Review of *Noncommutative Geometry* by Alain Connes

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This and the following article are expositions of *Noncommutative Geometry* by Alain Connes (Academic Press, 1994, 661 pages, $59.95 hardcover), reviewed in the *Bulletin* of the American Mathematical Society, v. 33 no. 4, October 1996, pp. 459–466.

"The correspondence between geometric spaces and commutative algebras is a familiar and basic idea of algebraic geometry. The purpose of this book is to extend the correspondence to the noncommutative case in the framework of real analysis."

Thus begins Connes' book *Noncommutative Geometry*. The central thesis is that the usual notion of a "space"—a set with some extra structure—is inadequate in many interesting cases and that coordinates may profitably be replaced by a noncommutative algebra. Here is a simple but central example. Suppose the compact Hausdorff space X is acted on by a discrete group Γ. If the action is sufficiently simple, one may put the quotient topology on the orbit space, X/Γ, thus capturing the orbit structure in an entirely satisfactory fashion. But as the action becomes more complicated, the quotient topology will fail to separate orbits, and in extreme cases like the action of $\mathbb{Z}$ by powers of an irrational rotation of the circle the quotient topology has no information whatsoever. The noncommutative approach is first to replace X by the algebra $C(X)$ of all continuous complex-valued functions on X (which, by the famous Gelfand-Naimark theorem, completely captures the topology of a compact Hausdorff space). Then replace the quotient space not by a smaller algebra such as $C(X/\Gamma)$ but by a *larger* one, the *crossed product* $C(X) \rtimes \Gamma$ consisting of sums $\sum_{\gamma\in\Gamma} f_\gamma \gamma$ (with $f_\gamma \in C(X)$) which are multiplied according to the rule $\gamma f \gamma^{-1} = f \circ \gamma^{-1}$. This algebra is noncommutative and should be completed in some fashion, but when this is done, it provides a powerful tool for the study of X/Γ. In the context of operator algebras there is nothing pathological about $C(X) \rtimes \Gamma$ which can be thought of as a *noncommutative desingularization* of X/Γ.

In our example the noncommutative algebra was obtained from a group acting on a commutative one, but once one admits the legitimacy of such algebras as "coordinate rings", one may expect to find new examples devoid of any commutative origin.

Connes explores many examples, as diverse as the space of Penrose tilings and the Brillouin zone in the quantum Hall effect. In so doing he develops a myriad of techniques, sometimes, as in the case of his Chern character, as necessary generalizations of known tools and sometimes, as in the case of his quantized calculus, of largely noncommutative inspiration.

The technical background of Connes' program is the theory of operator algebras on Hilbert space, closed in some topology. Why should operator algebras be the vessel of noncommutative geometry? After all, the Gelfand-Naimark theorem is not about operators, and, according to

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what we have said so far, it is little more than a curiosity that may faithfully represent $C(X)$ on a Hilbert space. There is a simple but telling reason for operator algebras. The first noncommutative algebras are the $n \times n$ matrix algebras $M_n(\mathbb{C})$, which are algebras of operators on finite-dimensional Hilbert space. So operator algebra is just "large matrix algebra". Or we could take a cue from quantum physics, where the classical observables—continuous functions on phase space—are replaced by algebras of operators, in general unbounded, on the Hilbert space of wave functions. According to von Neumann's remarkable insight, algebras of *bounded* operators, if sufficiently well understood, would in fact be enough to understand the unbounded ones. This led to "algebraic quantum field theory" initiated by Haag and Kastler in [14]. But we digress.

In algebraic geometry access to the power of the theory requires nontrivial results in pure commutative algebra. In noncommutative geometry, then, we expect to need a serious dose of operator algebraic results right at the start. Far more so indeed than algebraic geometry needs commutative algebra, since the intuition afforded by the set of points in a variety has disappeared and there is no reason to expect that all the interesting phenomena of noncommutative geometry arise as mere perturbations of commutative ones. Thus a substantial, but by no means dominant, portion of Connes' book is devoted to the abstract structure theory of operator algebras.

