum had either become nonexistent or degenerated into meaningless ritual. For those who went to school in the 40s or 50s, such a statement may come as a surprise, as not a few of us had been charmed by Euclidean geometry—the essence of proofs—into becoming mathematicians (cf. [23]). Yet Euclidean geometry is now perhaps the most vilified portion of school mathematics. What happened? The mathematics curriculum in the schools went through the New Math of the 60s and the Back-to-Basics Movement of the 70s and emerged oversimplified and dumbed down. Even synthetic Euclidean geometry has become corrupted by bad texts and bad teaching, so that a proof can often be mistaken for "one more thing to memorize" by the students.

Example: the first theorem in the popular geometry text [25] is: "If two angles are right angles, then they are congruent." This occurs on p. 24—up to that point there has been no attempt to build up students' geometric intuition—and the two-column proof consists of five steps. Without divulging the secret of how to stretch such a proof to so many steps, let me just mention that, according to the text, "a *right angle* is an angle whose measure is 90", and "*congruent* angles are angles with the same measure." In turn, "the *measure* of an angle is the amount of turning you would do if you were at the vertex, looking along one side, and then turned to look along the other side," but for the precise measure of "90 (degrees)", the text tells you to use a protractor (p. 9, line -9). To make matters worse, a teacher in this situation typically asks students to commit to memory the format of such a presentation for the purpose of exams (cf. [27], pp. 157–8). The whole point of having axioms and modus ponens in order to ascertain the truth of a statement has clearly fallen by the roadside in the intervening years.

The overriding characteristic of the traditional curriculum is its emphasis on learning algorithms by rote: mathematics becomes a set of algorithms to be memorized and regurgitated at exam time. For example, while some algebra texts of the 50s still gave proofs of the basic properties of polynomials, it would be difficult to find a standard algebra text of the 80s that takes the trouble to explain anything. Anyone who teaches freshman calculus regularly knows only too well the ill-effects of this kind of mathematics education. It does not add to our comfort to realize that many calculus courses in college also lend themselves to learning by rote, so that students often come out of such a course equating derivative with "the thing that changes x^n to nx^{n-1}" and nothing more. The accusation against this traditional curriculum is that it is arid, boring, and irrelevant. Students lose interest. The abysmal test scores of the late 70s through the early 90s together with massive drop-outs in K–12 math classes testify to its failure.

From a mathematical point of view, the main problem with the traditional curriculum is that it deals with the *how* of mathematics, but not with the *why*. The basic questions of why something is true and why something is important are allowed to remain unanswered. What we need is a curriculum that provides answers to these questions.

Proofs

A reasonable response to the absence of proofs in the traditional curriculum would be to give precise proofs of a set of select basic theorems, with rigor appropriate to the grade level, and to offer heuristic arguments whenever possible for the rest. The crucial point here is to help develop students' critical faculty by making them aware of the distinction between the two: a proof and a heuristic argument. On the one hand, logical deduction—proof—is the backbone of mathematics. If we are serious about mathematics education, we should aspire to making every high school student learn what a proof is. On the other hand, it would be a grave mistake to insist that *every* statement in elementary mathematics, up to and including calculus, be given a proof. There is no reason to impose the kind of training designed for future professional mathematicians on the average student. (In particular, ϵ-δ proofs may be best reserved for honors calculus.) What *is* important, however, is to give students adequate training in making logical deductions. This can be done by using what may be called *local axiomatics*; i.e., before the proof of a theorem, make clear what statements are assumed to be true and proceed to show how to use them in the proof. This shows students how to demonstrate the truth of a statement on the basis of explicit hypotheses. A reasonable mathematics *education* should aim for at least this much.

We turn now to the reform's response to the absence of *why*. The overall strategy of the reform is to supply motivation and heuristic arguments, but only motivation and heuristic arguments. In mathematics, heuristic arguments are used as preludes to proofs, but in the recent reform documents they are used as substitutes for proofs.

