Rigor in Calculus

Leonard Gillman

Rigor means disciplined thinking, the heart of mathematics. One prominent businessman recalled his college mathematics courses in the following way.

> I'm not out to convince anyone that calculus, or even algebra and geometry, are necessities in the hotel business. But I will argue long and loud that they are not useless ornaments pinned onto an average man's education. For me, at any rate, the ability to formulate quickly, to resolve any problem into its simplest, clearest form, has been exceedingly useful. It is true that you do not use algebraic formulae but in those three small brick buildings at Socorro I found higher mathematics the best possible exercise for developing the mental muscles necessary to this process.
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> In later years I was to be faced with large financial problems, enormous business deals with as many ramifications as an octopus has arms, where bankers, lawyers, consultants, all threw in their particular bit of information. It is always necessary to listen carefully to the powwow, but in the end someone has to put them all together, see the actual problem for what it is, and make a decision—come up with an answer. A thorough training in the mental disciplines of mathematics precludes any tendency to be fuzzy, to be misled by red herrings, and I can only believe that my two years at the School of Mines helped me to see quickly what the actual problem was—and where the problem is, the answer is. Any time you have two times two and *know* it, you are bound to have four[1].

Proof is the soul of mathematics and logic, distinguishing them from all other disciplines. Rigor does not require that every statement you make be accompanied by a proof, but only that the need for a proof be recognized: the *status* of the proposition as well as that of any supporting discussion be made clear. Tell whether the proposition is a theorem or a conjecture and whether the accompanying discussion is a proof or only a suggestion or an outline of it or just a sketch of the main idea. Rigor also demands emphasizing that theorems have hypotheses: after all, one of Bertrand Russell's definitions of mathematics was as the class of all propositions of the form P implies Q. Do not let students get away with saying that every continuous function assumes a greatest value and a least value.

The recognition that a theorem has hypotheses includes requiring students to verify them when invoking the theorem. But be sensible about it. Hypotheses that govern the universe of discourse, once they have been established in the students'

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[1] *Conrad Hilton,* Be My Guest, *Prentice-Hall Inc., 1957, p. 71.*

minds, should be taken for granted. A similar consideration applies to invoking a theorem in the course of a proof. A student who has just recently encountered the theorem should quote or refer to it by name, but by the time it has become assimilated into the local culture, a citation is no longer needed. Within a short time after encountering the Pythagorean theorem, the student, having written, "This shows that in triangle ABC, C is a right angle" may pass at once to "therefore $a^2 + b^2 = c^2$" without further ado.

Rigor also requires students to state definitions correctly, except for hopelessly complicated definitions such as that of limit.

None of this is intended to suggest that all mathematics courses have to be presented at the level of advanced calculus. On the contrary, one should always keep any discussion at a level appropriate to the audience. There is a favorite story in our home about the little boy next door asking his mother where babies come from. Having prepared long since for this very moment, she responded with a flawless rendition of her elegantly crafted reply and asked whether he had any questions. "Yes", he said, "how do you make bricks?" John Kelley put it best back in the New Math days: "Tell them the truth and nothing but the truth, but for God's sake don't tell them the whole truth."

It is fashionable in some circles to portray math as being easy. Now, I certainly believe that ordinary kids can learn standard school math if they work hard at it. But that is not the same as saying math is easy. The notion is ill-considered and dangerous. Consider the effect on students: passing an easy course offers nothing to take pride in, and failing it can shatter one's self-esteem.

Another fashion is to denigrate drill, usually referred to as "mindless" drill, suggesting that mindlessness is something intrinsic to a problem or set of problems. I would say it resides rather in the attitude of the student who sits there thinking about something else or the of teacher who says, "Do these ninety-seven problems" instead of "Do enough of these problems that you feel confident you don't need further practice, then do three more for good measure." Many writers have noted the similarity to music lessons and the recognized need for pupils to practice scales and other exercises. I myself have been practicing the piano for more than seventy years, but still heartily enjoy exercises and continue to invent new ones. Practice and drill are established strategies for honing one's skills. No pianist, from rank beginner to seasoned professional, ever inserts five-finger exercises into a recital program. No football player ever runs through a zigzag of automobile tires in the course of an actual game.

Mathematics teachers are virtually unanimous in vetoing (ε, δ) as a fit topic for beginning calculus students. Quantifiers are unfamiliar to students in any case, and the intimidating succession in (ε, δ) is well beyond their comprehension. So we try somehow to give them the *idea* of limit. Unfortunately, the typical attempt is just about as futile. When I was a student, the instructor told us that y will be as close as you please to L provided that x is close enough to a. The entire class, all 12 of us, stared at him with glazed eyes. When, many years later, I gave my freshman students the same description, the entire class, all 120 of them, stared at me with glazed eyes.

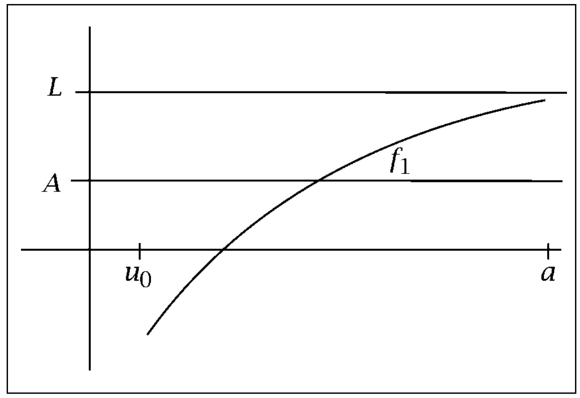

Figure 1.

