Groups and Physics—*Dogmatic Opinions of a Senior Citizen*

A. John Coleman

The editors of the *Notices* have invited me to ruminate publicly on two topics which have engaged my interest for sixty years. In so doing, I hope to convey some of the pleasure and excitement which I experienced while revisiting the relations between groups and physics. Both are huge subjects and their interactions, multifaceted.

In 1939–40 when I was a graduate student at Princeton, the original Fine Hall was one of the world's most revered shrines for mathematical devotees. We were told that this charming building had been modelled on a medieval manor house. Therefore, on either side of the main entrance a niche and pedestal had been provided to accommodate the obligatory statues of protective saints. Perhaps because of the abhorrence of idolatry in the Presbyterian tradition from which the university sprang, these niches were empty. Perhaps the university authorities were waiting for the Church to name patron saints for mathematics!

History records that one day as he approached the entrance, Lefschetz paused, turned to his companion and said, "Regardez, Claude, une place pour Saint Elie et l'autre pour Saint Hermann." He was gently poking fun at Claude Chevalley. We all knew that in Chevalley's pantheon, Elie Cartan and Hermann Weyl were the greatest mathematicians of the twentieth century, perhaps of any century, surpassed, if at all, only by Gauss.

I have no reason to disagree with Chevalley's judgment. Both Cartan and Weyl were prodigious mathematicians. Further, it was these two giants who revealed to a largely unappreciative world the power, depth, and beauty of the ideas of Sophus Lie, Wilhelm Killing, and Isai Schur about continuous groups and their representations and taught us how to apply them to physics and to diverse areas of mathematics.

Cartan

In his thesis [1], Cartan expounded and extended the basic work of Killing [2], who had classified the simple Lie algebras over the complex numbers. Of course, they did not call them "Lie algebras". This term became current only after 1935, proposed, I understand, by Nathan Jacobson and adopted rather hesitantly by Weyl.

It is important when one reads the early authors to realize that it was only in the period 1925 to 1940 that current fastidiousness about topological ideas emerged. The concept "continuous group" arose originally in connection with the theory of differential operators on sets of functions. Whenever the early authors talked about "infinitesimal groups", which we would now think of as Lie algebras, at the same time they had in mind an action by what we might call

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He feels privileged to have attended lecture courses by Coxeter, Brauer, Wigner, and Chevalley; to have corresponded briefly with Cartan; and to have heard lectures by Einstein, Dirac, and Weyl. He helped proofread the seminal "Theory of Lie Groups" (1946) by Chevalley.

a "global Lie group". They confidently assumed the existence of such a group as the solution of a set of differential equations. It was only in the mid-twenties with Schreier's definition of the *universal covering group* that subtleties which we now regard as crucial were recognized. At first, all "continuous groups" were groups of *transformations*. Thus in Lie, Killing, and Cartan before 1913, and as late as 1933 in the writings of other authors, one can see phrases such as "Consider a group G with symbols $X_a f \ldots$" [3, p. 109]. Here, X_a is a first-order differential operator, and f denotes an arbitrary function (assumed, without comment, as sufficiently differentiable to justify all subsequent arguments. Before Lebesgue, except for a set of measure zero, all "functions" were arbitrarily differentiable!). Although Hermann Weyl's course at Princeton in 1934–35 was entitled Continuous Groups, not Lie Groups, the notes for the second term, written by Weyl's young assistant, Richard Brauer, begin with a survey of Lie algebras. This is the first occurrence of this term that I have encountered. However, throughout the remainder of the notes, presumably out of deference for the ingrained habits of the master, Brauer refers to the same objects as "infinitesimal groups".

In 1896 in a paper [4] which has been little noted, Cartan applied Lie's approach to systems of partial differential equations associating different systems of PDEs to the various types of simple Lie algebras classified by Killing. Incidental to the main thrust of the paper, Cartan discussed in detail the finite permutation groups, *defined by Killing*, of the roots of the exceptional simple Lie algebras. About fifty years later, representations of these as orthogonal groups became known as "Weyl groups".

In 1913 Cartan [5] proved that irreducible finite-dimensional representations of simple LAs of rank n are parameterized by *n-tuples*, (p_i), of nonnegative integers which define the highest *weight* of the representation. This n-tuple is often called the *Dynkin index* of the representation by physicists. They are used to characterize the states of quantum systems and also certain elementary particles. Cartan showed that any irrep (p_i) occurs, with unit multiplicity, as the first addend in the decomposition into irreducibles of the tensor product $\otimes_{i=1}^{n}\Pi_i^{p_i}$ where Π_i is the *fundamental* representation defined by $\left(p_j\right)$ with $p_j = \delta_{ji}$. It was in this paper that *spinors* first appeared.

