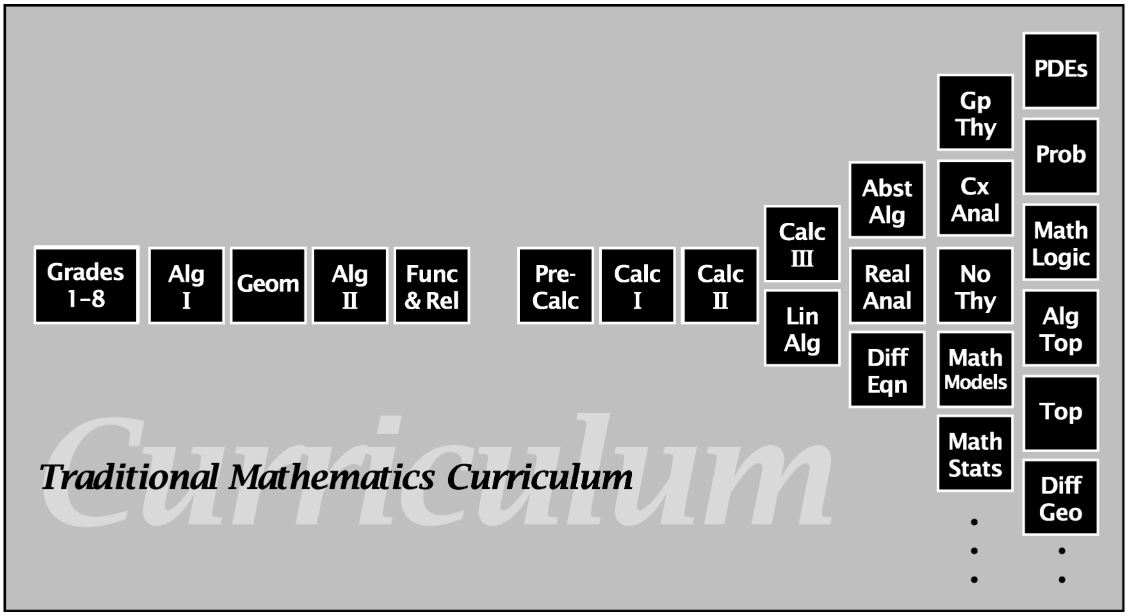
Do We Need Prerequisites?

Donal O'Shea and Harriet Pollatsek



Introduction

The schematic above represents the traditional mathematics curriculum from grade school to baccalaureate.

There are, to be sure, differences from region to region and from school to school: geometry may come after algebra, courses may be pushed down one year to allow high school seniors to take calculus. Some colleges offer additional mathematics courses in the first two years, such as Discrete Mathematics or Math for Poets, and many offer several strands of calculus. Moreover, as the implementation of the NCTM Standards proceeds, the dependence of K–12 mathematics courses, one upon the preceding, suggested by the diagram, is lessening. Nonetheless, the structure of the mathematics curriculum is largely that sketched: a sequence of courses, one purportedly building on the other, with little choice until the student's junior year in college.

We pause to list a few of the defects of this curricular structure. Although obvious, they are too seldom acknowledged, and the harm they work is almost always understated.

First, the whole structure creates a climate of fear. A student who does poorly in one course feels incapable of mastering further courses. Parents jockey desperately to keep their children from falling off the narrow track and push them ahead before they are ready. There is good reason for the fear. A student who falls off the track somewhere in high school and enters college requiring one or more precalculus courses has a very long road to negotiate before encountering the useful and appealing ideas in junior- and senior-level mathematics courses. Such students often find themselves effectively closed off from a mathematics major (or even a minor) and any other majors requiring substantial mathematics.

Nor does the curricular structure disenfranchise only those who fall off the track before college. Students who have conceptual difficulties but a high tolerance for work learn to make it

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through high school by concentrating solely on technique. This strategy may serve as a survival mechanism in the first few calculus courses, but ultimately the student is the victim of a cruel joke: all that hard work has left him or her without the conceptual tools needed to benefit from more advanced courses, much less pursue a scientific career.

