Recent Developments in the Cohomology of Finite Groups

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In this expository paper we describe some recent work in the cohomology of finite groups. In particular, we discuss techniques for calculation and how they apply to key examples from group theory, namely, the sporadic simple groups. We also mention recent theoretical results arising from these calculations and interactions between algebra and topology which occur in the context of finite group cohomology.

Definitions and Motivation

Let G denote a finite group. One of the basic tools for understanding G is its expression as a *group extension*

$$1 \to N \to G \to K \to 1$$

where N is a normal subgroup in G and K is the quotient G/N. An obvious problem is to try to understand all extensions of this type and indeed determine what different groups can arise in this way. Assume that N is abelian; if we choose a section $\phi : K \to G,\ k \mapsto g_k$, then the group structure on G implies that for $k, l \in K$ there must exist $n_{k,l} \in N$ such that $g_k g_l = n_{k,l} g_{kl}$. This defines a function $K \times K \to N$ satisfying certain "cocycle properties". Group cohomology arises when trying to understand equivalence classes of these functions in a *functorial way*. In particular, isomorphism classes of extensions as described above are in one-to-one correspondence with these equivalence classes of 2-cocycles, which can be regarded as elements in a "cohomology group", denoted by $H^2(K, N)$ (see [3], Chapter I). It was, however, the impetus from topology that eventually gave rise to a global definition of group cohomology using machinery from homological algebra, generalizing specific low-dimensional information. We refer the reader to the excellent historical account by S. Mac Lane [17].

Let us review the essential elements in the general definition (for complete details the reader may look at [14] or [3]). Let $\mathbb{Z}$ denote the integers with the trivial action of a finite group G. It is evident that we can map a copy of the group ring $\mathbb{Z}G$ onto it with a finitely generated kernel IG (this is, in fact, the *augmentation ideal*). Now taking generators for IG as a $\mathbb{Z}G$-module, we can map a free $\mathbb{Z}G$-module of finite rank onto IG. Continuing in this way, we obtain a sequence of free $\mathbb{Z}G$-modules and $\mathbb{Z}G$-maps

$$\cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0$$

such that at each stage the image of the homomorphism coming in is equal to the kernel of the homomorphism going out (it is said to be *exact*), and $F_0/\text{image}\,(d_1) = \mathbb{Z}$; i.e., F_0 maps onto $\mathbb{Z}$. Such an object is called a *free resolution* for $\mathbb{Z}$, and an obvious analogue can be constructed for any finitely generated $\mathbb{Z}G$-module taking the place of $\mathbb{Z}$.

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Now let A denote any $\mathbb{Z}G$-module. We can consider all G-homomorphisms from the F_i to A and put these together to obtain a cochain complex

$$\begin{aligned}&\mathrm{Hom}_G(F_0, A) \to \mathrm{Hom}_G(F_1, A) \to \cdots \\ &\quad \to \mathrm{Hom}_G(F_{i-1}, A) \to \mathrm{Hom}_G(F_i, A) \to \cdots .\end{aligned}$$

The cohomology of G with coefficients in A is defined as the cohomology of this cochain complex; that is, $H^i(G, A) = H^i(\mathrm{Hom}_G(F_*, A))$. Note that the exactness of $F_1 \to F_0 \to \mathbb{Z} \to 0$ implies the exactness of $0 \to \mathrm{Hom}_G(\mathbb{Z}, A) \to \mathrm{Hom}_G(F_0, A) \to \mathrm{Hom}_G(F_1, A)$, whence we see that $H^0(G, A) = \mathrm{Hom}_G(\mathbb{Z}, A) = A^G$, the submodule of invariants. Furthermore, if A has a trivial action, then a similar argument shows that $H^1(G, A) = \mathrm{Hom}_G(IG, A)$, where IG is the augmentation ideal in $\mathbb{Z}G$. This group is isomorphic to the group of homomorphisms from G to A. A slightly more complicated argument can be used to show that in fact $H^2(G, A)$ will recover the isomorphism classes of extensions described previously. If R is any commutative ring with 1, then we can define $H^i(G, M)$ for any finitely generated RG-module M using RG-resolutions of the trivial module R. It turns out that any two resolutions will give rise to the same cohomology groups.[1]