Another contribution of Cartan [6] was the classification of simple LAs over the *reals*. This, of course, is of key importance for physics, where we normally deal with real Lie groups and algebras. I have never seen a reference in the mathematics or physics literature to the fact that, as part of his argument in [6], Cartan obtained the classification of conjugate classes of what we now call *Cartan subalgebras* of the real LAs. This classification was later obtained independently by Kostant [7]. It is essential in the theory of harmonic analysis on real groups and therefore in studying the properties of many of the functions which physicists employ.

I shall comment below on Cartan's contribution to the theory of relativity.

Elie Cartan.

Weyl

The first book by Hermann Weyl on the *Riemann Surface* is familiar to many mathematicians. But his first papers (1908–13) were about integral and differential equations. A remark by the famous physicist H. A. Lorentz to the effect that the frequency distribution of black-body radiation should not depend on the shape of the body led Weyl to conceive his remarkable formula for the asymptotic distribution of the eigenvalues of a Laplace-type operator. This formula plays an important role for our understanding of the properties of solids.

Hermann Weyl.

There is a well-known story that shortly after Eddington wrote his book [8] an earnest Cambridge undergraduate, sharing with him the cucumber sandwiches at tea, said unctuously, "It is common knowledge that you are one of the three persons in the world who understand Einstein's theory." Seeing a cloud on Eddington's face and fearing he had somehow offended the

Photograph by Francis Simon courtesy of AIP Emilio Segré Visual Archives.

Erwin Schrödinger.

Photograph courtesy of AIP Emilio Segré Visual Archives.

Werner Heisenberg.

great man, the student spluttered an apology. Eddington responded graciously, "Don't worry. It is merely that I could not imagine who the third person might be." This is very much an English story, since by 1923 hundreds of Swiss and Germans had read Weyl's masterly exposition of the theory.

Einstein announced [9] the General Theory of Relativity early in 1916. Weyl quickly mastered the theory and gave a series of lectures on it in the summer of 1917 in the ETH in Zürich. These lectures were the basis of his book [10], which appeared in 1918 and which already by 1923 had attained its fifth edition, the preface of which concludes with the rather acerbic remark: "A French translation of the fourth edition has appeared which, in places, is so 'free' that I am compelled to disclaim any responsibility for its contents."

His 1918 paper on *Gravitation and Electricity* was reprinted in 1955 in [11], where, in an addendum, Weyl remarks, "This work marked the beginning of a search for a unified field theory which was taken up later by many others, but as it seems to me with no striking success." He then states that he was misled in trying to use *gauge invariance* to relate gravitation and electricity, but in view of the subsequent development of quantum mechanics we now realize that "just as co-ordinate invariance corresponds to energy-momentum so gauge invariance corresponds to the conservation of electric charge." The term "gauge invariance" has now become a politically correct phrase in Yang-Mills quantum field theories. It must be used even if nothing is being "gauged". This is a good example of the frequent occurrence in the literature of physics of what the British and Harvard philosopher A. N. Whitehead called "The Fallacy of Misplaced Concreteness".

Since the current scene is dominated by the search for a "unified theory of almost everything" and this search provides the motivation of many of the most exotic occurrences of group theory in the journal *Nuclear Physics B* and elsewhere, it is worth reporting a remark of Wolfgang Pauli. First, here is part of the last sentence of the famous article [12] which he began at the age of nineteen on invitation of A. Sommerfeld and which appeared in 1921 when he was twenty-one years old: "…new elements which are foreign to the continuum concept of the field will have to be added to the basic structure of the theories developed so far, before one can arrive at a satisfactory solution of the problem of matter."

As part of the celebration of fifty years of relativity, Pauli's article was reprinted so that he had an opportunity to add "Supplementary Notes" giving his opinion about various ideas which had arisen in the intervening period. In the preface, dated November 1956, he remarks about unified field theories: "I do not conceal from the reader my scepticism concerning all the attempts of this kind which have been made until now and about the future chances of success of theories with such aims." In particular he was quite critical of Einstein's attempts at a unified field theory and concluded: "These differences of opinion are merging into the great open problem of the relation of relativity to quantum theory, which will presumably occupy physicists for a long time to come. In particular a clear connection between the general theory of relativity and quantum mechanics is not yet in sight."

In my view, such a "clear connection" is no more visible today than when Pauli wrote these words.