Operator algebra theory splits almost disjointly into two halves: von Neumann algebras, initiated by Murray and von Neumann in the 1930s [22], and C*-algebras, initiated by Gelfand and Naimark in the 1940s [12]. Technically the split is according to the topology in which the algebras are closed-von Neumann algebras in the topology of pointwise convergence and C*-algebras in the norm topology. But morally the split is the noncommutative version of the split between measure theory and general topology. If functions on [0,1] are represented on $L^2([0,1], dx)$ as multiplication operators, the continuous functions form a C*-algebra and the L^∞ functions form a von Neumann algebra. Every von Neumann algebra is a C*-algebra, but it is seldom wise to think of it as such.

Let us first discuss von Neumann algebras, where Connes has been such a dominant figure for the last twenty-five years. Murray and von Neumann quickly recognized the importance of von Neumann algebras whose center contains only scalar multiples of the identity, which they called "factors", and von Neumann wrote a paper [30] reducing any von Neumann algebra to a factor by a "direct integral"—a continuous analogue of the direct sum of algebras. Any von Neumann algebra is $\int_X^\oplus M(\xi)d\mu(\xi)$ for some measure space (X, μ) with each $M(\xi)$ a factor. Factors themselves were classified in [22] according to algebraic properties of the projections (operators p with $p^2 = p = p^*$) which they contain. In a von Neumann algebra M we say $p \le q$ if $pq = p$ and $p \sim q$ if there is a $u \in M$ with $uu^* = p$ and $u^*u = q$. There is a direct analogy with cardinalities of sets, so a projection is called "infinite" if it is equivalent to a proper subprojection, and "$\le$" induces an order "$\lesssim$" on equivalence classes of projections in M. If M were the $n \times n$ matrices, we would have $p \sim q$ if and only if $rank(p) = rank(q)$. In a *factor*, $\lesssim$ is a total order. The factor's type is determined by the existence of infinite projections type II_∞ and III, and the type of the ordered set $\lesssim$ according to the following table (the Hilbert space is assumed separable).

Murray-von Neumann type	ordered set $\lesssim$
I_n	$\{0, 1, 2, \ldots, n\}$ (n=∞ is OK)
II_1	$[0, 1]$
II_∞	$[0, \infty]$
III	$\{0, 1\}$

Here is an enormously condensed summary of the work of Murray and von Neumann.

- Any type I factor is of the form $\mathcal{B}(\mathcal{H}_1) \otimes id$ in some tensor "factorization" $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ (where we use $\mathcal{B}(\,\cdot\,)$ to denote the algebra of all bounded operators).
- Type II_1 factors may be constructed as the commutant of the regular representation of a discrete group all of whose conjugacy classes are infinite.
- Type II_∞ factors are always of the form $\mathrm{II}_1 \otimes \mathrm{I}_\infty$ (i.e., infinite matrices over a II_1 factor).
- Type III factors are more difficult but were constructed in [30].
- Type II_1 factors afford a *trace*—a linear functional tr: $M \to \mathbb{C}$ with $\mathrm{tr}(x^*x) > 0$ for $x \ne 0$ and $\mathrm{tr}(xy) = \mathrm{tr}(yx)$. Type II_∞ factors have an infinite trace analogous to that on $\mathcal{B}(\mathcal{H})$, and type III factors have no trace whatsoever.
- There is, up to isomorphism, a single II_1 factor with the "hyperfinite" property; i.e., any finite set of elements can be arbitrarily well approximated by elements in a finite-dimensional subalgebra. And there are II_1 factors without this property.

The period from the mid 1940s to the late 1960s was a period of digestion of the results of Murray and von Neumann. Highlights include the discovery by Dixmier of a plethora of maximal abelian subalgebras of a II_1 factor [10], the work of Dye on orbit equivalence [11], establishing the hyperfiniteness of crossed products of abelian algebras by $\mathbb{Z}$, and the characterization by Sakai of von Neumann algebras as C*-algebras which are duals as Banach spaces [26]. Let me cite two results as being the ultimate refinement of the Murray-von Neumann ideas—the construction by McDuff of uncountably many II_1 factors [21] and the construction by Powers of a continuum of type III factors [25].

