Periodic Solutions of Nonlinear Partial Differential Equations

C. Eugene Wayne

"It seems at first that this fact [the existence of periodic solutions] could not be of any practical interest whatsoever… [however] what renders these periodic solutions so precious is that they are, so to speak, the only breach through which we may try to penetrate a stronghold previously reputed to be impregnable."

—Henri Poincaré

As the existence theory for solutions of nonlinear partial differential equations becomes better understood, one can begin to ask more detailed questions about the behavior of solutions of such equations. Given the bewildering complexity which can arise from relatively simple systems of ordinary differential equations, it is hopeless to try to describe fully the behavior which might arise from a nonlinear partial differential equation. Thus it makes sense to first consider special solutions, in the hope that through a more concrete understanding of them one may gain insight into the behavior of more general solutions. An extremely fruitful avenue of study in the theory of ordinary differential equations has been the construction of periodic orbits: in many circumstances they form a sort of skeleton on which more complicated solutions can be built. It was Poincaré who first realized this possibility, a discovery which prompted the remark quoted above. By a careful analysis of the periodic solutions that occur in the celestial mechanics problem of three gravitationally interacting planets and of the solutions asymptotic to these periodic orbits, he proved the existence of "chaotic" orbits in this system.

For the past thirty years or so there has been an active search for periodic solutions of partial differential equations, employing a variety of methods and motivated, at least in part, by the important role that periodic solutions play in understanding the behavior of ordinary differential equations. My goal in what follows is to describe a new technique for constructing such solutions which both highlights the differences between ordinary and partial differential equations and which also exhibits a surprising connection with problems in quantum mechanics. However, contrary to what Poincaré's quotation might suggest, these periodic solutions are not only of theoretical interest but also have many practical applications. As far as I am aware, the first study of periodic solutions of a nonlinear partial differential equation was in the early 1930s in the work of Vitt ([22], described in [13]), who considered these solutions in the context of problems of electrical transmission. Additional research was carried out in the '30s and '40s, often by physicists; not until the 1960s did mathematicians begin a fairly intensive study of the existence and properties of periodic solutions. (See, for example, [14, 21].) One problem that aroused particular interest was the structure of periodic standing waves on the surface of an inviscid, irrotational fluid. (See [21] and [6].) In particular, Paul Concus [7] pointed out the difference between the existence of periodic solutions for systems of ordinary differential equations and

---

C. Eugene Wayne is professor of mathematics at The Pennsylvania State University. His e-mail address is `wayne@math.psu.edu`. After September 1, 1997, he will be professor of mathematics at Boston University.

The author was supported in part by NSF Grant DMS-9501225.

partial differential equations as he explicitly examined the possible occurrence of "small denominators" in these partial differential equations, a problem which does not arise in the construction of periodic solutions for ordinary differential equations. All of these studies were based on deriving formal power series which were believed to approximate periodic solutions of the partial differential equations. However, while the work of Concus and others identified many important questions, the convergence or divergence of these series was not established, leaving open the question of whether or not such periodic solutions actually existed.

Like these earlier approaches to the construction of periodic solutions, the method I will describe is essentially perturbative in character. As an illustration of the sort of questions one encounters, consider the elementary system of two ordinary differential equations,

$$\ddot{x}_1 = -\omega_1^2 x_1, \tag{1}$$

$$\ddot{x}_2 = -\omega_2^2 x_2 \,. \tag{2}$$

Clearly, *all* solutions of this system of equations are periodic. Suppose that one now adds nonlinear terms to the equations; one would like to use the information about solutions of (1) and (2) to understand the solutions of

$$\ddot{x}_1 = -\omega_1^2 x_1 - \frac{\partial V}{\partial x_1}(x_1, x_2), \tag{3}$$

$$\ddot{x}_2 = -\omega_2^2 x_2 - \frac{\partial V}{\partial x_2}(x_1, x_2) \,, \tag{4}$$

where $V(x_1, x_2)$ has a Taylor series at the origin beginning with terms of at least order three. If one chose to add arbitrary nonlinear terms, it would be hopeless to make any general statement about the behavior of solutions of the perturbed equation. For instance, one can easily construct examples in which the resulting equations have no periodic solutions (except for the trivial solution $x_1 = x_2 = 0$), a finite number of periodic solutions, or an infinite number. However, the particular form of the nonlinear term in (3)–(4) (which insures that the resulting system of equations can be written as a Hamiltonian system) allows one to analyze small periodic solutions in some detail.