Quantum Mechanics

The New Quantum Mechanics supplanted "old" 1913 quantum mechanics of Nils Bohr in 1925–26 as a result of an extraordinary outburst of creative imagination by Erwin Schrödinger, Werner Heisenberg, Eugene Wigner, Max Born, Pascual Jordan, and Paul Adrian Maurice Dirac. On their heels the mathematicians Hermann Weyl and John von Neumann followed to give the new theory an air of mathematical legitimacy.

At its inception there were two rival presentations due to Schrödinger and to Heisenberg.

Schrödinger's was expounded in terms of differential equations of functions of the coordinates of particles; Heisenberg's, slightly earlier, in terms of infinite-dimensional matrices acting on infinite vectors whose components represented allowed values of an observable. Schrödinger proved that the theories were equivalent. (A rigorous mathematical proof was published only in 1931, see second paragraph of [32].) Based on this result of Schrödinger, von Neumann gave a complete account of the new theory on an abstract Hilbert space. Schrödinger had realized the Hilbert space as L^2, whereas Heisenberg had realized it as ℓ_2. But von Neumann went further, emphasizing the role of density operators, which he invoked in his discussion of measurement and quantum thermodynamics. This he expounded in his classic book [13], the mathematical rigour of which repelled many physicists. On the other hand, it persuaded many mathematicians to take quantum mechanics seriously and also gave an enormous thrust to the study of Hilbert space and operator theory.

My readers are perhaps aware that the state of a quantum system is "described" by an element ψ of a Hilbert space, $\mathfrak{H}$. A hermitian operator A, say, is associated to an observable, and then $\langle\psi|A\psi\rangle$ is interpreted as the expected value of the observable on the system in state ψ. In Schrödinger's realization the state of a system of N particles with positions $\mathbf{x}_i$ would be represented by a square-integrable function $\psi(\mathbf{x}_1, \dots \mathbf{x}_N, t)$. The evolution in time of an isolated quantum sytem is determined by a unitary operator $U(t) = \exp(iHt)$ where H is a constant hermitian operator called the *hamiltonian* of the system. The eigenvalues of H are interpreted as possible energy levels of the system. For example, in the case of an atom, if $H\psi_\alpha = E_\alpha\psi_\alpha$, then the frequencies $\nu_{\alpha\beta}$ of electromagnetic radiation which the atom can emit or absorb satisfy the condition $h\nu_{\alpha\beta} = E_\beta - E_\alpha$, where h is Planck's constant. It is perhaps the fact that the set of energy levels is discrete and that therefore these changes of energy come in discrete bits or quanta that justifies the name "*quantum* mechanics".

According to the fascinating account of the early history of quantum mechanics in George Mackey's annotations to the first volume of Wigner's *Collected Works*, it was von Neumann who pointed out to Wigner that if the hamiltonian of a quantum system is unchanged by some group of symmetries G, then the eigenspaces of H span representation spaces for G. Indeed, Wigner states this in the introduction of the second paper of Volume 1. We must thus credit von Neumann with introducing group representations into physics.

Eugene Wigner.

A system of identical particles is unchanged by permutation of the coordinates of its particles. It then follows from the physical assumptions of quantum mechanics that any state ψ of the system belongs to a one-dimensional representation of the symmetric group of permutations of the N particles. There are only two possibilities: 1) the identity representation and 2) the antisymmetric representation. This fact implies that quantum particles are divided into two exclusive categories, called *bosons* and *fermions*, according as they exhibit the property (1) or (2). This does not mean that only one-dimensional representations of the symmetric group play a role in quantum mechanics. Indeed, for practical calculations for electrons, when the *spin* variable which is dichotomic is isolated, representations for Young diagrams with two rows or two columns play an essential role. When, as for nucleons, isotopic spin is also considered, diagrams with up to four rows or four columns are frequently deployed.

It turns out that while photons are bosons, electrons are fermions. That electrons are fermions justifies what physicists call the *Pauli Principle*. For quantum systems of N identical particles the wave function is regarded as an element of a Hilbert space which is the tensor product of N examples of the Hilbert space for a single particle. If the latter were of finite dimension r, then the former would have dimension r^N. However, the wave function for bosons or fermions is restricted to a relatively small subspace of dimension s_N, say, of the N-fold tensor product. For both bosons and fermions, s_N is such that the limit of s_N/r^N is zero when r increases indefinitely. A basic principle of particle physics, proved by Pauli, is that the spin of a boson or a fermion is, respectively, an even or an odd multiple of the spin of an electron. For particles of spin s, representations of dimension $2s+1$ of $su(2)$ play an essential role. Sir Arthur Stanley Eddington remarked that if there were no particles with spin $1/2$, there would be no electrons; therefore, no molecules; therefore, no human beings! Undoubtedly, the connection be-