Many of the seeds of future developments were sown during this period, but the first idea to truly go beyond Murray and von Neumann was the Tomita-Takesaki theory, first proposed by Tomita. The trace property $\mathrm{tr}(x^*x) = \mathrm{tr}(xx^*)$ means that $*$ is an isometry on the Hilbert space completion $\mathcal{H}_{\mathrm{tr}}$ of a factor M with respect to the inner product $\langle x, y\rangle = \mathrm{tr}(y^*x)$. This implies a certain symmetry between a type I or II factor and its commutant. The absence of this symmetry had caused type II factors to be somewhat inaccessible. Tomita envisaged $*$ as an *unbounded* (anti) linear operator on $\mathcal{H}_\varphi$, where φ is any positive (i.e., $\varphi(x^*x) > 0$) linear functional on M and $\mathcal{H}_\varphi$ is the completion of M with respect to $\langle x, y\rangle = \varphi(y^*x)$. The $*$-operator is preclosed, so one may form the polar decomposition $S = J\Delta^{\frac{1}{2}}$ of its closure S. Here Δ is a positive definite unbounded operator, and J is an isometry which restores the symmetry between M and its commutant: JMJ = the commutant of M on $\mathcal{H}_\varphi$, where M acts by *left* multiplication. But of perhaps greater significance is the second part of the Tomita-Takesaki theorem, which asserts:

$$\Delta^{it} M \Delta^{-it} = M \qquad (t \in \mathbb{R}),$$

which shows the existence of a 1-parameter automorphism group on the von Neumann algebra, called the modular group or canonical flow, written $\sigma_t^\varphi(x) = \Delta^{it} x \Delta^{-it}$. The modular group is entirely invisible in the abelian case and trivial for the types I and II, whereas type III factors are *dynamic* objects.

It is at this point that the work of Connes enters the history. If one had to find a single word to sum up his vast contribution to von Neumann algebras, it would be the word "automorphisms". His first step answered the question: "How does the modular group σ_t^φ depend on φ?" He showed that, given φ and $\psi : M \to \mathbb{C}$ with $\varphi(x^*x) > 0$, $\psi(x^*x) > 0$ (called "states" because of the quantum connection), there is a continuous map $t \mapsto u_t$ from $\mathbb{R}$ to the unitary group of M such that $\sigma_t^\varphi(x) = u_t \sigma_t^\psi(x) u_t^{-1}$ and $u_t \sigma_t^\psi(u_s) = u_{t+s}$. This is a noncommutative analogue of the Radon-Nikodym theorem comparing two measures and can be made into a genuine extension of it by considering the important Kubo-Martin-Schwinger condition from quantum statistical mechanics. We see that if $\mathrm{Out}(M)$ is the quotient group of all automorphisms by inner ones, σ_t^φ yields an invariant, called $T(M)$, the kernel of the map from $\mathbb{R}$ to $\mathrm{Out}(M)$. Now $T(M)$ can be quite complicated, and Connes also defined an invariant $S(M)$ as $\bigcap_\varphi \mathrm{spectrum}(\Delta_\varphi)\setminus\{0\}$, which is a closed multiplicative subgroup of $\mathbb{R}^+$. A type III factor is said to be of type III_λ, $0 \le \lambda \le 1$, according to whether $S(M)$ is

a)	$\mathbb{R}^+$	type III_1
b)	$\{\lambda^n \mid n \in \mathbb{Z}\}$	type $\mathrm{III}_\lambda, 0 < \lambda < 1$
c)	$\{1\}$	type III_0

One way to obtain a type III_λ factor $(0 < \lambda < 1)$ is to take an automorphism of a II_∞ factor N, scaling the trace by λ, and form the crossed product $N \rtimes \mathbb{Z}$. Connes showed that *any* type III_λ factor is of this form and gave a similar decomposition in the III_0 case [4]. Takesaki further showed that a III_1 factor is the crossed product of a II_∞ factor by a trace-scaling action of $\mathbb{R}$ [27].