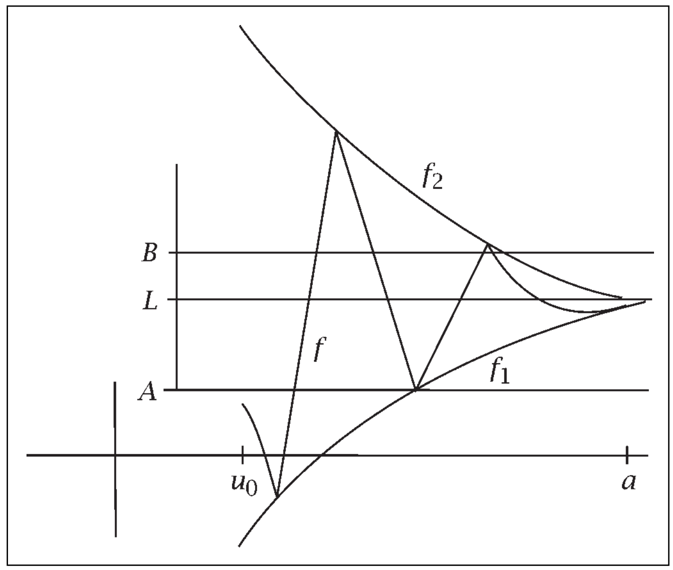

Figure 2.

I propose the following description of limit. It is equivalent to (ε, δ) yet is expressed low-key, in familiar terms. Concentrate for the moment on the limit from the left. Let f be a function defined on an interval (u_0, a), L a number, and f_1 a (strictly) increasing function on (u_0, a) whose graph lies everywhere below the level L but rises above every lower level. In detail, if A is any number less than L, then f_1 eventually rises above the level A (and once above, remains above (Figure 1)). It is here, in this relaxed form, that the quantifiers reside. A decreasing function f_2 is described correspondingly. We then say that

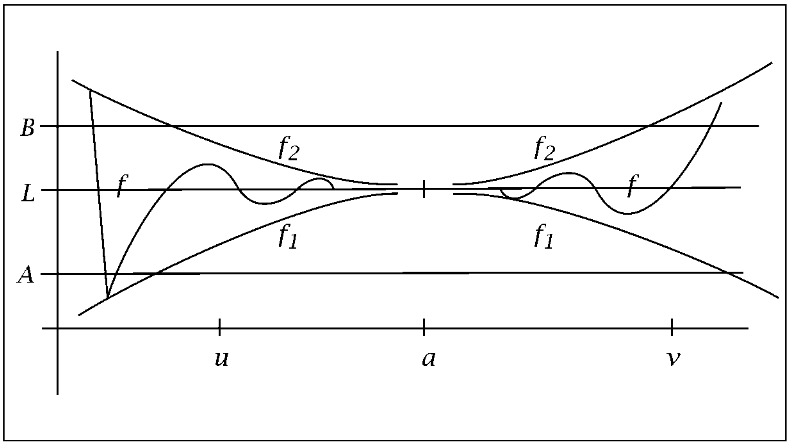

Figure 3.

$$\lim_{x \to a^-} f(x) = L$$

if there exist functions f_1 and f_2 as described satisfying $f_1(x) \le f(x) \le f_2(x)$ for all x in the interval (Figure 2). Thus, f "zeros" in on the value L.

The description of the limit as $x \to a^+$ is similar, but note that the lower bound f_1 is now technically a decreasing function. It is less confusing to say that f_1 increases as x moves from right to left (or as x moves towards a^+.) The corresponding remark applies to f_2. Finally, of course, we say that

$$\lim_{x \to a} f(x) = L,$$

provided that both one-sided limits exist and equal L (Figure 3).

The foregoing description would be awkward as an actual working definition of limit. But it is very useful for motivating the simple definition that if J is any open interval about L, then there is a punctured open interval $I \backslash a$ that f takes into J. In fact, this definition can be read off from Figure 3, with $(A, B) = J$ and $(u, v) = I$. (If you insist, you may stipulate that A and B be equidistant from L and choose u and v to be equidistant from a.) This shows that the description implies the (ε, δ) definition. Conversely, if f is defined on (u_0, a) and $\lim_{x \to a} f(x) = L$ according to (ε, δ), then the functions

$$f_1(x) = \inf_{(x,a)} f + (x - a)$$

and f_2, defined similarly, have the desired properties. (This proof is not meant for the students.)

The (ε, δ) mindset encourages cluttering the page with symbols that serve mainly to distract and invites unnecessary computation. In one of the better advanced calculus books, I read, "There is a number $\delta > 0$ such that for $a - \delta < x < a$, $f(x)$ has property P." Why not just, "$f(x)$ has property P on an interval (u, a)" or "$f(x)$ has property P near a^-"?

Moreover, the precision offered by (ε, δ) proofs can obscure the underlying ideas. I have before me a page from a top-selling calculus text containing a thirty-line, symbol-laden computational proof that if $f(x) \to L \neq 0$, then $1/f(x) \to 1/L$. At the end, the student has no idea of what went on. Are there instructors who actually present such proofs to a freshman class? In contrast, the proof based on order relations occupies only two lines (with an additional short explanation in case the tactic of replacing the challenge (A, B) by a suitable subinterval is not yet familiar). The reasoning is on the level that if $0 < x < y$, then $1/y < 1/x$:

> Say $L > 0$. Given $A < 1/L < B$, we may assume $A > 0$. Then $1/B < L < 1/A$. Since $f(x) \to L$, $1/B < f(x) < 1/A$ near a. Then $A < 1/f(x) < B$ near a. □