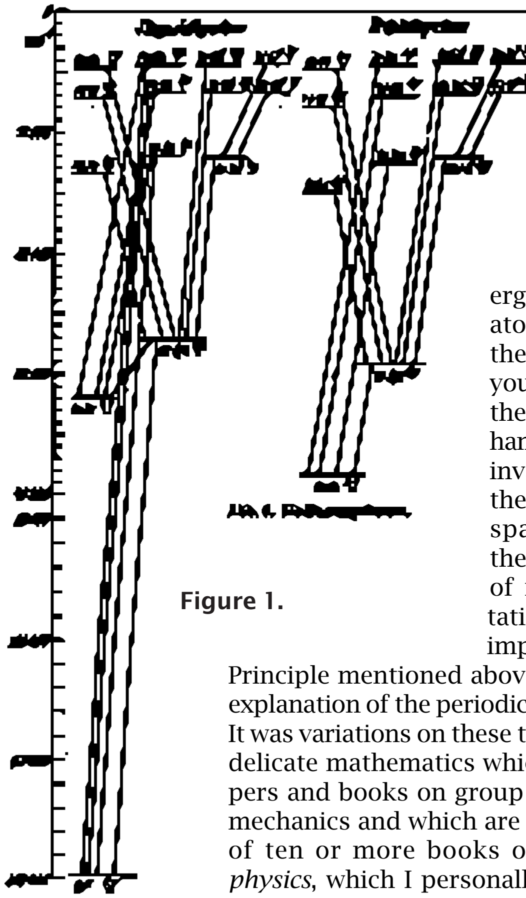


Figure 1.

tween spin and statistics is one of the most important and intriguing mysteries of physics.

Furthermore, the energy levels of an isolated atom do not depend on the direction from which you look at it! This has the consequence that the hamiltonian of an atom is invariant with respect to the rotation group in 3-space. A knowledge of the possible dimensions of irreducible representations of SO(3), plus the implications of the Pauli Principle mentioned above, quickly leads to an explanation of the periodic table of the elements. It was variations on these themes involving quite delicate mathematics which filled the early papers and books on group theory and quantum mechanics and which are still the main subject of ten or more books on *group theory and physics*, which I personally own. They have all been greatly influenced by the original book of Hermann Weyl.

For the winter semester 1927–28 Weyl had planned to give a set of lectures on group theory. However, before he started, Schrödinger and Debye, who were the leading physicists in Zurich, accepted positions elswhere. In the event, Weyl felt it was his duty to "spring into the breach" and change his projected course into one on group theory and quantum mechanics. Based on Bohnenblust's notes of this course, Weyl's classic book [14] appeared in 1928.

The five chapters were entitled:

I. Unitary Geometry

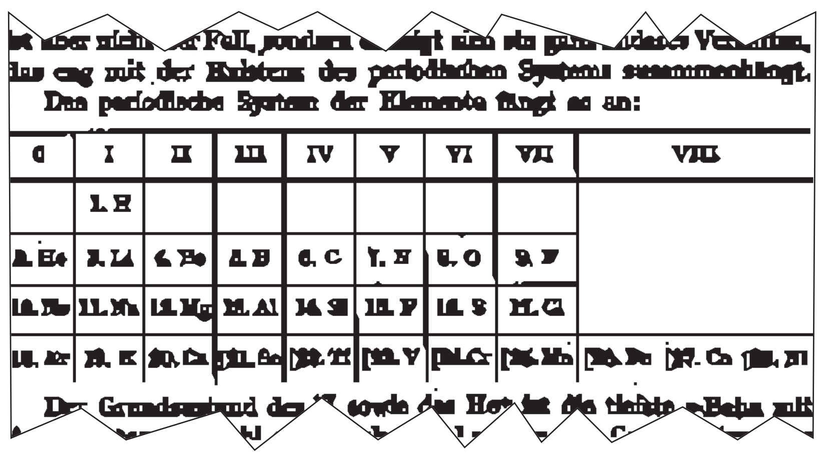
II.Quantum Theory

III. Groups and Their Representations

IV. Applications of the Theory of Groups to Quantum Mechanics: A. Rotation Group; B. The Lorentz Group; C. The Permutation Group; D. Quantum Kinematics.