The extraordinary conclusion of all this is that type III factors can be completely understood by knowing type II factors and their automorphism groups. So Connes began a penetrating study of automorphisms of type II factors, introducing entirely new techniques. The first step was to classify periodic automorphisms of the hyperfinite II_1 factor R, then arbitrary automorphisms of $R \otimes \mathcal{B}(\mathcal{H})$. In order to complete the classification of hyperfinite III_λ factors, $0 \le \lambda < 1$, a serious snag had to be overcome, for it was not known that a hyperfinite II_∞ factor was automatically isomorphic to $R \otimes \mathcal{B}(\mathcal{H})$. Connes solved this in [5] by introducing a new approach to hyperfiniteness: injectivity of the von Neumann algebra M, meaning that M has a Banach space complement in $\mathcal{B}(\mathcal{H})$. The reviewers consider [5] to be one of the highlights of twentieth-century mathematics. The result (injective) $\Longleftrightarrow$ (hyperfinite) and work of Krieger allowed Connes to complete the classification of hyperfinite III_λ factors, $0 \le \lambda < 1$. Haagerup showed uniqueness in the III_1 case some ten years later using a condition discovered by Connes.

The ideas involved in the injective factors theorem spawned a vast number of new results on factors: in particular, an invariant $\chi(M)$ — an abelian group which Connes has shown to be quite arbitrary, thereby constructing in a natural way an uncountable family of II_1 factors and, for example, II_1 factors not isomorphic to their tensor products with themselves and II_1 factors distinct from their opposite algebras (see [6]). Such problems were long-standing ones that had been considered intractable before Connes. His book contains a detailed account of this story.

Having thus dispatched injective von Neumann algebras, Connes began his investigation of noncommutative geometry. The first insight was the existence of a *canonical* von Neumann algebra associated to a smooth foliation of a manifold. The idea is to form the algebra of sections of the bundle over the leaf space which associates to each leaf its canonical L^2-space of half-densities. These sections must be measurable in some sense, and that is where the interest lies: if the leaves wrap around themselves a lot, the leaf space is like the orbit space for an irrational rotation, already discussed, and one may not be able to find sections arbitrarily. The notion of measurability is with respect to the transverse measure class of Lebesgue measure; and Connes saw that if the leaves wind together in an ergodic way, then his von Neumann algebra is a factor. Using noncommutative measure theory, Connes showed how to obtain real-valued Betti numbers for a foliation with holonomy-invariant transverse measure and gave a Gauss-Bonnet formula with purely geometric consequences. This set the stage for a full generalization of the Atiyah-Singer index theorem to foliations and more general structures.

At this stage noncommutative measure theory begins to become insufficient, and the next step towards noncommutative geometry is noncommutative topology. In the case of foliations this allows us to study individual leaves rather than just the average leaf.

As we have said, a C*-algebra—abstractly just a Banach $*$-algebra with $\|x^*x\| = \|x\|^2$—is the noncommutative version of a locally compact Hausdorff space. So it is natural that it took longer before the interesting examples in C*-algebras became clear, since we have to sift among the noncommutative versions of the sometimes tiresome pathological spaces of point-set topology. (By contrast, there is essentially only one measure space, so that any von Neumann algebra should be interesting.) The founding work was done by Gelfand and Naimark, who showed that the C*-algebra axioms actually characterize norm-closed sub $*$-algebras of $\mathcal{B}(\mathcal{H})$. A hardy group of pioneers—notably including Dixmier, Kadison, and Kaplansky—kept the theory of C*-algebras growing during a period when their future importance was not yet appreciated. But the first major structural result was that of Glimm [13], who showed that either a C*-algebra is quite simple, in that all its representations generate type I von Neumann algebras, or there is a non-type I representation and all hell breaks loose, there being representations of all kinds whose complete classification is impossible. Examples of type I C*-algebras are abelian ones and the algebra of all compact operators. The C*-algebra of almost any nonabelian discrete group is not of type I. Something like a classification of type I C*-algebras could be envisaged, but such a thought would be folly away from type I.