The curriculum also cheats the very able student who has understood everything but who chooses not to pursue a mathematics major. Such students rarely have time to complete the Calculus I-II-III and Linear Algebra sequence. Those who do almost certainly do not have time to take any other mathematics courses. This is a shame: many of the ideas in the junior–senior-level courses would serve these students well in their chosen professions and stimulate their imaginations. To compound the intellectual shortchanging, the standard Calculus-Linear Algebra sequence was never designed to meet the needs of students who would not continue with mathematics courses.

What, then, of mathematics majors? Here, too, we think the traditional curricular structure is less than optimal. Students who major in mathematics but do not go on to graduate school in mathematics or science do not encounter the range of ideas embraced by modern mathematics until their junior and senior years. They seldom see these ideas used and rarely have sufficient time for real mastery. What remains is only the hazy notion that modern ideas require years and years of calculus and preliminary study. Many of these students become school mathematics teachers and unwittingly propagate the narrow view of mathematics they have experienced.

The mathematics majors who go on to graduate school are the only ones to reap the reward for their years of preparatory study. In the course of doing their doctoral research, they will experience mathematics as a living field: one in which miracles happen, but in which our ignorance dwarfs our knowledge; one where deep problems lurk, but one where even modest insights bring great joy. But even among such students, us included, the curriculum they survived haunts them. The narrow road with course supposedly built upon course becomes an unexamined part of the way we think about teaching mathematics. We internalize it and come to think of it as reflective of the way the discipline really is.

We forget that mathematics is a field in which newcomers can make substantial contributions. We look at the gaps in our own backgrounds and see, not the obvious lesson that one can proceed very far with such gaps, but rather our shortcomings. We have learned to enjoy building towering intellectual structures on narrow, but firm foundations. Our curriculum is an instance of such a structure, and we seldom ask whether we learned mathematics in spite of that structure, not because of it.

In short, we contend that the traditional hierarchical curricular structure brutalizes most who proceed through it. And we count the members of the college and university mathematical community, us included, among the brutalized. How else to explain the perpetuation of such an inhuman gauntlet. How else to explain that when we think of teaching some course—Galois Theory, say—we think first of the prerequisites, Linear Algebra and a semester of Abstract Algebra, that we will use to exclude students rather than how we might explain the material to students without these courses and what opportunities might arise along the way to teach something about vector spaces, groups, and fields. (If this seems impractical, would you tell an interested colleague, a statistician or an electrical engineer, who asked you to explain the elements of Galois theory to first take the prerequisite courses?)

This strategy (of concentrating solely on technique) may serve as a survival mechanism in the first few calculus courses, but ultimately the student is the victim of a cruel joke: all that hard work has left him or her without the conceptual tools needed to benefit from more advanced courses, much less pursue a scientific career.

We are not claiming that prerequisites are unnecessary. Rather, we submit that insufficient thought has been given to creating a curriculum which minimizes prerequisites and maximizes a student's options.

Towards a Broader Curriculum

It is easy to complain about prerequisites and a hierarchical curriculum, and less easy to see what to do about it. We list four challenges that we believe a department will need to confront in reworking offerings to increase access and student choice, and suggest strategies to meet those challenges.

1) Fix the precalculus/calculus sequence. The necessity for some such effort enjoys a growing consensus and needs little further comment. Students entering college have many different needs, and each department needs to ask itself how its calculus offerings meet those needs. At a minimum, no matter where students enter or leave the calculus sequence, the coursework should engage their interest, teach them some substantial mathematics, and enable them to use the ideas of calculus in contexts in which they are likely to need it. This suggests that departments consider replacing the monolithic standard calculus sequence with a variety of offerings. We find Peter Lax's argument that departments assure the highest-quality teaching by giving each instructor maximum flexibility to adopt the calculus book of his or her choice very persuasive. Since the choice of many of the topics and presentation of material in the traditional precalculus course is predicated on students continuing to a traditional calculus course, departments should look very hard at precalculus courses. The wasted resources at the precalculus level are phenomenal. Many schools have found success with a year-long course that covers the traditional precalculus material, together with much of the material in the traditional first- and second-semester calculus courses. Others find that some of the reform courses allow students with weak high school backgrounds to pursue technical and scientific careers.

