Efficient Methods for Covering Material and *Keys to Infinity*

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It seems that most mathematics teachers at all levels are in a hurry to make progress in covering testable material. For example, beginning algebra students are taught that there are ways of setting up word problems, methods for mixture problems, different methods for area problems, and other methods for distance problems instead of being taught that any mathematical word problem can be distilled to its essence by analyzing each sentence for its mathematical meaning. Clearly, a teacher gets testable results more quickly by ignoring the unity.

A recent article in *Science News* (Nov. 30, 1996, vol. 150, p. 341) described a new study of eighth graders that again showed East Asian students leading U.S. students in mathematics. Surprisingly, the article reports that students from better-scoring nations were found to spend less time in school and as much time watching television as U.S. students, but teaching techniques were quite different from those in U.S. schools. U.S. mathematics teachers taught many more topics than their counterparts. (That reminds us of our topics-crammed calculus books.) Japanese math classes were found to spend 40% of their time practicing routine problems vs. 96% in U.S. classes. Japanese teachers would first challenge students with a new problem, asking them to try to solve it from what they know. The teacher gradually led them to a method. U.S. teachers went directly to the method and then assigned many more examples.

The U.S. method does not sound so different from much of the teaching in college math classes, including advanced calculus. We present a theorem and go directly to a proof, possibly following some examples. It is often felt it wastes time to try to get the students to puzzle through the proof on their own, and, besides, there is a great deal of material to cover, so there is no time for such strategies.

An alternative approach to teaching was developed by R. L. Moore, and it has often been used for teaching general topology. It forces the students to find all the proofs and then present them in class. Gian-Carlo Rota has written a delightful, strongly worded book with many historical insights. He writes [1] of a class at Yale taught by Nelson Dunford (one of the authors of the outstanding Dunford and Schwartz *Linear Operators* trilogy). Rota writes, "The core of graduate education in mathematics was Dunford's course in linear operators. Everyone who was interested in mathematics at Yale eventually went through the experience, even such brilliant undergraduates as Andy Gleason, McGeorge Bundy and Murray Gell-Mann. The course was taught in the style of R. L. Moore: mimeographed sheets containing unproved statements were handed out every once in a while, and the students would be asked to produce proofs on request. Once in a while some student at the blackboard fell silent and the silence sometimes lasted an unbearable fifty minutes, since Dunford made no effort to help. Given the normal

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teaching of 12 hours a week for full professors, I suspect that he wanted to minimize his load." Most proponents of the Moore method would of course ameliorate the silence! In fact, this method of teaching is not easy on the professor, with a need to produce tasks that will keep most of the class in the game, but it does produce students who are tremendously enthusiastic about mathematics and are able to prove theorems. Of course, this method is rarely followed because less material is covered, but possibly some compromise in approaches could be reached.

After learning algebraic topology from a text in grad school, one of us (J. Y.) heard a special lecture by John Milnor and was amazed when he drew pictures and related homology to them, showing that there was geometric intuition in the subject, not just algebra. Geometric intuition does not help significantly in covering material—that is, in getting through the algebraic proofs of standard theorems in that field—so it is often largely omitted from our courses.

There is a craze in middle schools, high schools, and colleges to accelerate excellent students to allow them to cover even more material. From the beginning of high school onward, half the students taking math courses each year take no more after that year. Perhaps for some this justifies the emphasis on covering material while we still have the students in our classes, but it seems crazy to take excellent students who could otherwise handle the material and accelerate them to the point where they cannot keep up or at least cannot enjoy it. When J. Y.'s seventh-grade class was split into those ready for algebra and those not, he was selected for the slow half, perhaps leaving him with a bias that the selection is based more on the ability to manipulate expressions with precision than on mathematical potential.

More generally, when a graduate student in mathematics passes the qualifying exam(s) for writing a Ph.D. dissertation and is then expected to create new theorems, he is likely never to have been asked in any class to create a new theorem, no matter how trivial. Students in classes could be asked questions such as, given two examples of some phenomenon, find a theorem that includes both cases; or, given a theorem with unnecessary hypotheses, find a cleaner, more elegant theorem. We show our emphasis on covering material when we expect our students to create new publishable theorems when they have never been asked to create simple, elementary theorems.