Let us fix as coefficients a field $\mathbb{F}_p$ of characteristic p which divides the order of G. If G happens to be a finite p-group, then it is possible to construct *minimal resolutions* P_* of the trivial module $\mathbb{F}_p$ which are directly related to the cohomology data. In fact, we have that in this situation $H^*(G, \mathbb{F}_p) \cong \mathrm{Hom}_G(P_*, \mathbb{F}_p)$; i.e., the cochain complex has zero coboundary maps. From this it is evident that computing minimal resolutions is an important aspect of group cohomology. Indeed, computer-assisted calculations of minimal resolutions can provide substantial information on the low-dimensional cohomology of a finite p-group of reasonable size (see [10] or check out the Web site `www.math.uga.edu/~jfc/groups/cohomology.html`).

At this point things look a bit artificial to say the least, and no doubt any geometrically inclined reader feels unhappy. It is time to explain what is *really* going on here! The fact is that homological constructions originate in topology, and group cohomology is no exception. It turns out that we can construct a contractible *topological space EG* (in fact, a cell complex) with a free G-action (i.e., no isotropy). The complex of singular or cellular chains on EG will then give rise to a free $\mathbb{Z}G$ resolution of $\mathbb{Z}$ as before. Hence we can use this topological construction to compute group cohomology. A direct consequence of this is the following: if BG denotes the orbit space EG/G, then the singular cohomology of BG coincides with group cohomology. Using the homotopy invariance of homology, we in fact have that if BG is *any* path-connected topological space with contractible universal cover and fundamental group G, then $H^*(G, \mathbb{F}_p) \cong H^*(BG, \mathbb{F}_p)$, where the term on the right is the usual singular cohomology ring. This isomorphism makes the cohomology of finite groups an interesting object in both algebra and topology. Aspects of this are summarized in the following diagram of interactions:

Group Theory
Representation Theory
Homological Algebra
Number Theory
K-Theory

↗

COHOMOLOGY
OF FINITE
GROUPS

↘

Classifying Spaces
Group Actions
Characteristic Classes
Homotopy Theory

A classical example of this is given by the following result, which combines work of P. Smith and R. G. Swan (see [3], pg. 146).

Theorem. A finite group G acts freely on a finite complex X homotopy equivalent to a sphere if and only if every abelian subgroup in G is cyclic.

In this example the topological notion is connected to the group-theoretic hypothesis via the concept of groups with *periodic cohomology*. The point is that a finite group G satisfies the group-theoretic condition above if and only if there exists an integer $d > 0$ such that $H^i(G, \mathbb{Z}) \cong H^{i+d}(G, \mathbb{Z})$ for all $i > 0$. Given a free G-action on the n-sphere $\mathbb{S}^n$, one can in fact take $d = n + 1$.

Another important instance stems from Quillen's foundational work on algebraic K-theory [22]. Given a perfect group S (i.e., S is equal to its commutator subgroup $[S, S]$), one can construct a space BS^+ from BS by attaching 2- and 3-dimensional cells and such that $\pi_1(BS^+) = \{1\}$ but, $H_*(BS^+, \mathbb{Z}) \cong H_*(BS, \mathbb{Z})$. The higher homotopy groups of BS^+ are very interesting invari-

[1] *We should mention that the traditional* bar resolution *(see [3], Chapter II) provides a functorial construction for producing a free resolution of the trivial module $\mathbb{Z}$ in terms of free modules constructed from n-tuples of elements in G; however, it is far too large to be used in actual computations.*

ants that in some instances contain important information about the stable homotopy group of spheres. More generally, the plus construction is an indication of the deep relationship between finite groups and stable homotopy theory which has many other manifestations (we refer to [18] for more on this). The cohomology of groups is necessarily a key ingredient in all of this, and computations play an important part.