V. Representations of the Symmetric and the Unitary Groups

Like his 1918 book on *Space, Time and Matter*, this book was an instant success. Already in 1930, when he moved to Göttingen after spending 1928–29 at the Institute in Princeton, he had produced a revised edition for which Chapter V was extended from 70 to 110 pages. In this transformation of Chapter V his avowed object was to avoid the use of transcendental methods (i.e., invariant integration) in favour of a thoroughly "elementary" approach. Personally, I found this revised chapter very difficult to understand, and I have never met anyone else who claimed to have read all of it carefully. It contains what Wigner in 1940 was to call the "Method of Frobenius", including the Frobenius Reciprocity Theorem applied in detail to the analysis of tensor representations of $GL(n,\mathbb{C})$ and the Schur theory of the symmetric group algebra as commutant. He developed this theory in order to estimate the energy levels of quantum systems involving several identical particles. This analysis was applied by Weyl and Wigner to understand the periodic table for atoms and now plays a role in unravelling the varieties of "elementary" particles. A key to this understanding is what physicists call "exchange energies", for which formula (8.18) in Chapter V of the English edition of Weyl's book provides an estimate.



0	I	II	III	IV	V	VI	VII	VIII
	1. H							
2. He	3. Li	4. Be	5. B	6. C	7. N	8. O	9. F	
10. Ne	11. Na	12. Mg	13. Al	14. Si	15. P	16. S	17. Cl	
18. Ar	19. K	20. Ca	21. Sc	22. Ti	23. V	24. Cr	25. Mn	26. Fe 27. Co 28. Ni

Figure 2.

Figures 1 and 2 are taken from van der Waerden's 1932 textbook on group theory methods in quantum mechanics. Figure 1 interprets the observed spectrum of helium as transitions between the energy levels partitioned into two systems according as the total spin of the two electrons is 0 or 1. In each system the levels are labeled by irreps of su(2). Figure 2 is the beginning of the Periodic Table of the elements as organized according to the Pauli Principle taking into account irreps of su(2) and the spin 1/2 of the electron.

I have never seen a reference to the fact that Chapter V contains one of the first substantial discussions of what we now refer to as "induced representations". Perhaps it was because Weyl's "elementary methods" were so obscure that few people have noticed this. It may well be that for contemporary mathematicians the transcendental methods which he used in the first edition would be more accessible.

The mathematical elegance and profundity of Weyl's book was somewhat traumatic for the English-speaking physics community. In the preface of the second edition in 1930, after a visit to the USA, Weyl wrote, "It has been rumoured that the 'group pest' is gradually being cut out of quantum physics. This is certainly not true in so far as the rotation and Lorentz groups are concerned;...." In the autobiography of J. C. Slater, published in 1975, the famous MIT physicist described the "feeling of outrage" he and other physicists felt at the incursion of group theory into physics at the hands of Wigner, Weyl et al. In 1935, when Condon and Shortley published their highly influential treatise on the "Theory of Atomic Spectra", Slater was widely heralded as having "slain the Gruppenpest". Pages 10 and 11 of Condon and Shortley's treatise are fascinating reading in this context. They devote three paragraphs to the role of group theory in their book. First they say, "We manage to get along without it." This is followed by a lovely anecdote. In 1928 Dirac gave a seminar, at the end of which Weyl protested that Dirac had said he would make no use of group theory but that in fact most of his arguments were applications of group theory. Dirac replied, "I said that I would obtain the results without *previous* knowledge of group theory!" Mackey, in the article referred to previously, argues that what Slater and Condon and Shortley did was to rename the generators of the Lie algebra of $SO(3)$ as "angular momenta" and create the feeling that what they were doing was physics and not esoteric mathematics. In this context, footnote 1 of [32] is interesting.

On the other hand, chemists were apparently more open, so that well before the appearance of [15] in 1944 they had integrated group theory ideas into their modes of thinking and happily discussed "*sigma* and *pi* bonds". With the announcement by Murray Gell-Mann and Yuval Ne'eman in 1960–61 of the "eight-fold way" as the clue to classifying hadrons, high-energy physicists, at least, have seen the light. Any current issue of the journal *Nuclear Physics B* contains a succession of Coxeter-Dynkin graphs and papers exploiting the properties of the corresponding simple Lie groups. Further, practitioners of the theory of condensed matter, which is currently one of the most active and economically important branches of physics, now constantly use representations of the crystallographic groups in order to study the Brillouin Zone, which is a key tool for understanding the energy spectrum of Bloch electrons and unravelling the properties of solids.