Many students are awarded a Ph.D. degree after writing perhaps a hundred-page dissertation that is in no way ready for publication. The student has never written a publishable paper and does not realize how difficult it will be to turn the dissertation into a few publishable pages. The student has learned little about making the many severe choices necessary when writing papers and has been taught a style that is inappropriate for a research mathematician. Once the degree has been received, who is going to teach the student to write? And who is going to pay the student to turn the dissertation into papers? To change the culture without alienating many faculty members, one could simply require that the advisor certify before the dissertation defense that "The main ideas of the dissertation have been submitted for publication in a paper or papers written by the student."

The advisor often says there is no time to write papers in addition to making "progress" toward the degree. But more and more advisors are taking an alternative approach of asking students to work at writing papers and at the end to staple them together and call it a dissertation. Of course, some minor changes in the document will be necessary, but the student's career will not be in jeopardy. The student need not be the only author, and a loophole is intentionally left that the paper(s) need not be accepted for publication. This is in fact a statement of an approach or objective rather than a true requirement, since anything can be submitted for publication.

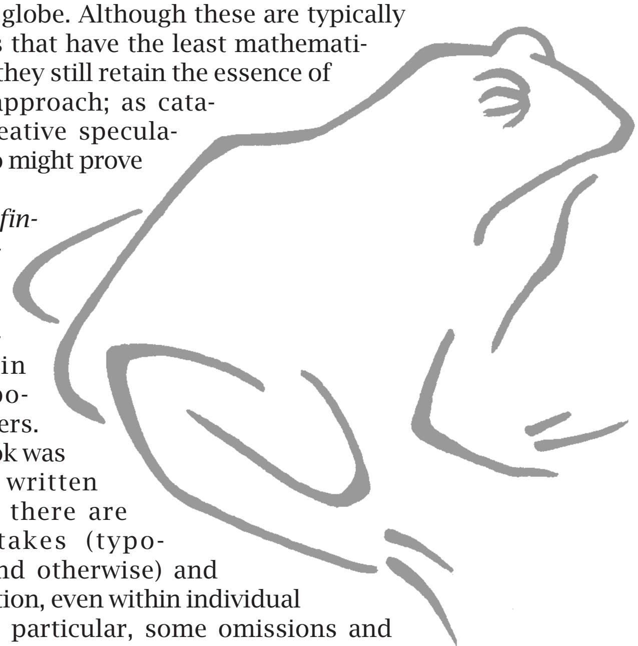
Since the roots of the problems described above run so deep, it is imperative that potential solutions (such as the Moore method) be implemented early in students' careers—and not just for students planning to become mathematicians. A good first step would be to deemphasize algorithmic problem solving and concentrate instead on more open-ended problems. Such problems would require not only technical proficiency for their solution but also a willingness to think about mathematics creatively and flexibly. This would benefit future math Ph.D. candidates, of course, but it would also ameliorate the math illiteracy and math phobia so common in our society.

A potential source for such stimulating, open-ended math problems is Clifford A. Pickover's new book, *Keys to Infinity*. As a frequent contributor to the "Brain Bogglers" section of *Discover* magazine, Pickover has ample experience with clever math-style puzzles. In book format, though, he can give his interests free rein, and each chapter in *Keys to Infinity* is akin to a Brain Boggler expanded and developed in considerably more detail. Although no readers of the *Notices* will find their mathematical muscles stretched by *Keys to Infinity*, they might find some of the problems intriguing enough to pass along to students. We think readers will find that *Keys to Infinity* fosters a flexible and creative approach to mathematics, an approach similar in spirit to the Moore method and the teaching methods successfully implemented in countries such as Japan.