Calculational Methods

Unfortunately, computing the cohomology of a finite group can be quite difficult. In this section we will attempt to describe some of the more useful calculational tools available. One of the main features about group cohomology is the importance of understanding the contributions from subgroups. Given $H \subset G$, one can induce a restriction map $H^*(G, \mathbb{F}_p) \to H^*(H, \mathbb{F}_p)$ which carries a lot of information. In order to assemble the information from subgroups, we need to organize the combinatorial information in a suitable way. This was first done by K. Brown in a topological context via the construction of "subgroup complexes". The idea is to take a collection of subgroups in G and use inclusions to devise a partially ordered set with interesting properties. The two most important complexes associated to finite groups in this context (see [2], pg. 170) are the poset of all nontrivial p-subgroups in a finite group G (denoted $S_p(G)$) and the poset of all nontrivial p-elementary abelian subgroups in a finite group G (denoted $A_p(G)$). Both of these posets are endowed with a G-action via conjugation, and their realizations $|S_p(G)|$, $|A_p(G)|$ are finite complexes (i.e., compact spaces assembled by gluing simplices) with an action of G which is compatible to this "simplicial" structure.

The fact that G is finite allows us to make use of Sylow's theorems in our computations. Namely, let $\mathrm{Syl}_p(G)$ denote a p-Sylow subgroup of G. Then the first basic fact is that the inclusion $\mathrm{Syl}_p(G) \subset G$ induces an embedding of mod p cohomology $H^*(G, \mathbb{F}_p) \hookrightarrow H^*(\mathrm{Syl}_p(G), \mathbb{F}_p)$. This is proved by making use of a map in the other direction (called the transfer) which when composed with the restriction yields multiplication by the index $[G : \mathrm{Syl}_p(G)]$ on $H^*(G, \mathbb{F}_p)$, which is a mod p isomorphism. Obtaining the exact image of the restriction requires understanding the intersection of the p-Sylow subgroup with its conjugates in G, but this can be done very efficiently using a double coset decomposition for G in terms of $\mathrm{Syl}_p(G)$ (see [3], pg. 78). In simpler terms, this is the familiar fact that $H^*(G, \mathbb{F}_p)$ restricts injectively into the cohomology of a p-Sylow subgroup, and the image can be computed as the "stable elements" in $H^*(\mathrm{Syl}_p(G), \mathbb{F}_p)$. For example, if $\mathrm{Syl}_p(G)$ is normal in G, then this result simply says that there is an isomorphism $H^*(G, \mathbb{F}_p) \cong H^*(\mathrm{Syl}_p(G), \mathbb{F}_p)^K$ where $K = G/\mathrm{Syl}_p(G)$ has an action on the cohomology induced by conjugation.

Next we have a landmark result due to Quillen [23] which tells us that the p-elementary abelian subgroups can be used to understand most of the mod p cohomology of G. The restriction maps corresponding to elementary abelian subgroups can be assembled to define a map

$$H^*(G, \mathbb{F}_p) \longrightarrow \lim_{E \in A_p(G)} H^*(E, \mathbb{F}_p).$$

In this expression the limit consists of the sequences of cohomology classes $x_E \in H^*(E, \mathbb{F}_p)$ indexed by elements in $A_p(G)$ which are compatible with respect to inclusion and conjugation. In other words, if $E_1 \subset E_2$, then $x_{E_1} = res(x_{E_2})$, and if $E_2 = gE_1g^{-1}$, then $x_{E_1} = c_g^*(x_{E_2})$, where c_g^* is the map induced by conjugation on cohomology. Quillen's result tells us that the kernel of the map above consists entirely of nilpotent elements and that a sufficiently high power of any element in the limit is in the image of the map. This result provides critical input for cohomology calculations.