But Connes' work, and hence his book, has had little to do with the structure theory of an abstract C*-algebra. Rather, he has looked for the interesting examples and how to understand them in a *geometric* framework.

In the commutative world, the most immediate and powerful tools are homology and the fundamental group. Unfortunately, these do not have straightforward noncommutative generalizations. It is topological K-theory—the study of vector bundles—that has been the most fruitful, as it passes immediately to the noncommutative world. A vector bundle on a manifold is, by a result of Swan, the same thing as a projection in a matrix algebra over the smooth functions on the manifold. So vector bundles are generalized by the algebraic K_0-group (equivalence classes of projections in matrix algebras) of a C*-algebra. Thus, geometry as approached by vector bundles does admit a noncommutative version. However, this merely represents a starting point in the quest to build a full-fledged extension of the "commutative" homological apparatus. In both algebraic geometry and algebraic topology, the contravariant K-group based on vector bundles goes hand in hand with a covariant counterpart, and this duality mutually enhances their power. A decisive step in the direction of defining the K-homological group in operator-theoretic terms was taken around 1969 by Atiyah [1], who proposed a construction of the Whitehead dual to K-theory based on axiomatizing the concept of elliptic operator. His ideas were taken up and fully developed by Brown, Douglas, and Fillmore [3], who turned them into a theory of C^*-algebra extensions, and by Kasparov. The latter then succeeded in merging, in the general framework of C^*-algebras, the contravariant K-functor with the covariant *Ext*-functor into a bivariant KK-functor, endowed with a remarkable intersection product [15]. Besides Kasparov's theory, two other fundamental results, dealing with crossed products, provided a catalyst for the quest to understand the K-groups of noncommutative C^*-algebras: the Pimsner-Voiculescu [23] exact sequence for crossed products by $\mathbb{Z}$ and Connes' Thom isomorphism [7] for crossed products by $\mathbb{R}$.

The newly developed K-theoretical apparatus led to the formulation of a host of exciting generalizations of the Atiyah-Singer index theorem [2], extending the scope of the fertile interaction between topology and geometry on one side and functional analysis on the other. These developments were synthesized by Baum and Connes into a powerful guiding principle known as the Baum-Connes conjecture. Under the form of an analytical analogue for the assembly map of surgery theory, it provides a geometric description to the K-group of a C^*-algebra representing a "noncommutative space".

This constitutes the main theme of Chapter II, where the principle is formulated in the language of smooth groupoids and is richly illustrated. The special cases treated in detail include: the tangent groupoid of a manifold as an embodiment of the Atiyah-Singer index theorem, the orbit space of a group action, the leaf space of a foliation and the longitudinal index theorem, the case of Lie groups and the equivariant index theorem on homogeneous spaces of Lie groups.

With all the advances made on the K-theoretical flanck in the early 1980s, the lack of a cohomological companion to K-theory remained a serious impediment. In the absence of effective computational devices, potentially interesting generalizations of the index theorem would run the risk of becoming little more than tautology. Noncommutative topology, reared in the rarefied environment of topological K-theory, was now in need of its own, down-to-earth homology theory which would allow making concrete calculations. The breakthrough came in 1981 with the discovery by Connes of *cyclic cohomology* and of the spectral sequence relating it to Hochschild cohomology [8]. Far from being serendipitous, Connes' discovery came as a logical outcome of his inspired search for a purely algebraic de Rham-type theory for noncommutative algebras, which should pair with the *Ext*-functor and with its "even" mate in a manner similar to the Chern character.

To elaborate, an abstract elliptic operator, or *Fredholm module*, over a unital $*$-algebra $\mathcal{A}$ over $\mathbb{C}$ consists of a pair $(\mathcal{H}, F)$, where $\mathcal{H}$ is a separable Hilbert space on which $\mathcal{A}$ acts by a $*$-representation π and $F = F^*$ is a bounded operator on $\mathcal{H}$ such that $F^2 - I$ together with the commutators $[F, \pi(a)]$, $\forall a \in \mathcal{A}$, are "small", i.e., compact. The prototypical model consists of an elliptic (0^{th} order) pseudodifferential operator on a closed smooth manifold M, together with an approximate inverse (i.e., parametrix). In that case, $\mathcal{A} = C^\infty(M)$, and the commutators $[F, \pi(a)]$ are not just compact but of Schatten p-class: $\text{Trace}\,|[F, \pi(a)]|^p < \infty$, for any $p > \dim M$. The latter condition, viewed as a "finite dimensionality" axiom, can be incorporated into the general definition of a *p-summable Fredholm module*. Furthermore, there is no essential loss of generality in requiring $F^2 = I$.