It follows from a result of Emmy Noether, which now appears as "Noether's Theorem" in virtually all standard graduate texts in classical mechanics, that if the properties of a physical system are invariant with respect to a continuous group, then to each infinitesimal generator of the group there corresponds a conservation law. There was an allusion to this in the previous quotation from Weyl. In particular, if the system is invariant under translation, then the linear momentum is constant—this is close to Newton's First Law. Again, if the hamiltonian does not depend explicitly on time, then energy is conserved. If the hamiltonian is invariant under rotation, angular momentum is conserved.

With the observation, about 1960, that eight of the heavy elementary particles could be classified by the weights of the adjoint representation of $SU(3)$, an unexpected new role for applications of group theory in physics appeared. Previously, the groups which played a role in quantum physics were the isometries of spacetime or subgroups thereof, the permutation group, lattice groups, or space and time inversion. But currently the real excitement surrounds the *internal symmetry groups* of nuclear and elementary particle physics. When I was young, there were *two* elementary particles: *electrons* and *protons*. Then came *neutrons* and *neutrinos* in the early 1930s. By 1960 instead of just two basic forces, electromagnetic and gravitational, there were two new forces named the *weak* and the *strong* force with mysterious properties, and there were 24 known particles, called *hadrons*, which interact by means of this strong force, which is rather short range but holds the nucleus together. As of August 1994 as well as 12 leptons which interact by the weak force, 129 hadrons had been identified. Mathematicians who seek insight into current developments in particle theory can probably do no better than to read the clear, well-written exposition [31] by Anthony Sudberry.

In a conversation I had in Göttingen with Werner Heisenberg on October 5, 1947, about Eddington's *Fundamental Theory* (which, to my chagrin, he dismissed as of little interest), he said that he had "little faith in the then-current meson theories which seemed to appear at the rate of a new one in each issue of *Physical Review*". He continued, "A satisfactory theory would predict an infinite number of elementary particles whose masses would follow from some systematic context, such as an eigenvalue problem."

Since it now seems that the only limiting factor on the discovery of new particles is the power of the accelerators, the first part of Heisenberg's belief was apparently well founded. Though hitherto no eigenvalue problem for the masses has been discovered, based on the representation theory of the direct product $U(1) \otimes SU(3) \otimes SU(6)$, a number of rather ad hoc formulas for the masses of the elementary particles have been proposed.

A topic which has become very important in the recent physics literature—scarcely addressed by Weyl or his immediate successors—is that of *broken symmetry*, in which the relation of the representations of a group and its subgroups is of central importance in understanding a physical situation. Of the many books on group theory and physics with which I am familiar, only that of Lyubarskii [16] discusses *phase transitions*. His discussion is thorough and clear. As an example, for a crystal of a particular symmetry type he exhibits the possible structures of lower symmetry to which a second-order phase transition would be possible. A similar type of analysis is used by Ozaki [17] to show that the solution of the Hartree-Fock-Bogolubov equations for a triclinic lattice state is capable of a transition to any one of twenty-four symmetry-breaking phases of the paramagnetic state and to seventeen of the ordinary Bardeen-Cooper-Schrieffer state.

Finally, without pretending to have covered all of the many applications of group theory in quantum mechanics, we mention briefly the idea of *dynamical* or *noninvariance groups*. If the eigenspaces of constant energy are reducible, this is generally taken as evidence that one has not used the *complete* symmetry group of the hamiltonian. If all the energy eigenspaces span irreducible representations of the symmetry group, one still has the problem of finding the eigenvalues. For numerical calculations the common approach is to choose a finite orthonormal basis of functions with which to perform an approximate calculation. How to choose an appropriate basis set? If G is the complete symmetry group of the hamiltonian, it has proved helpful sometimes to find a group G' such that $G \subset G'$ and to consider an irreducible representation of G' which when restricted to G decomposes into irreducible representations, including the ones you wish to study. In [18] Wulfmann illustrates this approach to symmetry breaking by applying it to the atoms in the first row of the periodic table. It is also widely used in classifying elementary particles. There are several clusters of eight or ten elementary particles, each of which, to a fair degree of accuracy, can be related to representations of $SU(3)$. Some light on the mutual relations of these clusters has been obtained by regarding $SU(3)$ as a subgroup of $SU(4)$ or of $SU(6)$.

Most of the books with titles such as *Group Theory and Physics*—and they are legion—treat only quantum mechanics. In this respect Weyl's book [14] was less pretentious, claiming only to deal with quantum mechanics and actually covering essentially all that was known about quantum mechanics when it was written. Even the 2-volume, 1,000-page treatise of Cornwell [19], which is very well written, does not touch on all the topics described above. I doubt that any one person would be capable of covering all current applications of group theory to quantum mechanics.