We should also mention here another result which is useful for calculations and which is due to P. Webb [24]. He shows that the cohomology of G can be computed using an alternating sum with the cohomology of the stabilizers of either one of the G-complexes above.

From Quillen's result it is apparent that we are interested in understanding the image of the restriction map to the cohomology of an elementary abelian subgroup. This can often be quite complicated, but under special conditions there exist techniques to compute this (see [3], pg. 113). In fact, in many cases it is possible to show that the image of the restriction map to $H^*(E, \mathbb{F}_p)$ is exactly the ring of invariants $H^*(E, \mathbb{F}_p)^{N_G(E)}$, where $N_G(E)$ denotes the normalizer of E in G.

We can now roughly outline an approach for computing $H^*(G, \mathbb{F}_p)$:

Step 1: Calculate $H^*(\mathrm{Syl}_p(G), \mathbb{F}_p)$ using minimal resolutions and spectral sequences. In many cases it is now feasible to use a computer to find low-dimensional generators for the cohomology of a p-group. A typical p-group will fit into many extensions, each of which provides a Lyndon-Hochschild-Serre spectral sequence. Classical methods from algebraic topology can be used here.

Step 2: Understand the structure of $A_p(G)$. Once again, computer packages such as MAGMA make this feasible.

Step 3: Calculate $\lim\limits_{E \in A_p(G)} H^*(E, \mathbb{F}_p)$ and the restriction map from $H^*(G, \mathbb{F}_p)$. Here a basic ingredient is given by the invariants $H^*(E, \mathbb{F}_p)^{N_G(E)}$.

Step 4: Determine the radical of $H^*(G, \mathbb{F}_p)$, i.e., the nilpotent elements. This is perhaps the trickiest step and requires a good hold on $H^*(\mathrm{Syl}_p(G), \mathbb{F}_p)$.

Having described these computational steps, we need to point out that they obviously will not work for a sufficiently large and complicated group. However, in the next section we will see that recently they have been successfully applied to examples identified as interesting from the point of view of group theory, the so-called sporadic simple groups.

Examples

In order to understand and develop theoretical results in group cohomology, it is fairly clear that the computation of interesting examples is desirable. In this section we will consider the mod 2 cohomology of certain specific examples which are important in group theory, namely, simple groups. Here we can make use of information provided by the Classification Theorem for Finite Simple Groups, which is an enormous achievement of finite group theory (see [16]). Briefly stated, there is now a complete list of all finite simple groups, divided into three types: alternating groups, finite groups of Lie type (and twisted versions of them), and the so-called sporadic simple groups (there are exactly 26 of these). From our point of view we can make use of this to single out important examples and obtain calculations for large classes of simple groups.

We will now present examples of cohomology calculations for sporadic simple groups with coefficients in $\mathbb{F}_2$, the field with two elements (note that by the Feit-Thompson Theorem, any nonabelian finite simple group is of even order). Coefficients will be suppressed.[2]

Example. The Mathieu group M_{12} (see [8]): this is a group of order 95,040. Let $H = \mathrm{Syl}_2(M_{12})$; this is a group of order 64 which can be expressed as a semidirect product $Q_8 \times_T D_8$, where D_8 denotes the dihedral group of order eight and Q_8 the quaternion group of order eight. There are subgroups $W_1, W_2 \subset M_{12}$, both of order 192 intersecting along H such that the configuration of subgroups

$$\begin{array}{ccc} \mathrm{H} & \longrightarrow & W_2 \\ \big\downarrow & & \big\downarrow \\ W_1 & \longrightarrow & M_{12} \end{array}$$

determines $H^*(M_{12})$ as the intersection $\mathrm{res}_H^{W_1}(H^*(W_1)) \cap \mathrm{res}_H^{W_2}(H^*(W_2))$ in $H^*(H)$. We obtain generators for $H^*(M_{12})$ in degrees