With these amendments, the pairing $\langle [F], [e] \rangle$ between the K-homology class represented by $(\mathcal{H}, F)$ and the K-theory class $[e] \in K_0(\mathcal{A})$ of an idempotent $e^2 = e \in \mathcal{M}_q(\mathcal{A})$, given by the Fredholm index, can be expressed as

$$\langle [F], [e] \rangle = (-1)^n \text{Supertrace}\Big(e[F, e]^{2n}\Big), \quad \forall n > \frac{p+1}{2}.$$

Regarding $de = [F, e]$ as a *quantized differential*, the right-hand side bears a striking resemblance to the formulae expressing the Chern classes of a bundle in terms of curvature. The caveat is that the above expression starts making sense precisely at the stage where the usual one becomes trivial, i.e., when the degree of the noncommutative "curvature form" surpasses the dimension of the "space". The polarized form of the right-hand side,

$$\begin{aligned} &\tau_F(a^0, a^1, \dots, a^{2n}) \\ &= (-1)^n \text{Supertrace}\Big(a^0[F, a^1] \dots [F, a^{2n}]\Big), \\ &a^j \in \mathcal{A}, \end{aligned}$$

encodes the defining features of the cyclic cohomology theory and typifies the quintessential cyclic cocycle. As an element of the cyclic cohomology group $HC^*(\mathcal{A})$ (which is a graded module over the polynomial ring $HC^*(\mathbb{C})$), the cohomology class of τ_F represents the *Chern character* of $(\mathcal{H}, F)$ in Connes' sense. Using this character-pairing as a guiding principle, Connes was able to uncover the fundamental properties and operations of cyclic cohomology, starting with a crucial *periodicity operator* implicit in the above discussion. For methodological reasons, these topics are covered in the book in reverse chronological order: cyclic cohomology in III.1–3 and the Chern character of Fredholm modules in IV.1.

It is worth noting that in the framework of noncommutative spaces the integrality of the index pairing acquires new potency. A suggestive example is the elegant argument (cf. IV.5) given to the Pimsner-Voiculescu theorem [24] that settled a long-standing conjecture of Kadison, asserting that the reduced C^*-algebra of a free group has no nontrivial idempotents. The relevant "space" is the "dual" to a free group Γ; i.e., the reduced C^*-algebra $C_r^*(\Gamma)$, whose canonical trace $\tau(\sum_{g \in \Gamma} a_g g) = a_1$, happens to be precisely the Chern character of a geometric 1-summable Fredholm module. Thus, $\tau(K_0(C_r^*(\Gamma))) \subset \mathbb{Z}$.

The relationship with the index theory of elliptic operators on manifolds is quite transparent. It does involve, though, a highly nontrivial step: the identification of the periodic *continuous cyclic cohomology* of the algebra $C^\infty(M)$, equipped with its Fréchet space topology, with the de Rham homology of M (cf. III.2.α).

As often happens with an idea whose time has come, cyclic cohomology was quickly recognized as relevant to other branches of mathematics.

Inspired by Connes' work and motivated by algebraic K-theory, Loday and Quillen [18] developed the dual theory, cyclic homology, and related it to the Lie algebra homology of matrices over a ring. Starting from a similar standpoint, B. Tsygan [28] arrived on his own at the concept of cyclic homology as an additive version of algebraic K-theory.