2) Add entry-level courses that go somewhere. We believe there is a need for serious freshman-level, noncalculus mathematics courses that will expose students to a variety of mathematical ideas and that will prepare them for further study in mathematics. Such courses should increase the mathematical maturity of students and provide them with solid technical and conceptual tools that will stand them in good stead later in life and in subsequent mathematics courses. They can also attract more students to the study of mathematics. Topics from statistics, geometry, algebra, and number theory lend themselves particularly well to developing exploration-based courses that are open to entering freshmen with high school algebra and provide the student with a rich store of examples on which to base further study. Even with a minimum of technique, such courses can build mathematical maturity and whet the appetite for further study.

3) Increase access to advanced mathematics courses. Departments should rework existing junior–senior-level mathematics and statistics courses to eliminate as many prerequisites as possible, thereby making them accessible to students in other majors. We believe that imaginative rethinking of upper-level mathematics courses can go a long way to alleviating some of the narrowness in the mathematics curriculum.

4) Add a program for students planning a Ph.D. in mathematics. The traditional curriculum was designed with the prospective Ph.D. student in mind, and broadening the curriculum is not without cost. In order to make sure that students bound for graduate school are as prepared as those in previous generations, departments will probably find that they need to make some special arrangements for students planning to continue into graduate studies. For example, departments might require such students to do a senior honors thesis or participate in some research experience or seminar. If there are a lot of students who continue on to graduate study, a department might want to consider separate junior- and senior-level honors courses. However it is done, we think that it makes more sense educationally to make special arrangements for the few students who are bound for graduate school than to have the relatively few such students drive the curriculum for the vast majority of other students.

Some Examples

We now discuss some of the tactics we have tried at Mount Holyoke to implement the strategies above and some of the things that we have learned. We do not claim that we have the answers. In fact, we do not even claim that everything we are doing is wise. But we do hope that some of our experiences may prove useful at other institutions and may spur others to their own, perhaps more imaginative and successful efforts.

Regarding the calculus/precalculus sequence, we have done three things. First, we have been heavily involved in the National Science Foundation-funded Five College Calculus in Context program. This program restructures the standard calculus sequence so that the main ideas emerge from real scientific and mathematical questions and so that students who terminate their mathematical studies with any course in the sequence carry away a substantive understanding which will enable them to use the ideas of calculus in contexts in which they are likely to need it. Second, we created a year-long precalculus-calculus sequence, funded by the Dana Foundation, aimed at allowing students with weak high school backgrounds to pursue technical and scientific careers. The course covers the traditional precalculus material, together with most of the material in the traditional first- and second-semester calculus courses. Third, we encourage every instructor to adopt the calculus book of his or her choice.

We undertook two programs aimed at creating introductory courses. The first was funded by a grant from the Sloan Foundation under its New Liberal Arts program. The outcome is a course, Case Studies in Quantitative Reasoning, which provides students with some basic techniques in data analysis, statistical inference, and modeling. The course is now taken by 20 percent of entering first-year students. The second was supported by the NSF and the NECUSE program (New England Consortium for Undergraduate Science Education) of the Pew Charitable Trusts and has put in place three freshman mathematics courses—Geometry, Algebra, and Number Theory—and a sophomore-level course, Laboratory in Mathematical Experimentation. The geometry, algebra, and number theory courses are open to entering freshmen with high school algebra.

Although we did not realize it at the time, the main innovation on which most of our curricular efforts would turn was the introduction of our Laboratory in Mathematical Experimentation course—the Lab, as students call it. On the one hand, the Lab can be taken by any student who has completed our introductory Algebra, Geometry, Number Theory, or Calculus I. On the other, taking the Lab (and sometimes also Linear Algebra) opens the door to several of our reworked advanced courses (see below) for a student who has not taken the calculus sequence.

In the Lab, students carry out five to seven investigations ranging across mathematics and statistics and accessible to beginning college students. The instructor selects topics from a list including stopping times for the Euclidean algorithm for rational integers and for Gaussian integers, graph-coloring and chromatic polynomials, iteration of linear maps in the plane, difference sets, and a randomized response technique for surveys asking sensitive questions.

Each investigation invites students to observe and look for patterns and encourages them to establish language to describe, conjecture, and analyze the phenomena under study. Projects lead to ideas which students will encounter in later courses and supply a repertoire of concrete examples to nourish intuition. In most, the student discovers some things she or he believes to be true and wants to prove but cannot.