As the title and Pickover's background suggest, *Keys to Infinity* is a smorgasbord of math puzzles organized around the theme of infinity. The first chapter, "Too Many Threes", may serve as an example. Pickover considers the problem of how many numbers (integers) contain the number 3. Though his discussion is rife with hyperbole (e.g., "some [colleagues] nearly fainted when they heard the correct answer"), perhaps some students will be surprised to learn that the proportion of numbers containing 3 approaches 1 as the number of digits increases without bound. In addition to a standard argument (proportion of 3s $= 1 - (\frac{9}{10})^n \to 1$ as $n \to \infty$, where n is the number of digits in the number), Pickover includes a nice computer simulation showing how the proportion is near 1 even for modest values of n. In addition, at the end of the chapter he presents a more challenging puzzle, the search for so-called "super-3" numbers: "integers i such that, when raised to the power of 3 and then multiplied by 3, contain three consecutive 3s." Though hardly of much mathematical importance, such a problem might provoke some valuable thinking: e.g., how might one go about searching for such numbers?

As the previous paragraph hints, Pickover has a penchant for computation, and *Keys to Infinity* emphasizes the use of computer simulations in investigating mathematical puzzles. In the hands of an interested (and computer-literate) reader, numerical simulations are a flexible tool for investigating intriguing problems, and Pickover includes a source code (in C and BASIC) for the programs he employs. (Unfortunately, the code is not included on a disk or available by anonymous ftp, so readers have to key it in themselves.) Pickover's use of simple computer graphics is particularly nice. For example, simple plots nicely complement a chapter on fractal "batrachions" (meaning "froglike"), sequences of integers that "hop" around from integer to integer in complicated ways. His favorite example is defined by the rules $a(n) = a(a(n-1)) + a(n - a(n-1))$ and $a(1) = a(2) = 1$. Even such a simple recursive sequence has exceedingly complex and unpredictable behavior—and yet, as Pickover notes, "there is an incredible amount of hidden structure." Plots of $a(n)/n$ vs. n show this, as the batrachion "hops from one value of 0.5 to the next along very intricate [fractal-like] paths." Near the end of the chapter, Pickover introduces some other batrachions, together with a litany of open-ended questions about their behavior.

Most of the other chapters are written in the same spirit as the two examples mentioned, though many veer quite far away from the theme of infinity. Computer simulations recur throughout, as do extended transcripts of Pickover's e-mail correspondence regarding a variety of speculative questions. One example is "Infinity World", where Pickover considers what the various repercussions would be if the world were an infinite strip of repeated Mercator-type Earths instead of a globe. Although these are typically the chapters that have the least mathematical content, they still retain the essence of Pickover's approach; as catalysts for creative speculation, they too might prove valuable.



Keys to Infinity has considerable merit, but some warnings are in order for potential readers. First, the book was apparently written quickly, as there are many mistakes (typographical and otherwise) and much repetition, even within individual chapters. In particular, some omissions and notational gaffes (such as using three different letters to represent the golden ratio) might confuse novice readers. (Jokes such as the song "$\aleph_0$ Bottles of Beer on the Wall" will also be lost on the uninitiated, since Pickover never defines $\aleph_0$ or discusses transfinite cardinals—a golden opportunity missed.) Second, the book definitely does not comprise a coherent narrative, and a strategy of hopping around from section to section (in the spirit of the batrachion) is highly recommended—in fact, Pickover himself recommends this strategy in the preface. Finally, Pickover makes no pretensions of mathematical rigor, so *Keys to Infinity* will do nothing to further an understanding of mathematical proof. Nevertheless, the spirit of the book is at times infectious, and many students will likely find themselves excited by mathematics in a new way. At the least, *Keys to Infinity* is a welcome counterbalance to the algorithmic, "cook-book" approach to doing mathematics—and to teaching mathematics.

Reference

[1] CLIFFORD A. PICKOVER, *Keys to infinity*, John Wiley & Sons, Inc., New York, 1995.

[2] GIAN-CARLO ROTA, *Indiscrete thoughts*, Birkhäuser, Boston, 1997.