The role of cyclic cohomology as a noncommutative de Rham theory of currents has found spectacular applications in connection with one of the central problems in topology, the Novikov conjecture. In its best-known form, this conjecture states that the "higher signatures" of a non-simply connected oriented manifold M, formed out of the L-classes of M and the cohomology of $\pi_1(M)$, are homotopy invariants. When the manifold is simply connected, this amounts to a consequence of Hirzebruch's signature formula, the homotopy invariance of the L-genus.

For manifolds M whose fundamental group $\Gamma = \pi_1(M)$ is free abelian, Lusztig [19] devised a seminal method of proof, based on the Atiyah-Singer index theorem for families of elliptic operators. The family involved is that of signature operators on M twisted by flat line bundles, hence parametrized by the Pontryagin dual $\hat{\Gamma}$. The index bundle of this family, which belongs to $K^*(\hat{\Gamma})$, is a homotopy invariant, and the Atiyah-Singer theorem allows the identification of its Chern character as the higher signature.

If Γ is not abelian, the Pontryagin dual ceases to exist as an ordinary space, but the group C^*-algebra $C^*(\Gamma)$ remains a natural substitute for $C(\hat{\Gamma})$. Moreover, work of Mishchenko and Kasparov gave a precise meaning to the "noncommutative" incarnation of the above index bundle, as an element of $K_*(C^*(\Gamma))$, and established its homotopy invariance. Various K-homology/cohomology devices, in the context of C^*-algebras, have been successfully used by Mishchenko [20], Kasparov [16], Kasparov-Skandalis [17], and quite recently by Higson-Kasparov to prove the validity of the Novikov conjecture in a wide variety of cases, such as when Γ is a discrete subgroup of a Lie group (Kasparov) or when Γ is amenable (Higson-Kasparov). All these proofs circumvent the last step of Lusztig's original argument, which can only be restored by means of cyclic cohomology.

The cyclic cohomological approach (due to Connes) is treated in III.4–5. Its salient features are the following. First, one constructs a refined variant of the Mishchenko-Kasparov signature, as an element of the K-group of the group ring $\mathcal{R}\Gamma$, where the ground ring $\mathcal{R}$ consists of infinite matrices of rapid decay. Moreover, any (normalized) group cocycle gives rise canonically to a cyclic cocycle in $HC^*\left(\mathcal{R}\Gamma\right)$. The result of the pairing between this cocycle and the generalized signature in $K_0\left(\mathcal{R}\Gamma\right)$ is then identified, by means of a "higher index theorem", as the higher signature corresponding to the given group cocycle. This could have been the end of the story, except that the homotopy invariance of the generalized signature is known to take place only in $K_0(C^*(\Gamma))$. The transition from the first K-group to the second requires a suitable analogue of the Harish-Chandra-Schwartz algebra. The latter is known to exist when Γ belongs to the remarkable class of hyperbolic groups in Gromov's sense, which therefore satisfy the Novikov conjecture.

Another remarkable application of cyclic cohomology, in its capacity of de Rham homology of closed currents on the space of leaves, is to the theory of foliations (cf. III.6–7). To begin with, it allows the introduction on a transversely oriented foliation $(V, \mathcal{F})$ of a fundamental cyclic cohomology class $[V/\mathcal{F}]$. As in the case of the Novikov conjecture, the topological invariance is not automatic and raises in fact a major issue. Connes' solution involves a new technique—the passage to a longitudinal frame bundle of leafwise metrics in a manner which preserves the pairing of the transverse fundamental class with the K-theory. A number of interesting and purely geometric consequences flow out of this. The inherently noncommutative nature of a leaf space manifests itself with a vengeance in the surprising interaction between the differential-geometric and the measure-theoretic aspects. The most striking example is the equality

$$GV = i_\delta \, \frac{d}{dt} \, [V/\mathcal{F}]$$

relating the Godbillon-Vey class of a codimension 1 foliation to the time evolution of the transverse fundamental class under the modular automorphism group of the corresponding von Neumann algebra.