Students write a report at the end of each project. We ask them to explain their experimental strategies (what examples they chose and why) and to support their conjectures with arguments based on empirical evidence, on mathematical analysis, and, when possible, with proof. We have found that students who observe a pattern in experimental evidence are strongly motivated to *know* whether the pattern will hold in future examples: they really understand the function of a proof, and they are eager to learn how to construct proofs of their own. In fact, we feel that students can hardly begin to appreciate the meaning of "proof" unless they have had concrete experience with examples which sometimes lead to correct conjectures and sometimes to false ones.

The Lab requires either Calculus I or one of our freshman courses, although most students take it after Calculus II. The reason that the freshman geometry, algebra, and number theory courses work as prerequisites is that these courses also require significant investigation on the part of students. In the geometry course, for example, students investigate what parts of Euclid carry over to the sphere; in the algebra course, students investigate solutions to quadratic equations in finite fields.

Students who take the Lab course develop a good feeling for convergence and for different rates of convergence and different types of asymptotic behavior. This serves them well in calculus, if they have not taken it, and in real analysis.

We have been teaching the Lab each semester since 1989 with good results. The course is very popular and, we believe, does a good job of allowing students to confront major ideas in an active way and make them their own. The substantial writing (and rewriting) they do improves the quality of the proofs they write in this and subsequent courses. We currently require the Lab of all mathematics majors. (*Laboratories in Mathematical Experimentation: A Bridge to Higher Mathematics*, a student laboratory manual for the sixteen investigations developed for the Lab, plus an instructor's manual, are forthcoming from Springer-Verlag.)

Building on our experience with the Lab and with the generous support of the Department of Education's FIPSE (Fund for the Improvement of Post Secondary Education), our department reworked seven of our junior–senior-level courses in order to reduce the prerequisites to at most two semesters of college-level mathematics: Differential Equations with Modeling, Analytic Number Theory, Mathematical Statistics, Lie Groups, Polyhedral Differential Geometry, Theory of Equations, and Symmetry Groups in Geometry and Physics. Our FIPSE grant supported sabbatical visitors whose teaching provided release time for the faculty member developing an advanced course. In addition, a visitor attended and critically commented on each new course under development. (This collegial help proved so valuable that now it is something we try to do for each other.)

By reducing the prerequisites, we hoped to attract students who were not mathematics majors but who would enjoy and be able to use some of the ideas encountered in traditional junior–senior-level courses. (Of course, we also hoped that appealing electives available early in a student's college years might attract more majors.)

We began with our own and others' experience that systematic and thoughtful use of computers often allows the introduction of relatively advanced, but exceedingly useful, ideas at an early stage in an undergraduate's career. As a result, most of our revised courses use computers in an essential way. Happily, the increased computer use also accords well with the way in which mathematics is increasingly practiced outside the classroom.

Normally we offer each course at least once every three years so that students have the option of a more accessible advanced course each semester. Details of the courses are available from the authors or by consulting the department's Web page.

Besides accessibility, other goals guided our development efforts. We sought topics that exhibit some of the range of mathematical ideas currently being investigated and applied. In particular, we wanted students to experience mathematics as a subject created and used by people. We also wanted to give students the opportunity to use the tools, technological and conceptual, that enable them to work with these ideas.

We also considered pedagogical issues in designing these courses. From the Lab and our calculus courses, we learned that working in groups strengthens students' learning of mathematics. Similarly, we found that writing is a useful tool; like group work, it forces a student to express his or her ideas clearly and to organize them coherently. Thus, all the new courses require careful writing. Finally, we wanted our new courses to be exploratory to some degree, and some of the writing we ask of students is a description and analysis of their explorations. Beyond the justifications mentioned in the description of the Lab, we note that successful exploration is not possible without taking risks. We work to create an atmosphere in which students feel free to be adventurous in their thinking. For women especially, encouragement to let go of safe, cautious strategies and try bold ones seems important.

We have also been mindful of the pitfalls in reducing prerequisites. There is a very real danger of ending up with shallow courses that neither stretch students' minds nor prepare them well for further study or for using the ideas in other contexts. We have tried to avoid these pitfalls by building courses around substantial, explicit examples. Learning a few significant examples thoroughly and exploring their implications not only prepares a student well for more general study (and enriches the student who has already had a more general course) but also leaves the student in command of some important ideas. Very often, the computer is the tool that enables us to bring these examples to the desired level.