$$2,\ 3,\ 3,\ 3,\ 4,\ 5,\ 6,\ 7$$

with 14 relations among them. Taking the $4-, 6-, 7-$dimensional generators D_4, D_6, D_7, we in fact have that they generate a polynomial subalgebra $\mathbb{F}_2[D_4, D_6, D_7]$ *in* $H^*(M_{12})$ over which $H^*(M_{12})$ *i*s free and finitely generated. A convenient way to express cohomology information is via a Poincaré series. Let $p(t) = \sum_{i=0}^{\infty} dim_{\mathbb{F}_2} H^i(M_{12}) t^i$, then from the calculation above it follows that

$$p(t) = \frac{1+t^2+3t^2+t^4+3t^5+4t^6+2t^7+4t^8+3t^9+t^{10}+3t^{11}+t^{12}+t^{14}}{(1-t^4)(1-t^6)(1-t^7)}.$$

How can we interpret this Poincaré series? To do this, we must recall that the exceptional Lie group G_2 is a 14-dimensional manifold which admits a classifying space BG_2 such that its mod 2 cohomology is precisely $\mathbb{F}_2[D_4, D_6, D_7]$. Now an appropriate homomorphism $M_{12} \to G_2$ might reveal the numerator in terms of the fiber of the map induced on classifying spaces, but unfortunately no such homomorphism exists. However, this works topologically: it is possible to construct (2-locally) a map $BM_{12}^+ \to BG_2$ such that the cohomology of the base injects precisely as the polynomial algebra described above and such that its fiber X has the homotopy type of a 14-dimensional Poincaré duality complex with Poincaré series equal to the numerator of $p(t)$ (see [18]). Note that this explains the palindromic nature of this polynomial. This example illustrates how group cohomology can have substantial geometric content which is not at all apparent from the original group.

Example. The O'Nan group O'N (see [4]): this is a group of order > 460 billion. However, the quotient space $|A_2(O'N)|/O'N$ can be schematically represented by Figure 1. We have labelled the stabilizers of the simplices using group theory notation.

The cohomology of O'N is detected on two abelian subgroups, $M \cong (\mathbb{Z}/4)^3$ and $N \cong \mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2$; in fact, $H^*(\mathrm{O'N})$ has 12 gen-

[2] *Note that the cohomology will have a ring structure, as it is the singular cohomology of a topological space.*

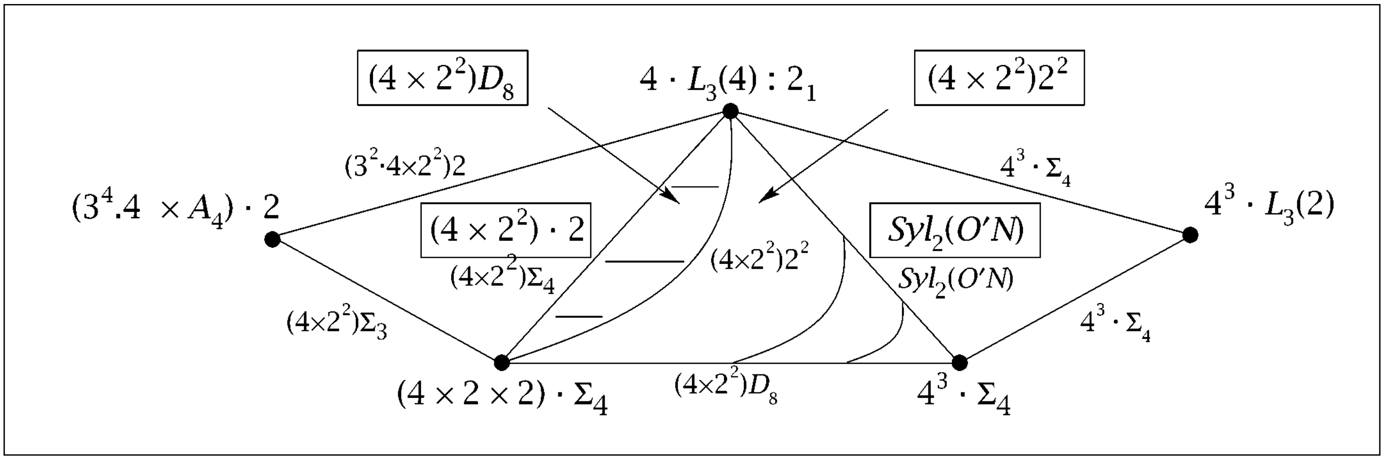

Figure 1.