Classical differential geometry began with a local calculus and evolved to be able to treat global issues. In stark contrast, the noncommutative version has progressed from global to local! In fact, the very meaning of the notion of "local" in noncommutative geometry was problematic before the advent of Connes' *quantized calculus* (cf. Chapter IV). While it is not surprising that the formalism of quantum mechanics should play a foundational role, it took Connes' extraordinary insight to realize that it can be actually turned into an organized collection of "local" tools, ready to assume the part of the infinitesimal calculus. Put in succinct dictionary form, the quantized calculus starts from the familiar quantum-mechanical interpretation of observables, with the replacement of *complex variables* (resp. *real variables*) by *operators* (resp.

selfadjoint operators) on a (separable) Hilbert space $\mathcal{H}$ and continues with the substitution of *compact operators* for classical *infinitesimals*, the *order* of a quantum infinitesimal T being measured by the *rate of decay* of its characteristic values $\mu_n(T)$. Motivated by the Fredholm module picture, the *quantized differential* is defined by the commutator with a fixed bounded operator $F = F^*$, $F^2 = I$ that splits $\mathcal{H}$ into two isomorphic orthogonal subspaces:

$$dT = [F, T].$$

Finally, the *integral* of a *measurable variable* $T \geq 0$ is defined as the logarithmic divergence of the trace of T,

$$\frac{1}{\log N} \sum_{0}^{N-1} \mu_n(T) \longrightarrow \int T \ ,$$

the meaning of measurability being that the right-hand side is independent of the underlying limiting procedure.

With the stage thus set, a bewildering array of applications of the new calculus follows in short order. They range from formulae for the Minkowski content of Cantor sets and for the Hausdorff measure on Julia sets—as applications of the one-variable quantized calculus—to computing a 4-dimensional analogue of the Polyakov action of 2-dimensional conformal field theory and making sense of the "area" of a 4-dimensional manifold passing through a canonical quantization of the calculus on conformal $2n$-manifolds.

The sixth and last chapter could lead to a revolutionary change in geometric thinking, comparable to the transition from classical to quantum mechanics. The notion of a *spectral space* X, in Connes' sense, is a quantum version of a Riemannian space. Thus, the *coordinates* of X are given by an involutive—although not necessarily commutative—algebra $\mathcal{A}$ of bounded *operators* on a Hilbert space $\mathcal{H}$. The *quantized length element* of X is an "infinitesimal" of the form

$$ds = D^{-1} \ ,$$

where $D = D^*$ is an unbounded selfadjoint operator on $\mathcal{H}$ such that the commutators $[D, a]$ are bounded $\forall a \in \mathcal{A}$. While ds no longer commutes with the coordinates, the algebra they generate does satisfy nontrivial commutation relations.

In this picture, a Riemannian (spin) manifold M is described (cf. [9]) by the algebra $C^\infty(M)$ represented by multiplication operators on the L^2-space of sections of the spin bundle over M and with the quantized length element given by the Dirac propagator $ds = D^{-1}$ (the ambiguity created by the possible zero modes is inconsequential). The commutation relations are

$$[[f, ds^{-1}], g] = 0 \ , \ \forall f, g \in C^\infty(M)$$

with an additional relation, of degree $n = \dim M$, arising from the homological nature of the volume form on M. An example of how classical geometric features are recovered from this perspective is the "dual" formula giving the geodesic distance

$$\text{dist}(x, y) = \sup \left\{ f(x) - f(y);\ \|[D, f]\| \leq 1 \right\} ,$$

which retains its meaning for discrete spaces as well.

As the mathematical underpinning for Einstein's general relativity theory, Riemann's geometry (in its Lorentzian guise) is the geometry of the space-time at macroscopic scale. By placing it on equal footing with the geometry of discrete spaces, in a manner directly compatible with the formalism of quantum mechanics, Connes' spectral approach gains the ability to reach below the Planck scale and attempt to decipher the fine structure of space-time. This is precisely what the author sets out to do in the very last section, devoted to a geometric explanation of the best available phenomenological model of particle physics, the Standard Model.

The book does have its faults. Some parts require much more mathematical sophistication than others. The index should be much more detailed. And the publishers should be censured for binding the book in a way that will not withstand the many readings the book requires. But as a book of genuine vision, *Noncommutative Geometry* renders insignificant such minor quibbles.

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