For instance, the analytic number theory course treats one theorem in theoretical detail (Dirichlet's theorem on primes in progressions) but introduces the statements of others (e.g., the prime number theorem, the prime number theorem for progressions, and Littlewood's theorem) through numerical experimentation. Students are asked to be quite independent as they work with the computers to find their own paths toward the correct statement of a theorem. This is a substantial adjustment for students accustomed to having results given to them in lectures. However, after some initial discomfort (much less for students who have been through the Lab), they rise to the challenge.

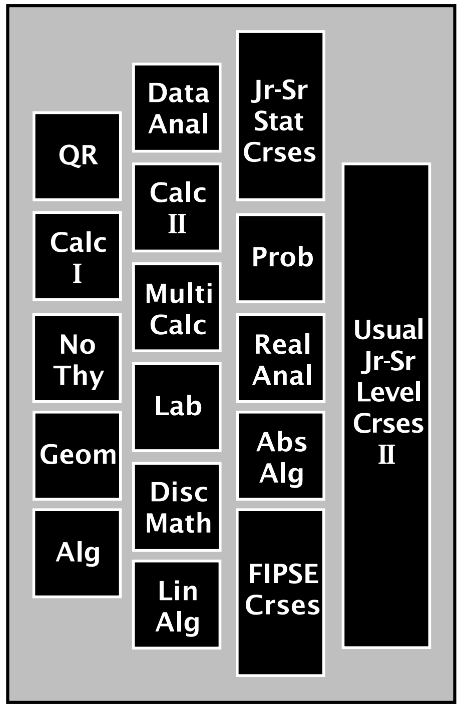
For another example, the polyhedral differential geometry course asks students to construct models of polyhedra and to investigate lines, angles, polygons, and areas on them. The students work toward the Gauss-Bonnet theorem for polyhedra and polyhedral analogues of other differential geometric results. In addition, students investigate polygonal knots, equivalence, and Rademacher moves and examine the invariance of knot groups and knot polynomials.

Finally, in order to attend to the needs of students planning to pursue advanced degrees in mathematics, we now do three things. First, we expect such students to do a senior honors thesis. Secondly, we encourage students contemplating further study in mathematics to spend at least one summer at an REU (Research Experiences for Undergraduates) site. We insist that the site students choose be one, like our own, in which they will learn some mainstream mathematics. [Our REU site is supported by the NSF and NECUSE—for more details see the book *Models for Undergraduate Research in Mathematics* (L. Senechal, ed., MAA Notes 18) and the article by O'Shea therein.] In the last few years we have had some truly outstanding senior theses based on work begun in REU programs. Third, we are in the process of instituting a senior seminar in which students will go to the literature and report back on one or two papers of interest to the student and chosen with the help of the instructor. These strategies work well for us because we are a relatively small department: each year we graduate between twenty and twenty-five mathematics and statistics majors, and perhaps two will attend graduate school in mathematics.

How Does It Work?

Here is how our curriculum looks now (See diagram next page).

How does all this work? Is the curriculum outlined above an improvement over the standard curriculum? The short answer is that we think so, but we really do not know. We do offer a number of tentative observations and conclu-



Any two courses in the same column can be taken concurrently or independently. One or two courses in a column serve as prereq for a course in the column to the right. A single course in the first column gives access to the second column, and from anywhere in column *i* it is always possible to move to some course in column *i* + 1.

sions, organized about the strategies mentioned above.

We will not say too much about calculus, except to remark that some of the reform courses can be taken successfully by students with weak algebra skills. This has allowed us to drop all sections of traditional precalculus. We now advise many students with weak high school algebra backgrounds to take the first semester of Calculus in Context or the quantitative reasoning course because we feel that either is a much better terminal mathematics course than precalculus. Unfortunately, students who go into these courses with weak algebra skills come out with weak algebra skills. Whatever else it did, the traditional course did improve the high school algebra and precalculus skills of successful students. To serve students who do wish to continue to a second semester of mathematics, we offer an intensive high school algebra review during January term (in fact, since skills are easy to teach, we have senior mathematics majors run it).

The geometry, algebra, and number theory first-year courses seem to work well insofar as we have three or four students in each class go on to take further mathematics courses. We have even had a couple of mathematics majors (one now in graduate school), and many more minors, begin their study of mathematics with one of these courses. The downside, of course, is that one often wonders whether something practical like calculus or quantitative reasoning would not be a better choice for students who will take only a single mathematics course. We have no answer for this. The mathematical maturity of students in the alternative first-year courses does seem to increase markedly. However, even if one could actually measure this, it is not clear that the same would not have resulted from a calculus course.