This strategy presents a new set of problems of its own. For example, when a seductively phrased heuristic argument, in reality very far from a proof, is presented without further comments, it is perilously close to a deception. The argument offered for the Fundamental Theorem of Calculus on p. 171 of [10] is a good illustration: Given F defined on $[a, b]$, partition the latter into n equal subdivisions $x_0 < x_1 < \cdots < x_n$ and let the length of each subdivision be Δt. Then for n large, the change of F in $[t_i, t_{i+1}]$ is approximately $\Delta F \approx$ Rate of change of $F(t) \times$ Time $\approx F'(t_i)\Delta t$. Thus the total change in $F = \sum \Delta F \approx \sum_{i=0}^{n-1} F'(t_i)\Delta t$. But the total change in $F(t)$ between a and b can be written as $F(b) - F(a)$. Thus, letting n go to infinity: $F(b) - F(a) =$ Total change in $F(t)$ from a to $b = \int_a^b F'(t)dt$.

Problems of a different kind arise when heuristic arguments based on appeals to technology are made with increasing frequency: the computer or calculator begins to assume the role of arbiter of pure reason. Thus, in the eighth grade textbook of the widely used Addison-Wesley series [6, p. 396], students are told that if a number is not a perfect square or a quotient of perfect squares, then its square root is an irrational number (which is defined to be a nonrepeating and nonterminating decimal). Why? Because one can check this on a calculator. Or consider the reason offered for $\frac{d}{dx}\sin x = \cos x$ on pp. 101–3 of *Derivatives* in [5]: graph the function $f_h(x) = (\sin(x+h) - \sin x)/h$ for successively smaller values of h (e.g., 0.5, 0.1, 0.001, …) using the computer and observe that this graph approaches that of $\cos x$ as $h \to 0$.

Of course the strategy also has momentary lapses. At times, no argument of any kind is offered. Thus the precalculus text [22] defines the inverse of a square matrix R as a matrix S so that $RS = I$ (p. 259) but immediately states: "It is also true that $SR = I$. … The inverse of R is symbolized by R^{-1}, so that $R^{-1} = S$ and $S^{-1} = R$." The inherent logical difficulties behind these statements (Why does $RS = I$ imply $SR = I$? Why is such an S unique? Why is $S^{-1} = R$?) are never mentioned, much less resolved. The beginning algebra text [3] tells students (p. 676) that the quadratic formula is "one of the most important formulas in mathematics," but does not see fit to offer any derivation for it. The geometry text [28] never explains why the three altitudes of a triangle meet at a point, among many things.

When it does happen that a heuristic argument is in fact a correct proof, the problem then becomes one of credibility gap: if *no distinction is ever made* between a correct proof and a heuristic argument, which may be difficult or impossible to upgrade to a proof, how to convince students that *this time around* it is really true? So at the end, students are simply left with the mistaken belief that every piece of reasoning they have encountered is valid. At least the de facto absence of proofs in the traditional curriculum did not mislead students into the illusion that they know the reason for anything, but the reform manages to do otherwise. For example, the comment in *Derivatives* of [5] about the above heuristic argument for $\frac{d}{dx}\sin x = \cos x$ is (p. 103):

> How sweet it is. Math happens.

In other words, the students are explicitly asked to believe that, thanks to the computer, they have witnessed mathematics at work.

Precision and Technical Skills

Precision is a defining characteristic of our discipline. For ease of discussion, let us artificially separate precision into the following two categories:[2] conceptual precision (definitions, theorems, and proofs) and formal precision (symbolic computations and algorithms). Since proofs have already been discussed, we now concentrate on formal precision.

The traditional curriculum is driven by algorithms-without-explanations. By overly simplifying mathematics in this fashion, this curriculum acquires several virtues: it has built-in precision, it brings computational skills to the forefront, it sets a clear goal for students (always strive to produce a correct answer), and finally, it lets teachers know unambiguously what to

[2] *We note for emphasis that in reality there is no such separation.*

teach. Its weaknesses are that, especially in unskilled hands, it can easily degenerate into mindless number crunching and symbol pushing, so that students end up not learning even the computational skills. These weaknesses are correctable: supply the motivation and reasoning behind the algorithms, and replace some of the routine drills with exercises that make a greater demand on students' conceptual understanding.