erators, one each in dimensions 3 through 14. For this group the Poincaré series is of the form

$$\frac{q(t)}{(1-t^8)(1-t^{12})(1-t^{14})},$$

where $q(t)$ is a palindromic polynomial of degree 31.

Let us recall that $H^*(G,\mathbb{F}_p)$ is *Cohen-Macaulay* if it is free and finitely generated over a polynomial subalgebra. Using the explicit information provided by the classification theorem for finite simple groups combined with the examples above and other (substantially easier) examples, we have [6].

Theorem. Let G denote a finite simple group which does not contain a subgroup isomorphic to $(\mathbb{Z}/2)^4$. Then the cohomology ring $H^*(G,\mathbb{F}_2)$ is Cohen-Macaulay.

As one would expect, the situation for rank 4 and beyond is much more complicated, although surprisingly many groups are tractable.

Example. The Mathieu group M_{22}: we have $|M_{22}| = 2^7\cdot 3^2\cdot 5\cdot 7\cdot 11$; its 2-Sylow subgroup S can be characterized as the largest 2-group occurring as the symplectic automorphism group of a $K3$ surface. This is proved by first observing that if a 2-group P acts on a $K3$ surface as above, then a central involution $z\in P$ must have 8 fixed points. Hence the quotient $P/<z>$ will permute these 8 fixed points, and this in fact provides an embedding into the alternating group A_8. Comparing orders provides 2^7 as an upper bound on the order of P. On the other hand, there is only one possible central extension of this type which can occur, namely, S. Finally, it can be shown that in fact S acts symplectically on the $K3$ surface described by the equation

$$X^4+Y^4+Z^4+T^4+12XYZT=0$$

in $\mathbb{P}^3$ (see [21] for complete details).

The group S is rather remarkable in that it occurs as the 2-Sylow subgroup of several interesting groups: M_{22}, M_{23} (the next Mathieu group), McL (the McLaughlin group), $U_4(3)$ and $\tilde{A}_8$ (the double cover of A_8). None of these groups have Cohen-Macaulay mod 2 cohomology.

In M_{22} there are exactly three conjugacy classes of maximal elementary abelian 2-subgroups, V_3, V_4, and W_4 (subindex denotes rank). The invariants in their cohomology under the actions of the respective normalizers are

$$H^*(V_3)^{L_3(2)} = \mathbb{F}_2[D_4,D_6,D_7]$$
$$H^*(V_4)^{\Sigma_5} = \mathbb{F}_2[d_3,d_5,d_8,d_{12}](1,x_6,x_8,x_9,x_{10},x_{12},y_{12},x_{14},x_{15},x_{16},x_{18},x_{24})$$
$$H^*(W_4)^{A_6} = \mathbb{F}_2[d_3,d_5,d_8,d_{12}](1,x_9,x_{15},x_{24}).$$

One can in fact show that $H^*(M_{22})$ restricts onto each of these invariant rings. Using Quillen's Theorem, we obtain (see [5]) that there is an exact sequence

$$0\to Rad\ H^*(M_{22})\to H^*(M_{22})\to H^*(V_3)^{L_3(2)}\oplus H^*(V_4)^{\Sigma_5}\oplus H^*(W_4)^{A_6}$$

with explicit image and such that

$$Rad\ H^*(M_{22})\cong \mathbb{F}_2[d_8,d_{12}](a_2,a_7,a_{11},a_{14}).$$

An interesting point to note is that the low-dimensional cohomology is quite sparse. In fact, Milgram has used the above to compute $H^*(M_{23})$, showing that $H_i(M_{23},\mathbb{Z})\cong 0$ for $1\le i\le 4$ [19]. This is the first known instance of such a highly connected finite group. An obvious question here is whether or not there exists a fixed integer $K>0$ such that for any finite group G if $H_i(G,\mathbb{Z})=0$ for $1\le i\le K$, then $G=\{1\}$.