And there are applications of group theory to other parts of physics. Let us turn briefly to some of these with which I happen to be familiar.

Relativity, Crystals et al.

Of course, the Lorentz group springs immediately to mind. Well before Einstein entered the lists it had been remarked that the equations of Maxwell for electromagnetic theory were invariant under the Lorentz group [20]. In fact, they are invariant under the larger *conformal* group. It is now generally admitted (apart from a few Einstein sycophants) that the term *relativity* and, more importantly, the current dogma that a physical law should have a formulation which is valid for all coordinate systems were proposed by Poincaré about 1895. It may well be that he was not the first to do so.

There are many ways available of motivating acceptance of the key role which the Lorentz group plays in modern physics. The approach which I prefer is set forth by Alfred North Whitehead in [21]. First he argues that the formulas for changing coordinates from one system to another moving uniformly with respect to the first will be linear. The assumption that the set of transformations *is a group* then implies the existence of an absolute constant k such that the qualitative nature of the transformations will be radically different depending on whether k is negative, zero, or positive. Since $k = 0$ corresponds to the form familiar in newtonian mechanics, we infer that if $k \neq 0$, it will be very small. Negative k implies bizarre physical phenomena, so we are left with the choice $k = 0$, or $\dot{k} > 0$, but very small—equal, say, to c^{-2}, where c is a large real number. Newtonian mechanics is invariant under the transformations if $k = 0$, whereas Maxwell's theory is invariant for $k > 0$ with c equal to the speed of light. In the latter case we obtain what Whitehead, writing in 1919, refers to as "the Larmor-Lorentz-Einstein theory of electromagnetic relativity". In [20] Whittaker refers to the "Poincaré-Einstein Theory". Readers of the *Mathematical Intelligencer* are familiar with the controversy over this question of name, which was rekindled by the article of Jeremy Gray [22].

I like the argument in [21], even though it is rather long and turgid, because it illustrates the power of the group concept. By merely assuming, as seems obvious, that the set of transformations among inertial systems is a group, we are able to infer the existence of an important physical constant which is independent of the observer. It was Einstein who was able to deduce the radical new consequences of the proper choice for k and forced us to revolutionize our understanding of space and time. It was Dirac's insistence that quantum mechanics be consistent with special relativity which forced the realization that particles and antiparticles exist in duality. For readers who actually look up Whitehead's argument, I should remark that my k is the reciprocal of his corresponding constant.

Paul A. M. Dirac (left) and Richard P. Feynman. Photograph taken by A. J. Coleman at the International Conference on Relativistic Theories of Gravitation, Warsaw, Poland, July 25–31, 1962.

Photograph courtesy of the AIP Emilio Segrè Visual Archives.

While the requirement that an equation such as Dirac's be invariant under the action of a particular group will not in general determine the equation uniquely, for Einstein's gravitational equations Cartan showed that this is almost the case in his remarkable paper [23] in 1922. Einstein was looking for a symmetric tensor G_{ij}, which, when set equal to the energy-momentum tensor for local matter and radiation, would determine the structure of space-time. It had to be determined by the ten components g_{ij} of the metric and their derivatives. Since he wanted to obtain Newton's theory in first approximation and the equation for the newtonian potential involves the Laplacian, Einstein postulated that G_{ij} should contain at most second-order derivatives of g_{ij} and that these occurred linearly. But—and this was very important—being a democrat at heart, Einstein insisted that his equation be covariant, that is, independent of the coordinate system. Cartan proved a posteriori that Einstein had discovered the *only possible* set of equations consistent with his desiderata. As a mathematician I regret to admit that he did this, as subsequent interchange [24] with Cartan demonstrates, with minimal or zero understanding of the marvelous subject we call group theory! Indeed, it has been not infrequent that the intuition of a physicist has outpaced the ratiocinations of mathematicians.

However, Cartan went further. He was working in what we now call the *principal bundle* two or three decades before this idea was formalized. He showed that the components of the Riemann curvature tensor $R_{ij,k\ell}$ could be regarded as spanning a representation of the Lorentz group at each space-time point with irreducible components of dimension 1, 9, and 10. The first two were combined in the Ricci tensor, which Einstein needed. But the 10-dimensional tensor was new. It is now called the *Weyl tensor*, since it may be used to characterize the 1918 theory in which Weyl attempted to unify the gravitational and electromagnetic fields.