The reform responds by promoting what it calls "*process over product*". It stresses qualitative reasoning (hence heuristic arguments, as described above) and motivation. It also introduces the idea of looking at counterexamples or conjectures in connection with a new concept or theorem. These are welcome changes. Not welcome is the reform's downplaying of symbolic computations, precise definitions, neat formulas, and precise answers.

The advisability of these decisions is debatable. Consider the following statement on p. 125 of the *Standards* [18]: "the 9–12 standards call for a shift in emphasis from a curriculum dominated by memorization of isolated facts and procedures to one that emphasizes conceptual understandings, multiple representations and connections, mathematical modelling, and mathematical problem solving." In and of itself, this sentiment cannot be faulted—trying to lead students away from memorization towards understanding. Yet when there is no simultaneous emphasis on basic technical skills throughout the whole document, statements such as this in [18] open the door to texts and curricula which make believe that one can be technically deficient (not knowing *precise* definitions or not equipped with symbolic computational skill, say) and still achieve conceptual understandings, make multiple connections, and solve problems. How much can a student understand about second-degree polynomials without knowing the quadratic formula? Quite a bit for a history major, perhaps, but not nearly enough if one wants to do exact sciences or mathematics. The fact must be faced that, in mathematics, one cannot have understanding without technique. The two are intertwined.

An excellent illustration of the shortcomings of the "process over product" approach is in the treatment of arithmetic series and geometric series in the introductory algebra text [30]. It goes through the detailed method of summing these series for concrete cases but relegates the two well-known formulas to two exercises (pp. 191 and 399). All of a sudden, formulas seem to have fallen into disgrace. In the reform curriculum this is a severe case of throwing out the baby with the bath water. Similarly, we have the absence of any mention of convergence tests for infinite series in the calculus text [10], and of the binomial theorem in the precalculus text [22], and of the geometric series in [22] and [12].

Another aspect of "process over product" is the downgrading of the importance of getting a single correct answer. One way is to demonstrate that mathematics is not "a domain of single right answers". To this end, the NCTM Teaching Standards have a teacher posing the following problem (p. 45 of [19]): if 30 points were scored in a basketball game without a single foul shot, how were the 30 points scored? (There are 2-point shots and 3-point shots in basketball.) From a mathematical standpoint, this problem is not correctly formulated: the teacher could have asked the students either "to list all the possible ways the 30 points were scored" (in which case there would be a unique answer), or to show "how such an imprecise question in daily life could be translated into a precise mathematical problem." It is clear that this is where a firm mathematical direction from the teacher would help to clarify the situation. Instead, the discussion in [19] makes a point of *not* making such a clarification (cf. the marginal remarks on pp. 46–7). In the meantime, even some good teachers are led to believe that problems which have a unique correct answer are bad for students (p. 122 of [32]; one can find in [32] a more extended discussion as well as further examples of such problems).

The slighting of technical skills in calculus presents additional educational problems, however. For example, the text [10] is written to be accessible to students with weak algebraic background. To the question whether reform calculus was "passing students through calculus with at best a rudimentary knowledge of algebra," a reformer's reply was that "we were doing this long before calculus reform" [17]. Should the calculus reform not be interested in *improving* on this aspect of mathematics education instead of accepting the status quo? Educators have long recognized the unfortunate fact that the prestige of any K–12 mathematics curriculum hinges on its ability to prepare students to pass calculus. By sending out a signal that being weak in algebra is acceptable in calculus, the reform in effect sanctions the continued decline in school students' symbolic manipulative skill, especially when this signal is reinforced by [10], at present a bestseller in calculus. The devastating impact this has on school mathematics education as a whole will be long lasting, because the students of today will be the teachers of tomorrow and weak teachers tend to produce even weaker students. This is not the kind of "improvement" expected of a reform.