Lyapunov [17] originally derived sufficient conditions to insure that equations like (3) and (4) have periodic solutions which are close to those of the linear equations (1)–(2). As a motivation for what follows, let me sketch a proof of the existence of periodic solutions of (3)–(4), which is somewhat different from standard demonstrations. Any periodic solution must be of the form

$$x_1(t) = \sum_{n \in \mathbf{Z}} e^{in\Omega t} \hat{x}_1(n), \tag{5}$$

$$x_2(t) = \sum_{n \in \mathbf{Z}} e^{in\Omega t} \hat{x}_2(n) \,. \tag{6}$$

If we substitute these forms of the solutions into (3)–(4), we find that the Fourier coefficients $\hat{x}_j(n)$ must satisfy

$$-n^2\Omega^2 \hat{x}_1(n) = -\omega_1^2 \hat{x}_1(n) + \hat{V}_1(\hat{x}_1, \hat{x}_2)(n) \,, \quad n \in \mathbf{Z} \,, \tag{7}$$

$$-n^2\Omega^2 \hat{x}_2(n) = -\omega_2^2 \hat{x}_2(n) + \hat{V}_2(\hat{x}_1, \hat{x}_2)(n) \,, \quad n \in \mathbf{Z} \,, \tag{8}$$

where $\hat{V}_j$ is the result of inserting the expansion of x_1 and x_2 in Fourier modes into the nonlinear terms in (3)–(4) and expanding the resulting expression as a Fourier series. Exchanging two coupled, nonlinear, ordinary differential equations for infinitely many coupled, nonlinear, algebraic equations may not seem like progress, but this form of the problem turns out to be well suited to the application of the Lyapunov-Schmidt method (see [5]). Define the diagonal, linear operator with matrix elements $L(n, j) = \omega_j^2 - n^2\Omega^2$, and let $\hat{x} = (\hat{x}_1, \hat{x}_2)$ and $\hat{V} = (\hat{V}_1, \hat{V}_2)$. Then (7) and (8) can be combined as

$$(L\hat{x})(n, j) = \hat{V}(\hat{x}) \,. \tag{9}$$

If $\omega_1 \neq n\omega_2$ for all integers n—that is to say, if the two unperturbed oscillators are *nonresonant* and if the frequency Ω of the periodic solution is close to ω_1—then the diagonal, linear operator with matrix elements $L(n, j) = \omega_j^2 - n^2\Omega^2$ will have two small diagonal elements when $(n, j) = (\pm 1, 1)$ and all other diagonal elements are bounded strictly away from zero. (In defining the nonresonance condition, I assumed that $\omega_1^2 \leq \omega_2^2$; otherwise one must also insure that $\omega_2 \neq n\omega_1$. Below we will encounter systems of equations with infinitely many frequencies ω_j^2. In that case I will always assume that we have ordered the equations so that $\omega_j^2 \leq \omega_k^2$ if $j \leq k$, for similar reasons.)

More precisely, let P_1 be the projection onto the two-dimensional space spanned by the coefficients $\{\hat{x}_1(\pm 1)\}$, and let $Q_1 = P_1^{\perp}$. (We can take the orthogonal projection in the Hilbert space of ℓ^2 sequences of Fourier coefficients.) Now rewrite (9) as a pair of equations by applying P_1 and Q_1 to both sides of this equation. Defining $\hat{y} = P_1 \hat{x}$ and $\hat{z} = Q_1 \hat{x}$, one has