Other rank 4 examples which have been calculated recently include the McLaughlin group McL [7], the Janko groups J_2, J_3 [11], and the

Lyons group [2]. This last group is rather remarkable in that its cohomology is actually "detected" on elementary abelian subgroups, meaning that for Step 3 (as described in the section "Calculational Methods") we obtain an injective map.

In fact, looking at the list of sporadic groups, we see that only two rank 4 sporadics are left: the Higman-Sims group HS and Conway's group Co_3. Currently (in a joint project with Carlson and Milgram) we are working on the mod 2 cohomology of the Higman-Sims group HS. Computer-generated low-dimensional classes play an important role in the calculation. The situation for Co_3 seems rather more complicated.

Essential Cohomology

Examples are important for testing conjectures and hence can provide a major thrust for theoretical developments. One of the basic facts emanating from the previous calculations is the use of "detecting subgroups" in cohomology. More precisely, a collection of proper subgroups $H_1, \dots, H_n$ in G detects the cohomology of G if the map induced by restrictions

$$H^*(G, \mathbb{F}_p) \longrightarrow H^*(H_1, \mathbb{F}_p) \oplus \cdots \oplus H^*(H_n, \mathbb{F}_p)$$

is injective. In the case of complicated p-groups, finding computable detecting subgroups is an important aspect of most calculations. Another important collection of examples is given by the finite symmetric groups, whose mod 2 cohomology is detected on elementary abelian subgroups (see [3], Chapter VI, and the more recent [15], where complete calculations are announced). This naturally leads us to consider groups with *undetectable* cohomology classes, i.e., those which restrict to 0 on every proper subgroup. We shall say that an element $x \in H^*(G, \mathbb{F}_p)$ is *essential* if $res^G_H(x) = 0$ for all proper subgroups $H \subset G$. We denote the set of all essential classes by $Ess^*(G)$. A key problem is to characterize those groups such that $Ess^*(G) \neq 0$ (see [12], page 438, where it appears as a problem in J. F. Adams's problem list, suggested by M. Feshbach). We should point out that in fact these groups are "universal detectors" for group cohomology. Indeed, a moment's thought reveals that any cohomology class must restrict nontrivially to a subgroup with nonzero essential cohomology, unless of course the group itself is of this type.

Recently the following has been established (see [1]):

Theorem. Let G denote a finite group. Then the following two statements are equivalent:

(1) $H^*(G, \mathbb{F}_p)$ is Cohen-Macaulay, and $Ess^*(G) \neq 0$.

(2) G is a p-group such that every element of order p in G is central.

This result was motivated by examining numerous calculations, which seems to be a new approach in a subject which until recently had few interesting examples. The proof requires using methods developed by Duflot [13] and Benson-Carlson [9] for Cohen-Macaulay cohomology rings. Preliminary evidence also indicates that it relates to the existence of free group actions on products of spheres. More specifically, given a p-group G of rank k, there is a canonical action of G on $(\mathbb{S}^{2(|G|/p)-1})^k$ such that the action is free if and only if every element of order p in G is central. The case of groups of lower depth with nontrivial essential cohomology can be studied using this perspective, which seems related to the problem of making a group of rank k act freely on a finite complex homotopy equivalent to a product of k spheres.