Relevance

A curriculum of elementary mathematics should achieve a balance between theory and applications and, ultimately, a balance between the abstract and the concrete. A majority of students learn mathematics to be good citizens, not to become professional mathematicians. For this they need to learn both the cultural aspect of mathematics and its utility.

For many students, a major defect of the traditional curriculum is its seeming irrelevance. The result is substantial dropouts in mathematics classrooms across the nation.

Part of this feeling of irrelevance stems from the poor integration of theory and applications in the traditional curriculum. Word problems, too often mishandled in the classroom, lend themselves too easily to solutions-by-template. This then leads to the danger of learning by rote not only the theories but also their applications. A second difficulty is that the applications used have not caught up with the explosion of the more recent applications of mathematics (especially discrete mathematics) in everyday life. A third difficulty is the paucity of "internal applications" within mathematics: e.g., how to use an algebraic technique in geometry or vice versa.

This perception of irrelevance was in fact one of the main targets of the reform from the beginning. "Mathematics has become a critical filter for employment and full participation in our society. We cannot afford to have the majority of our population mathematically illiterate: Equity has become an economic necessity" (p. 4 of [18]). The battle cry is "Mathematics for all!" To this end the reform promotes curricula which center around so-called real-world problems. Thus [10] states in the preface: "Formal definitions and procedures evolve from the investigation of practical problems" (p. vii). Also, one finds in [18]: "Problem solving must be central to schooling" (p. 4). The resulting more realistic alignment of the reform curricula with the real world is a definite improvement.

But again, improvement comes at a price. The literal insistence on using *real-world* problems as core brings in new and sometimes surprising pedagogical issues. For example, do the messy real-world details obscure the basic mathematics, thereby obstructing the very mathematical skills the students should be learning? Another is whether one person's real-world experience may not be another's drivel. Consider some of the applications in [22]: why short tennis players should use a spin serve, why it would be advantageous to have elevators go to certain floors but not all floors, locating a hot dog stand for student convenience, the number of barbers needed in a given town, how to relate the counter reading of a tape being wound in a cassette machine to the amount of time left on the tape, etc.

Perhaps the most serious issue faced by a problem-oriented curriculum is that of *mathematical closure*, or rather the absence thereof. The problems are only a means to an end — the vehicle to facilitate the learning of mathematics — but not the end itself. Therefore, the solutions of problems in such a curriculum need to be rounded off with a mathematical discussion of the underlying mathematics. If new tools are fashioned to solve a problem, then these tools have to be put in the proper mathematical perspective: their purely technical developments should be addressed and their place in the overall mathematical structure clarified. Otherwise, the curriculum lacks mathematical cohesion. Moreover, if care is given to the distillation of the key mathematical idea of a solution from its original (real-world) context and its subsequent applications to entirely different situations, students will become convinced of the need for precision and abstraction. Without appearing to minimize the difficulty of achieving this kind of mathematical closure in a problem-oriented setting, it must be said that none of the reform texts I have consulted for this article is entirely successful in this regard. In fact, many of the mathematical transgressions in these texts are directly traceable to this obsession with real-world applications at the expense of abstract considerations.

The *NCTM Standards* [18] do not ever mention mathematical closure. (See [31] for a more extended discussion of this fact.)

Is It an Improvement?

Is the current reform an improvement over the traditional curriculum? Looking over the facts, we see that almost every improvement brought about by the reform is accompanied by some pronounced liabilities and that the two curricula are flawed in complementary ways. The traditional curriculum teaches *how* but not *why*; the reform curriculum teaches a little bit of both, but at the end may succeed in teaching neither. However, education is not a purely intellectual enterprise. Are there perhaps other relevant social issues that need to be considered in evaluating the reform?

It has been suggested that given students' inability to master symbolic computations, as evinced by students' low mathematics achievements in K–12 and calculus, the reform should be given credit for doing the best job possible with the kind of students we have. What is left unsaid in this analysis is that a true reform needs to maximize curriculum, teacher qualification (especially in regard to knowledge of mathematics), and student effort at the same time. Instead, this NCTM-centered reform has thus far kept the last two constant while varying the first somewhat randomly. This does not seem a good strategy for optimization.

