From Matrix Mechanics and Wave Mechanics to Unified Quantum Mechanics

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Editor's note: Like many mathematicians of his generation, van der Waerden was extremely broad. In 1932 he published one of the first books on group theory and quantum mechanics. In the following remarkable paper, delivered in Trieste in September 1972 at a symposium celebrating Dirac's seventieth birthday, he clarifies the connection between the Heisenberg and Schrödinger formulations of quantum mechanics and earlier work by Cornelius Lanczos.

The story I want to tell you begins in March 1926 and ends in April 1926. Early in March two separate theories existed: matrix mechanics and wave mechanics. At the end of April these two had merged into one theory, more powerful than the two parents taken separately.

Wave mechanics was based upon three fundamental hypotheses:

A. Stationary states are determined by complex-valued wave functions $\psi(q)$, which remain finite everywhere in q-space.

B. The functions ψ satisfy a differential equation

$$H\psi = E\psi$$

in which the operator H is obtained from the classical Hamiltonian $H(p, q)$ by replacing every momentum p by

$$\frac{K}{i}\frac{\partial}{\partial q}, K = \frac{h}{2\pi}.$$

C. The eigenvalues E are the energy values.

To these three hypotheses, Schrödinger added Bohr's postulate:

D. $E_m - E_n = h\nu_{mn}$.

This theory was presented in Schrödinger's first and second communications on "Quantisierung als Eigenwertproblem" in *Annalen der Physik* **79**. The first communication was received on 27 January, and the second on 23 February 1926.

On the other hand, *matrix mechanics* was invented by Heisenberg in June 1925, and presented in a fully developed form in Dirac's first paper on quantum mechanics (received 7 November 1925) and also in the famous "three-men's paper" of Born, Heisenberg and Jordan (received 16 November 1925). This theory was based upon four mechanical hypotheses and two radiation hypotheses. The mechanical hypotheses are:

1. The behaviour of a mechanical system is determined by the matrices $\mathbf{p}$ and $\mathbf{q}$ (one matrix $\mathbf{q}$ for every coordinate q, and one $\mathbf{p}$ for every momentum p).
2. $\mathbf{pq} - \mathbf{qp} = (\frac{K}{i})\mathbf{1}$ if p belongs to the same coordinate q, otherwise equal to 0.
3. $H(\mathbf{p}, \mathbf{q}) = W$ = diagonal matrix, having diagonal elements E_n, the energy values.
4. Equations of motion:

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$$

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These hypotheses imply

$$p_{mn} = a_{mn} e^{2\pi i(\nu_m - \nu_n)t}$$

$$E_n = h\nu_n.$$

The radiation hypotheses determine the frequency and intensity of the radiation emitted or absorbed:

5. $E_m - E_n = h\nu_{mn}$.
6. The transition probabilities are proportional to the $|a_{mn}|^2$.

In his second communication, Schrödinger confesses that he did not succeed in finding a link between his own approach and Heisenberg's. This was written in February 1926, but in March he found the link. In his paper "Über das Verhaltnis der Heisenberg-Born-Jordanschen Quantenmechanik zu der meinen", received 18 March 1926, Schrödinger writes: "In what follows…the *inner connection* between Heisenberg's Quantum Mechanics and my own will be made clear. From the formal mathematical standpoint one may even say that the two theories are identical."

Now is this true? Are the two theories really equivalent in the formal mathematical sense?

Equivalence (or identity, as Schrödinger says) would mean

$$A, B, C, D \iff 1, 2, 3, 4, 5, 6.$$

Now what Schrödinger actually proves is

$$A, B, C \implies 1, 2, 3,$$

and of course

$$D \implies 5.$$

Moreover, if time-dependent functions ψ are allowed, satisfying Schrödinger's time-dependent differential equation, one can prove 4. However, hypothesis 6 can in no way be derived from Schrödinger's set of hypotheses.

The converse $\impliedby$ Schrödinger does not even attempt to prove. Yet he refers to his proof as *"Äquivalenz-Beweis"*, and he asserts confidently: "Die Äquivalenz bestecht *wirklich*, sie besteht *auch in umgekehrter Richtung*."

From the formal logical point of view, one may even say that it is impossible to derive A, B, C from $1, 2, 3, 4, 5, 6$, because in hypothesis A the notion "stationary state" occurs, which does not occur in $1, 2, 3, 4, 5, 6$.