Example. To illustrate the previous discussion, we will provide a simple example. Let Q_8 denote the quaternion group of order 8. It has a unique, hence central, element of order 2. One can verify (see [3]) that

$$H^*(Q_8, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, y_1, u_4]/(x_1^2 + x_1y_1 + y_1^2, x_1^2y_1 + x_1y_1^2).$$

Then $Ess^*(Q_8)$ is the ideal generated by x_1y_1 and x_1^2. Regarding Q_8 as a subgroup of the 3-sphere $\mathbb{S}^3$, it will act freely on it, and in fact the cohomology of the orbit space $\mathbb{S}^3/Q_8$ can be identified with $H^*(Q_8)/(u_4) \cong \{1, x_1, y_1, x_1^2, x_1y_1, x_1^2y_1\}$. These classes in turn can be identified with a basis for the cohomology as a free $\mathbb{F}_2[u_4]$-module. The Poincaré series for the mod 2 cohomology is

$$q(t) = \frac{1 + 2t + 2t^2 + t^3}{1 - t^4}.$$

The denominator describes the four-dimensional polynomial generator, whereas the numerator is precisely the Poincaré series for the mod 2 cohomology of the orbit space $\mathbb{S}^3/Q_8$.

Next we will discuss examples of groups with essential cohomology which arise from field theory but which surprisingly are related to certain topological questions. Let F denote a field of characteristic different from 2, and suppose that $|F^\bullet/(F^\bullet)^2| < \infty$, where $F^\bullet$ denotes the nonzero elements in F. Let F_q denote the quadratic closure of F and $G = Gal(F_q/F)$. The W-group of F has been defined in [20] as $U_F = G/G^4[G^2, G]$. Under our conditions, this is a *finite* 2-group. Moreover, if $|U_F| \neq 2$, all of its elements of order 2 are central if and only if F is not formally real.

Hence U_F is a group with essential cohomology in many interesting instances. One would of course like to compute its mod 2 cohomology; in fact, there is a "universal" W-group, $W(n)$, mapping onto any such U_F, where n is the dimension of $F^\bullet/(F^\bullet)^2$. This group will satisfy the 2-central property and can be described as a central extension

$$1 \to (\mathbb{Z}/2)^{n+\binom{n}{2}} \to W(n) \to (\mathbb{Z}/2)^n \to 1.$$

Recently, in joint work with Karagueuzian and Minac, we have shown that $H^*(W(n), \mathbb{F}_2)$ is free and finitely generated over a polynomial subalgebra on $n + \binom{n}{2}$ 2-dimensional generators. If $p_n(t)$ is the Poincaré series, then in fact we have that $p_n(t) = v_n(t)/(1-t^2)^{n+\binom{n}{2}}$, where $v_n(t)$ is the Poincaré series for an $n + \binom{n}{2}$-dimensional manifold constructed as follows. The group $E = (\mathbb{Z}/2)^n$ acts in a canonical and free way on $X_n = (\mathbb{S}^1)^{n+\binom{n}{2}}$ via rotations on the first n coordinates and complex reflections on the others. Then $v_n(t)$ is the Poincaré series for the orbit space X_n/E.

One would like to compute the cohomology of this orbit space, which is in fact equivalent to computing the group cohomology of the torsion-free discrete group $\Gamma_n = \pi_1(X_n/E)$. No general formula exists yet, but it is not hard to show that $v_2(t) = 1 + 2t + 2t^2 + t^3$ and $v_3(t) = 1 + 3t + 8t^2 + 12t^3 + 8t^4 + 3t^5 + t^6$. Note that, in particular, knowing $v_3(t)$ allows us to determine the Poincaré series for $W(3)$, a group of order 2^9.

Final Remarks

The real interest in the cohomology of finite groups lies in its interactions with other areas of mathematics. In this note we have described instances of this which arise from basic questions and examples in both algebra and topology. Enhanced computational ability has greatly increased the complexity and scope of the examples which can be understood in a meaningful way. Not surprisingly, this has already contributed to new theoretical insight. This trend will lead to an even better understanding of the cohomology of finite groups in the near future.

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