It has also been suggested that the deemphasis of the abstract in favor of the concrete, of symbolic computations in favor of technological supplements, and of precision in favor of qualitative reasoning are exactly what future users of mathematics (engineers, physicists, etc.) need. Evidently the feeling is that the few future mathematicians who need such training can get it in courses like elementary analysis. In the case of engineers, etc., good use of a tool is presumably not to be learned through forcing them to think abstractly and precisely.

Yet, again, there are good reasons for rejecting such an approach. One is that students who want to learn about mathematics per se, not only for its utility as a tool for science, should not be prevented from doing so. We should teach mathematics for what it is, unless and until we are willing to start labelling foundational mathematics courses as "minimal survival kits for the sciences".[3] In addition, the notion of what scientists need from mathematics is volatile. My own informal survey indicates that, while such opinions cover a wide spectrum, there is no disagreement on the need of versatility and flexibility in the use of mathematical tools. It is unlikely that such flexibility and versatility can be achieved in a curriculum without the kind of mathematical closure described above. Recently, the introduction of a textbook on mathematical physics [9] makes an eloquent plea for scientists to acquire such a rigorous training in mathematics:

> One might argue that although mathematics provides a very important tool to the scientist and engineer, this is not a sufficient reason for the arduous training in mathematics. After all, it is possible to use tools without detailed knowledge of their mode of functioning; it is possible to drive a car without any idea of the working of the internal combustion engine. Indeed, problems of a very well defined nature and limited scope are solvable by computer programs into which one has only to plug the data. But the situation of most engineers and scientists is not like that of the driver of a car, but rather like that of a worker detonating blast charges. Unless he has a good acquaintance with the properties of explosives, he is likely to come to grief.

[3] *The name* compumatics *has also been suggested in [8].*

It is time for us to restore mathematical balance to problem-driven curricula. Let us not deny our students the opportunity to acquire this kind of arduous training.

Summing Up

To the extent that the traditional curriculum is so seriously flawed, reform is way overdue. But if the preceding marshalling of facts means anything, it is that if this particular reform curriculum instead of the traditional curriculum were already in wider use across the land, then its serious defects would also be signaling the need for yet another reform. We cannot afford to experiment with a whole generation of our children when the odds are stacked against the present reform's long-term success.

While I have grave misgivings about other aspects of the reform, its pedagogical practices (cf. [31]), and its assessment strategies (cf. [2]), I have chosen not to discuss them here. This is due in part to considerations of space, but also because I believe (perhaps wrongly) that these are decisions that can more easily be reversed. What this article has presented then are aspects of the reform's curricular decisions reflecting an educational philosophy gone awry. These will not go away simply by the flipping of a switch. Corrections can be achieved only if the reform is revamped from the ground up.

Let us ask ourselves defining questions: Are we after a Band-Aid solution to a troubled curriculum? Are we to allow the issue of accessibility to override the basic integrity of the subject? In designing a curriculum, should we include only topics that can be learned without real hard work? Can we compromise the issue of student entitlement to the availability of mathematical knowledge? And, the question best addressed by the membership of the AMS: What kind of mathematics do we want to teach our students?

At the beginning of this article it was pointed out that the AMS has endorsed the vision of this reform as set forth in the *NCTM Standards* [18]. The question is whether, as members of the AMS, we believe we have been properly represented in this endorsement. There is a lengthy discussion of the *NCTM Standards* in the companion article [31], but this book is required reading for every mathematician who thinks mathematics should have something to do with mathematics education reform. You must decide for yourself if this is really a vision that *you* can support. Perhaps you have a better educational vision?

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References

[1] ARISE, Information material available (1993) from COMAP, Inc., Suite #210, 57 Bedford Street, Lexington, MA 02173.

[2] GEORGE E. ANDREWS, *Review of assessment standards for school mathematics*, submitted to *Notices Amer. Math. Soc.*

[3] RANDALL I. CHARLES et al., *Focus on algebra*, Addison-Wesley, Menlo Park, CA, 1996.