I am not sure that other readers of [24] would agree with my impression that it records an example of the *failure* of group theory to influence physics! If Einstein had had the mathematical background necessary to understand Cartan, he would not have spent the last decade of his life in his fruitless search for a unified theory and might have made additional revolutionary contributions to physics. However, before making depreciatory comments about Einstein's mathematical knowledge, it would be wise to note that Weyl admitted in 1938 that he found Cartan's writings quite difficult; and as Chern and Chevalley correctly state in their essay on Cartan's mathematical work [25], the rest of the mathematical world was not much better than Weyl!

Many books on group theory begin with a discussion of the crystallographic groups. Indeed, this is an ideal topic with which to introduce group ideas. As presented in some books by mathematicians, the uninformed reader might be left with the impression that it was a clever use of group theory which led to the analysis of crystal structure. However, the date 1784 is often taken as the moment at which understanding of the structure of crystals really began with the announcement by Abbé René Just Haüy (1743–1822) of his *Law of Rational Indices*. J. F. C. Hessel (1796–1872) established in 1830 that there are 32 symmetry classes. Auguste Bravais (1811–1863) described the 14 types of space lattices in 1850. This was achieved by mineralogists using, of course, ideas about symmetry, but well before the idea of "group" had even been formulated by mathematicians. When the 32 point-groups were combined with translations, including screw glides, etc., it was found that there are 230 distinct "space-groups". By one of the most remarkable instances of independent discovery on record, this was established independently by E. S. Fëderov (1853–1919) in Russia in 1890, by A. M. Schönfliess (1853–1928) in Germany in 1891, and by W. Barlow (1845–1934) in England in 1894. A clear and complete description of the space-groups can be found in [16] or [26].

Dimensional analysis is a very useful tool for engineers and physicists in making a preliminary sketch of a theory of any physical topic. I have seen, but cannot recall where, an exposition of dimensional analysis based on the theory of an abelian group on three generators: in this case, T, L, M representing units of time, length, and mass respectively. This was an application of group theory to a very basic level of physics.

We have already noted that "continuous groups" were invented by Lie in order to study differential equations. Certain notable results were achieved by Lie and his school. In particular, a theory was developed of when ordinary differential equations can be solved by quadratures. However, developments in algebra were so exciting in the early decades of this century that the attention of mathematicians was diverted from Lie's original aim. In the introduction to his book [27], Olver asserts that the 1918 theorem of Emmy Noether really opened up the possibility of applying Lie's methods to the analysis of shock waves, solitons, scattering theory, stability, relativity, fluid mechanics, elasticity, and any aspect of physics in which differential equations play a significant role. However, it is only in the period since 1960 that a major research effort has been deployed to exploit these possibilities. Stephani was led to write a book [27] about the use of symmetry groups in elucidating the structure of the solutions of differential equations when he realized this would help him to understand Einstein's equations.

Mathematicians unfamiliar with these recent developments could do no better than to page through the handbook [28] edited by Ibragimov for a vivid impression of the variety, scope, and profundity of the applications of Lie's method which have emerged in roughly the past thirty years. It has been fueled in large measure by the development of algebraic computing. This is needed in order to manage the extraordinarily complicated formulas which must be deployed. Programs such as MAPLEV have put enormous new power at the disposal of researchers. I quote, as an example, GRTensorII discussed in [29]. This is a package for performing all the component calculations for tensors, bases, null tetrads, etc., for anyone working in general relativity theory. It is being developed at too great a pace to be distributed by the printed word, so has been made freely available as shareware on the WWW at `http://astro.queensu.ca/~grtensor/GRHome.html`.

It can easily happen that the continuous groups that appear in this context are parameterized by one or more arbitrary functions. Since these are not finite-dimensional groups, they force us to explore the role of infinite-dimensional Lie groups in physics. This, again, is a huge area which has experienced very active exploration since the definition of Kac-Moody algebras simultaneously by Moody in Canada and Kac in Russia in 1967. K-M algebras soon found applications to the theory of solitons and numerous nonlinear equations such as that of Korteweg-de Vries. For these developments I refer the reader to Chapter 14 of [30] and to the article by Louise Dolan, "The Beacon of Kac-Moody Symmetry in Physics", in the *Notices*, December 1995.

Undoubtedly there are many significant applications of group theory both to quantum physics and to nonquantum physics to which I have not referred. My dogmatic opinions will surely call forth a flood of letters to the editor denouncing my egregious errors and/or insufferable lacunae. This could only stimulate a desirable continuing discussion of the interaction between physics and group theory—a topic which is wide-ranging, creative, exciting, and rapidly expanding.

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Note: "OC" is a reference to the Complete Works of Cartan in the 1984 edition published by Springer-Verlag.