After the publication of this paper, everybody accepted Schrödinger's conclusion that the two theories are "equivalent". Everybody except Pauli. He knew better.

On April 12, just after the publication of Schrödinger's first communication, but before his "equivalence" paper came out, Pauli wrote a very remarkable letter to Jordan, in which he established the connection between wave and matrix mechanics, in a logically irreproachable way, independent of Schrödinger. He never published the contents of this letter, but he signed the carbon copy (which is quite unusual), and he kept the letter in a plastic cover until his death. I am indebted to his widow, Franca Pauli, for giving me her consent to publish this letter.

PAULI'S LETTER

[This letter was probably written and typed at Copenhagen]

12th April 1926

Dear Jordan,

Many thanks for your last letter and for your looking through the proof sheets. Today I want to write neither about my Handbüch-Article nor about multiple quanta; I will rather tell you the results of some considerations of mine connected with Schrödinger's paper "Quantisierung als Eigenwertproblem" which just appeared in the *Annalen der Physik*. I feel that this paper is to be counted among the most important recent publications. Please read it carefully and with devotion.

Of course I have at once asked myself how his results are connected with those of the Göttingen Mechanics. I think I have now completely clarified this connection. I have found that the energy values resulting from Schrödinger's approach are always the same as those of the Göttingen Mechanics, and that from Schrödinger's functions ψ, which describe the eigenvibrations, one can in a quite simple and general way construct matrices satisfying the equations of the Göttingen Mechanics. Thus at the same time a rather deep connection between the Göttingen Mechanics and the Einstein-de Broglie Radiation Field is established.

To make this connection as clear as possible, I shall first expose Schrödinger's approach, styled a little differently. According to Einstein and de Broglie one can assign to any moving particle with energy E and momentum G, taking care of the relativity terms,

$$G = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}, E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

normed in such a way that the energy at rest is $= m_0 c^2$, hence $E^2 - c^2 G^2 = m_0^2 c^4$ an oscillation with frequency $\nu = E/h$ and wave length $\lambda = h/|G|$. (This assignment is invariant with respect to Lorentz transformations.) The phase velocity V is

$$V = \lambda\nu = \frac{E}{|G|},$$

hence the wave equation of de Broglie's radiation field

$$\Delta\psi - \frac{1}{V^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

assumes the form

$$\Delta\psi - \frac{G^2}{E^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{1}$$

Taking care of the relation

$$E^2 - c^2G^2 = m_0^2c^4$$

between energy and momentum, one obtains

$$\Delta\psi - \frac{E^2 - m_0^2c^4}{c^2E^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{2}$$

Now if we have a mass point moving in a field of force and if E_{pot} is its potential energy, the relation between energy and momentum becomes (taking care of the variability of the mass)

$$(E - E_{pot})^2 - c^2G^2 = m_0^2c^4$$

provided E is again normalized so that for the mass point at rest $E - E_{pot} = m_0c^2$. (For the hydrogen atom with relativistic correction one obviously has to put $E_{pot} = -Ze^2/r$). Substituting this into (1) one obtains instead of (2)

$$\Delta\psi = \frac{[E - E_{pot}(x,y,z)]^2 - m_0^2c^4}{c^2E^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{3}$$

The phase velocity now depends on position.

Schrödinger's approach is now as follows: *A quantum state of the system with energy E is only possible if a standing de Broglie-Wave without spatial singularities, depending on t like a sine function with frequency $\nu = E/h$, can exist in accordance with* (3).

So one has to replace ψ in (3) by a product of a new function $\bar{\psi}(x,y,z)$ depending only on position with the factor

$$e^{2\pi i\nu t} = e^{2\pi i(E/h)t}$$

thus obtaining

$$\psi = \bar{\psi}e^{2\pi i(E/h)t}$$

then

$$\frac{\partial^2\psi}{\partial t^2} = -\frac{4\pi^2}{h^2}E^2\psi$$

and one obtains

$$\Delta\bar{\psi} + \frac{[E - E_{pot}(x,y,z)]^2 - m_0^2c^4}{c^2K^2}\bar{\psi} = 0, \tag{4}$$

putting, as Schrödinger does, $K = h/2\pi$.