[4] *College preparatory mathematics, 1, 2, and 3*, CPM Educational Program, Sacramento, CA, 1994–95.

[5] BILL DAVIS, H. PORTA, and J. UHL, *Calculus & Mathematica: Approximations, derivatives, integrals, and vector calculus*, Addison-Wesley, Reading MA, 1994.

[6] ROBERT EICHOLZ et al., *Addison-Wesley mathematics* for grades 4–8, Addison-Wesley, Menlo Park, CA, 1995.

[7] JOHN EWING, *Mathematics: A century ago—A century from now*, *Notices Amer. Math. Soc.* **43** (1996), 663–672.

[8] EFIM A. GALPERIN, Letter to the editor, *Notices Amer. Math. Soc.* **43** (1996), 741–742.

[9] A. K. GHATAK, I. C. GOYAL, and S. J. CHUA, *Mathematical physics, differential equations and transform theory*, Macmillan India Ltd., New Delhi, 1995.

[10] D. HUGHES-HALLET et al., *Calculus*, Wiley, New York, 1994.

[11] *Interactive mathematics*, Glencoe/McGraw Hill, New York, 1995.

[12] *Mathematics Curriculum for 9–12*, Interactive Mathematics Program, 1991 version.

[13] Interactive Mathematics Program information material, available (1993) from IMP National Office, 6400 Hollis St. #5, Emeryville, CA 94608.

[14] *Baker's choice*, Interactive Mathematics Program, Key Curriculum Press, Berkeley, CA, 1995.

[15] CAROLE B. LACAMPAGNE, ed., *The Algebra Initiative Colloquium, Volumes I and II*, U.S. Department of Education, Washington, DC, 1995.

[16] *Mathematics framework for California public schools*, California Department of Education, Sacramento, CA, 1992.

[17] WILLIAM G. MCCALLUM, Posting in the e-net `calc-reform@e-math.ams.org` dated February 21, 1996.

[18] *Curriculum and evaluation standards for school mathematics*, National Council of Teachers of Mathematics, Reston, VA, 1989. Available at the Web site: `http://www.enc.org/online/NCTM/280dtoc1.html`.

[19] *Professional standards for teaching mathematics*, National Council of Teachers of Mathematics, Reston, VA, 1991.

[20] *Assessment standards for school mathematics*, National Council of Teachers of Mathematics, Reston, VA, 1995.

[21] *Geometry from multiple perspectives*, Curriculum and Evaluation Standards for School Mathematics Addenda Series, National Council of Teachers of Mathematics, Reston, VA, 1991.

[22] *Contemporary precalculus through applications*, The North Carolina School of Science and Mathematics, Janson, Dedham, MA, 1992.

[23] ROBERT OSSERMAN, *Are proofs in high school geometry obsolete?*, J. Math. Behavior, to appear.

[24] J. PELECH and J. PARKER, *The graphing calculator and division of fractions*, Math. Teacher **89** (1996), 304–305.

[25] R. RHOAD, G. MILAUSKAS and R. WHIPPLE, *Geometry for enjoyment and challenge*, McDougal, Littell & Co., Evanston, IL, 1991.

[26] C. SCHAUFELE and N. ZUMOFF, *Earth algebra*, preliminary edition, Harper Collins, 1993.

[27] ALAN H. SCHOENFELD, *When good teaching leads to bad results: The disasters of "well-taught" mathematics courses*, Educational Psychologist **23** (1988), 145–166.

[28] M. SERRA, *Discovering geometry*, Key Curriculum Press, Berkeley, CA, 1989.

[29] ZALMAN USISKIN, *Algebra and calculus for all?*, UCSMP Newsletter, no. 18 (Winter 1996), 7–14.

[30] ANITA WAH and HENRI PICCIOTTO, *Algebra*, teacher's edition, Creative Publications, Mountain View, CA, 1994.

[31] H. WU, *The mathematics education reform: What is it and why should you care?*, to appear.

[32] ____, *The rôle of open-ended problems in mathematics education*, J. Math. Behavior **13** (1994), 115–128.