A mixed blessing is that the alternative courses are challenging to teach. Student questions rapidly outrun the ability of the instructor to answer them, and, in order to be comfortable in class, the instructor needs a good knowledge of what is and is not known. This makes for classes that are intellectually engaging for the instructor. On the other hand, a runaway class with problems and conjectures flying left and right and no closure and no results leaves students with no real knowledge. Striking the right balance between exploration and conjecture, theorems and proof is often tricky.

These courses replaced a number of "freshman seminars" we offered many years ago. Those courses were much more in the "math-for-liberal-arts-students" mode. We feel that the new courses are much preferable: They engage students in doing mathematics and permit a student who catches fire and wants to take more mathematics to do so without going back and starting at the beginning of the calculus sequence.

We think of the Lab course, on the other hand, as an unqualified (and demonstrable) success. We require it of mathematics majors—it is a prerequisite for Real Analysis—and recommend, but do not require, that it be taken in the sophomore year. The result is that a number of majors take Abstract Algebra without first taking the Lab. Comparing this group with those who take the Lab first shows clearly that, with few exceptions, those who take the Lab first do better than those who do not. This bears out the strong anecdotal evidence that suggests the Lab really does make a substantial difference in students' mathematical maturity. In order to make room for the Lab course, we stopped requiring a fifth semester of calculus (Calculus on Manifolds) of mathematics majors. We have never really had serious doubts that the exchange was in the best interests of our students.

Our experience with the sophomore- and junior-level mathematics courses we reworked has convinced us that it is indeed possible to offer sophisticated topics with fewer prerequisites and neither overwhelm students nor cheat them with superficialities. Students are able to cope with quite advanced topics through their investigation of examples. The key elements here are the provision of opportunities for "pure" exploration not burdened by the instructor's specific agenda, lots of examples, and a variety of fairly routine but difficult computations. Needless to say, the computer is a key tool in most

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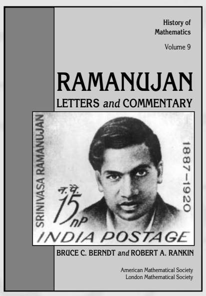

Bruce C. Berndt,
University of Illinois, Urbana
Robert A. Rankin,
University of Glasgow, Scotland

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of the explorations. We have found that having students work in groups on such explorations is usually more effective than having them work alone.

Unfortunately, while we have made a number of courses much more accessible, we have so far not succeeded in attracting significant numbers of nonmajors to them. On the other hand, our majors are enthusiastic about these courses, and some of them credit their existence as a factor in their decision to major in mathematics. Whether they take these electives before or after the traditional junior algebra and analysis, they get excited when they see connections between their classes.

Based on the student response, we do feel these courses have strengthened our major. Offering one each semester does mean that we have to offer one less traditional junior–senior-level mathematics course (this means that we offer two, instead of three, more traditional courses each semester), but we feel that the gains outweigh the costs.

Finally, because of the existence of strong REU programs, we do not feel that the preparation of the few students who go on to mathematics graduate school has suffered. On the contrary, we feel that our current students who are bound for graduate schools are actually better prepared than our students a decade ago.

In summary, we have bought into the broader, more accessible curriculum outlined above, and we invite you to consider doing the same. For us, the ideal curriculum would consist of courses which are independent but which nourish one another by increasing the students' repertoire of mathematical examples and experience. Every course should stand alone in the sense that it provides a student with examples and tools which will remain with him or her for life. Progression through the major should provide students with access to ever wider and deeper examples, an enhanced appreciation of what can go right and what can go wrong, a progressively more sophisticated sense of argument, and a growing respect for their own ideas and those of others. It should sharpen students' sense of wonder while providing them with the critical tools to distinguish dross from miracles.

For those who would argue that what we sketch here is a dumbing down of mathematics courses, we issue two challenges. The first is that you make it a principle that *every* mathematics course should increase students' options for further study of mathematics. The second is to thoroughly reexamine the prerequisites for the courses you teach and to consider how the course you want to teach might itself teach the prerequisite topic(s). Even if you retain the prerequisites, the exercise will keep you honest.