This is an eigenvalue problem for the possible values of $E = h\nu$. These ν are enormously large, because in E the energy of the electron at rest is included. The Frequency Condition now says that the light waves can formally be considered as difference-oscillations of the de Broglie-radiation. Planck's constant enters the theory only at that point where one passes from the energy of the states to the frequency of the radiation of de Broglie.

Neglecting relativistic corrections one obtains from (4) by putting $E = m_0c^2 + \bar{E}$ and expanding according to powers of $1/c^2$:

$$\Delta\bar{\psi} + \frac{2m_0}{K^2}(\bar{E} - E_{pot})\bar{\psi} = 0. \tag{5}$$

This equation is given in Schrödinger's paper, and he also shows how it can be derived from a Variation Principle.

Here is another remark for which I am indebted to Mr. Klein. The difference between the general Quantum Theory of periodic systems and Schrödinger's Quantum Mechanics based upon Equation (5) is, from the point of view of the de Broglie-Radiation, the same as the difference between Geometrical Optics and Wave Optics. Namely if the wave length of the de Broglie-Radiation is small, one can put in (5), as is well-known

$$\bar{\psi} = e^{i(1/K)S}.$$

If S/K is large, one now obtains from (5), according to Debye, the Hamilton-Jacobi differential equation for S. In this case $\bar{\psi}$ becomes a univalued point function only if the moduli of periodicity of S/K are integer multiples of 2π. This leads to the usual condition $\int pdq = nh$, which has been interpreted already by de Broglie from the point of view of the geometrical optics of his Radiation Field.

In reality, however, S/K is not large generally, so one has to stick to (5) and to use the mathematics of Wave Theory to integrate this equation.

Next comes my own contribution, namely the connection with the Göttingen Mechanics. For the sake of simplicity I shall consider a one-dimensional problem and use Cartesian coordinates (in the three-dimensional case and with arbitrary coordinates everything goes just so, also if gyroscopic terms are added). So let the wave-equation be given as

$$\frac{d^2\psi}{dx^2} + \frac{2m}{K^2}[E - E_{pot}(x)]\psi = 0$$

(compare (5), the bars are omitted).

Now let $E_1, E_2, \ldots, E_n, \ldots$ be the eigenvalues, $\psi_1, \psi_2, \ldots, \psi_n, \ldots$ a complete set of eigenfunctions. For these we have

$$\int_{-\infty}^{+\infty}\psi_n\psi_m dx = \begin{cases} 0 & \text{for } n \neq m \\ 1 & \text{for } n = m \end{cases}$$

The first equation (orthogonality) follows from Green's formula, the second means a normalization of the multiplicative constants in the ψ_n. Any arbitrary function of x can be expanded in a series with respect to the ψ_n. Now one considers in particular the expansion of $x\psi_n$

$$\text{(I)} \qquad x\psi_n = \sum_m x_{nm}\psi_m(x); \qquad x_{nm} = \int_{-\infty}^{+\infty} x\psi_n\psi_m dx$$

One also puts

$$\text{(II)} \qquad (p_x)_{nm} = iK\int_{-\infty}^{+\infty} \frac{\partial\psi_n}{\partial x}\psi_m dx; \qquad iK\frac{\partial\psi_n}{\partial x} = \sum_m (p_x)_{nm}\psi_m(x)$$

(i = imaginary unit, $K = h/2\pi$). Now $x_{nm} = x_{mn}$ is real, $(p_x)_{nm} = -(p_x)_{mn}$ purely imaginary. It can be shown without difficulty, that the matrices for x and p_x thus defined satisfy the equations of the Göttingen Mechanics. Namely

$$\mathbf{p}_x\mathbf{x} - \mathbf{p}x_x = -iK,$$

$$\frac{1}{2m}\mathbf{p}_x^2 + \mathbf{E}_{pot}(x) = \mathbf{E} \quad \text{(Diagonal matrix)}$$

From the rule of multiplication it follows that the matrix belonging to any function $F(x)$ of x is just given by

$$F_{nm} = \int_{-\infty}^{+\infty} F(x)\psi_n\psi_m dx.$$

I shall not write out the calculations in detail; you will be able to verify the assertion easily.

I have calculated the oscillator and rotator according to Schrödinger. Further the Hönl-Kronig-formulae for the intensity of the Zeeman components are easy consequences of the properties of the spherical harmonics. Perturbation theory can be carried over completely into the new theory, and the same thing holds for the transformation to principal axes, which in general is necessary if degeneracies (multiple eigenvalues) are cancelled by external fields of force. At the moment I am occupying myself with the calculation of transition probabilities in hydrogen from the eigenfunctions calculated by Schrödinger. For the Balmer lines finite rational expressions seem to come out. For the continuous spectrum the situation is more complicated: the exact mathematical formulation is not yet quite clear to me.

As regards Lanczos, my considerations have only very few points of contact with his ideas. He considers a problem for which the eigenvalues are the reciprocal energy values, whereas here the eigenvalues are just the energy values. In his exposition certain functions depending, like Green's function, on two points, play an essential role; such functions are not used here. On the whole I feel that Lanczos' approach has not much value.

About the physical significance of the expressions (I) and (II) I do not know much. In any case they seem to be connected with the idea that the ordinary light waves are different oscillations (beats) of de Broglie's radiation. The fact that in (I) and (II) no indefinite phases occur is due to the trivial reason that in passing from (3) to (4) the periodic factor depending on time has been suppressed. If this factor is taken into account, one obtains in x_{nm} and $(p_x)_{nm}$ besides $\exp[(2\pi i/h)(E_n - E_m)t]$ a phase factor $\exp[2\pi i(\delta_n - \delta_m)]$ in which δ_m and δ_n are the phases of eigenvibrations belonging to E_m and E_n. In principle, in the Göttingen theory as well as in de Broglie's statement of the quantum problem, no description of the motion of the electron in the atom in space and time is given. In the latter theory this is clear from the fact that outside the domain of validity of Geometrical Optics it is impossible to construct "rays" in de Broglie's Wave System that can be considered as orbits of particles. The problem of the asymptotic linkage with the usual pictures in space and time for the limiting case of large quantum numbers remains unsolved. Yet it is a definite progress to be able to see the problems from two different sides. It seems one also sees now, how from the point of view of Quantum Mechanics the contradistinction between "point" and "set of waves" fades away in favour of something more general.

Cordial greetings for you and the other people at Göttingen (especially Born, in case he is back from America; please show him this letter).

Yours, (carbon copy signed) W. Pauli

COMMENTS ON PAULI'S LETTER

Pauli's wave equation (1) is called in the letter "the wave equation of de Broglie's radiation field." It is not given in de Broglie's thesis, but it is very easy to derive it from the given expressions for ν and λ. It is valid for plane waves, i.e. for a free election.

Equation (3) is essentially the Klein-Gordon equation. It is true that the magnetic terms are missing, but Pauli expressly says in the course of his letter that "everything goes just so if gyroscopic terms are added," which shows that Pauli, who was at that time thinking very hard about the anomalous Zeeman effect, knew perfectly well how to handle magnetic fields. He omitted the magnetic terms only "for the sake of simplicity."

We know from Schrödinger's letters that he also tried the Klein-Gordon equation, but he gave it up because it did not yield the right fine-structure of the hydrogen atom.

The Klein-Gordon equation was discovered independently by Schrödinger, by Pauli, by Klein and Gordon, and by at least two other people.[1]

To Pauli's orthogonality relations

$$\int_{-\infty}^{+\infty} \psi_n \psi_m dx = \begin{cases} 0 & \text{for } n \neq m \\ 1 & \text{for } n = m \end{cases}$$

we may remark that in the one-dimensional case the eigenfunctions are single and real, so that complex conjugate factors ψ_n^* are not needed.

The paper of Lanczos, to which Pauli refers at the end of his letter, was published in *Zeltschrift für Physik* **35** (received 22 December 1925). I feel it has more value than the contemporaries suspected. Let us use our hindsight and start with Schrödinger's equation, which I shall write as

$$-\Delta\psi + V\psi = E\psi \qquad (6)$$

leaving out all numerical factors. Lanczos considers a finite domain in q-space, so let us enclose our atom in a large sphere of radius R. As a boundary condition we may assume $\psi = 0$ on the boundary. Since the zero-point on the energy scale is quite arbitrary, we may suppose that it lies below the lowest energy value. It follows that zero is not an eigenvalue.

Under these assumptions, the boundary value problem

$$-\Delta\psi + V\psi = u,$$
$$\psi = 0 \text{ on the boundary}$$

can be solved by means of Green's function $K(P, Q)$ as follows;

$$\psi(P) = \int K(P, Q)u(Q)dQ. \qquad (7)$$

Replacing $u(Q)$ by $E\psi(Q)$, and dividing by E, one obtains

$$\int K(P, Q)\psi(Q)dQ = \frac{1}{E}\psi(P). \qquad (8)$$

This integral equation is equivalent to Schrödinger's equation (6). Its eigenvalues are just $1/E$, the reciprocal energy values.

Now this is just the kind of integral equation Lanczos considers. He does not specify what kind of function $K(P, Q)$ is, but he does say that the eigenvalues of his integral equation are the reciprocal energy values.

Now let us hear what Schrödinger says about the paper of Lanczos. In a footnote on p. 754 of his "equivalence" paper he writes:

> Similar ideas are exposed in an interesting paper of Lanczos, which already contains the valuable insight that Heisenberg's atomic dynamics is capable of a continuous interpretation. For the rest, the paper of Lanczos has less points of contact with mine than one might think at first sight.
>
> The determination of the system of formulae, which Lanczos leaves quite undetermined, *cannot* be found in the direction of identifying the symmetrical kernel $K(s, \sigma)$ with Green's function of our wave equation… *For this function of Green, if it exists, has as its eigenvalues the quantum levels themselves.*

This is an error of Schrödinger, for which I have no explanation. We have just seen that the eigenvalues of Green's kernel $K(P, Q)$ are $1/E$ and not E.

Schrödinger continues: "On the contrary, the kernel of Lanczos is required to have as its eigenvalues the *reciprocal* quantum levels."

Schrödinger just missed the point.

If Lanczos' kernel $K(P, Q)$ is identified with the Green's function of Schrödinger's differential equation, its eigenfunctions $\phi_1, \phi_2, \ldots$ are Schrödinger's eigenfunctions.

Besides the integral operator K defined by the kernel $K(s, \sigma)$:

$$K\psi(s) = \int K(s, \sigma)\psi(\sigma)d\sigma,$$

Lanczos introduces two more integral operators **p** and **q**:

$$\mathbf{p}\psi(s) = \int p(s, \sigma)\psi(\sigma)d\sigma,$$
$$\mathbf{q}\psi(s) = \int q(s, \sigma)\psi(\sigma)d\sigma$$

in such a way that

$$\mathbf{pq} - \mathbf{qp} = \frac{1}{2\pi i}\mathbf{1}. \qquad (9)$$

Since **p** and **q** are supposed to be integral operators, the unit operator **1** must also be an integral operator

$$\mathbf{1}\psi(s) = \int E(s, \sigma)\psi(\sigma)d\sigma.$$

This implies, as Lanczos says, that $E(s, \sigma)$ is zero for a $\sigma \neq s$, and that

$$\int_{-\infty}^{+\infty} E(s, \sigma)d\sigma$$

is equal to 1. Hence, Lanczos' function $E(s, \sigma)$ is just Dirac's function $\delta(s - \sigma)$.

Lanczos concludes that the functions $p(s, \sigma)$ and $q(s, \sigma)$ cannot be everywhere finite. In fact, if one wants to reach complete agreement between Lanczos, Schrödinger, and Pauli, one has to assume

[1] *Jagdish Mehra has informed me that the other people were V. Fock (*Z. Phys. *38.* ***242*** *(1926), 39.* ***226*** *(1926)); H. van Dungen and Th. de Donder (*Compt. Rend. Acad. Sci. Paris*, July 1926); and J. Kudar (*Ann Physik *81.* ***632*** *(1926)).*

$$q(s, \sigma) = s \cdot \delta(s - \sigma),$$
$$p(s, \sigma) = -\frac{h}{2\pi i}\delta'(s - \sigma).$$

Next, Lanczos defines the matrices corresponding to the operators **p** and **q**:

$$p_{ik} = \int p(s, \sigma)\phi_i(s)\phi_k(\sigma)dsd\sigma$$
$$q_{ik} = \int q(s, \sigma)\phi_i(s)\phi_k(\sigma)dsd\sigma$$

and proves that the matrices **p** and **q** satisfy the Born-Jordan condition

$$\mathbf{pq} - \mathbf{qp} = \frac{h}{2\pi i}\mathbf{1}.$$

From this analysis we see that Lanczos' approach had more points of contact with the ideas of Schrödinger and Pauli than these two suspected. His weakness was that he was not able to specify his functions $K(s, \sigma)$, $p(s, \sigma)$ and $q(s, \sigma)$.

Let us now return to Pauli's letter. Pauli says at the end: "The problem of the asymptotic linkage with the usual pictures in space and time for the limiting case of large quantum numbers remains unsolved."

More light on this problem was shed by the study of the behaviour of wave packets. A most interesting contribution was a little-known paper of Ehrenfest, in which he proved that the centre of gravity of a wave packet moves according to the classical law: force = acceleration × mass, provided the force exerted upon the electron by the electromagnetic field is calculated by integrating the Lorentz force over the charge density $-e\psi^*\psi$. Another important contribution was, of course, Heisenberg's Uncertainty Principle, which was also derived from the study of the behaviour of wave packets in q-space and p-space.

A quite new point of view was Born's interpretation of $\psi^*\psi$ as a probability density, proposed in connection with his study of collisions. Dirac extended Born's probability interpretation to much more general measurements. However, in this lecture I wanted to restrict myself to what happened in March and April of 1926, so I shall stop here.

Editor's notes: 1. Unknown to van der Waerden, Cornelius Lanczos, whom he had never met, was in the audience. When the moderator introduced them, van der Waerden was visibly pleased and exclaimed, "Oh, This is marvelous. I didn't know that you were here at this symposium or that you would come to this lecture."

Later in the discussion period, Lanczos made a remark about Einstein's approach to quantization. van der Waerden then took the opportunity to initiate the following exchange:

van der Waerden (to Lanczos): Did you know all this to which I have referred in my paper? Were you aware of these connections?

Lanczos: You are absolutely right. You rehabilitated my work. Pauli was a vicious man, as everybody knows. Anything which didn't agree with his ideas was wrong, and anything was right only if he made it, if he discovered it, which is all right for such a great man. He could allow himself such viciousness, but I am very grateful to you for pointing out what you have.

I certainly was aware of these connections, but as you pointed out, the weakness was that I didn't specify the kernel function and the special functions. At that time, you know, after the matrix mechanics it looked as if you couldn't do anything with the continuum, and one would have to operate with the discontinuum. Everything is discontinuous in the matrices. Now I was very much interested at that time in integral equations, and actually the integral equation which I used was not the Schrödinger equation, but the inverse equation, because only a differential equation can be changed to an integral equation, as you pointed out. And that is the reason why the energies actually come in with a reciprocal, so that the Green's function which I had is basically the Schrödinger equation with the source of a delta function. If you have a source, and that source happens to be a delta function, then this function does have a physical significance, whereas it looks after Pauli's criticism that it has no physical significance. But it does have a physical significance.

van der Waerden: Yes, but after you read Schrödinger's paper, did you realize that this was the case?

Lanczos: Afterwards, it was too trivial. I mean, it was no longer of interest, because Schrödinger came along and he did it. As it often happens, it is the second man who hits the nail on the head and not the first one.

2. Cornelius Lanczos emigrated from Hungary to the U.S. in 1931 to take a position at Purdue University, and joined the AMS in 1934. In 1952 (while Schrödinger was acting director) Lanczos moved to the Dublin Institute for Advanced Study where he remained (including serving as director for a time) until his death in 1974 at athe age of 81.

W. Moore's biography *Schrödinger's Life and Thought* (Cambridge University Press, 1989) indicates that Schrödinger and Lanczos had considerable social as well as scientific contact during the years they were both in Dublin. He also writes (p. 450) "Erwin was quite fond of him (Lanczos) but this feeling was not reciprocated."

3. Independently of Schrödinger or Pauli, Carl Eckart, (then a young American at Caltech) established the relationship between the Heisenberg and Schrödinger formulations of quantum mechanics in a paper published in *Phys. Rev.* **28**, (1926) 711–26. See K. Sopka *Quantum Physics in America*, vol. 10, The History of Modern Physics (American Institute of Physics, 1988).