$$(P_1 L\hat{y})(n, j) = (P_1 \hat{V}(\hat{y}, \hat{z}))(n, j) \,, \tag{10}$$

$$(Q_1 L\hat{z})(n, j) = (Q_1 \hat{V}(\hat{y}, \hat{z}))(n, j). \tag{11}$$

The point that allows one to solve (11) with relative ease is that because of the observation that the eigenvalues of L are bounded away from zero if $(n, j) \neq (\pm 1, 1)$, $Q_1 L$ has bounded inverse, so that given $\hat{y}$, one can solve (11) by the implicit function theorem and obtain $\hat{z} = \hat{z}(\hat{y})$. Inserting this solution into (10), we obtain a pair of equations (recall that the range of P_1 is two-dimensional),

$$(12) \qquad (P_1 L\hat{y})(n, j) = (P_1 \hat{V}(\hat{y}, \hat{z}(\hat{y})))(n, j) , \quad (n, j) = (\pm 1, 1) \ .$$

For Ω close to ω_1 these two equations can be solved "by hand", and the resulting set of Fourier coefficients $\hat{x} = (\hat{y}, \hat{z}(\hat{y}))$ are the Fourier coefficients of a periodic solution of (3)–(4).

To sum up, this argument shows that there exists an $r_0 > 0$ and a smooth curve $\Omega(r)$, defined for $0 \leq r \leq r_0$, such that for r in this range there exists a periodic solution of (3)–(4) with amplitude r and frequency $\Omega(r)$. (One can measure the amplitude of the solution in a number of ways; for definiteness, use the ℓ^2 norm of the set of Fourier coefficients.)

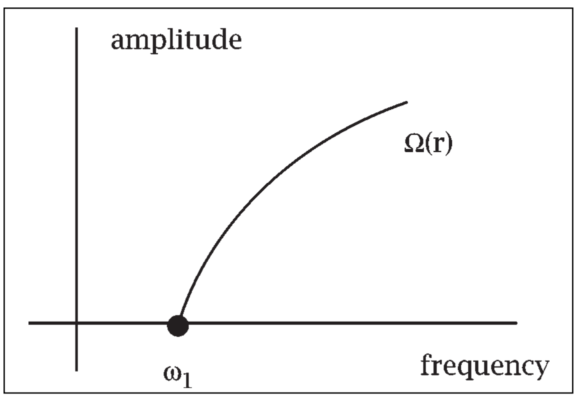

Figure 1. The bifurcation diagram for a periodic orbit of a system of ordinary differential equations.

This result is illustrated by Figure 1, which plots the frequency Ω of the periodic solution as a function of its amplitude r. Note that as the amplitude approaches zero, the frequency tends toward the frequency of the linear problem. The bifurcation curve may bend either to the right (as shown) or to the left, depending on the details of the nonlinear terms in the equations, but except in rare cases it will have some nonzero curvature, and hence we will obtain a family of periodic solutions of varying frequency whose frequencies fill an interval.

Despite the existence of results like these for systems of ordinary differential equations, rigorous proofs of the existence of periodic solutions in nonlinear partial differential equations were known only for certain special equations until the work of Paul Rabinowitz [20]. Rabinowitz considered problems of the form:

$$(13) \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(u) \, , 0 < x < \pi \ , \quad u(0, t) = u(\pi, t) = 0.$$

He proved that under certain (fairly weak) conditions on the nonlinear term $f(u)$, (13) has periodic solutions $u(x, t) = u(x, t + T)$ for any period T which is a rational multiple of π, the length of the x-interval. Note that in light of our discussion of periodic solutions for ordinary differential equations this restriction to solutions of rational period is quite unusual. In the ordinary differential equations case we found a whole interval of allowed periods, including both rational and irrational T.

Rabinowitz proved the existence of periodic solutions by constructing a functional on the space of functions which are periodic in time with period T and which satisfy Dirichlet boundary conditions in space. The critical points of the functional are periodic solutions of (13); and although the functional is not well behaved, being in particular unbounded from both above and below, Rabinowitz succeeded in proving the existence of critical points. The restriction to rational periods arises at an intermediate point in the argument, where it is necessary to invert the d'Alembertian operator $\Box = \partial_t^2 - \partial_x^2$ on the space of functions periodic in time with period T and satisfying Dirichlet boundary conditions at $x = 0$ and $x = \pi$. It is easy to compute the spectrum of $\Box$ acting on such functions; and one finds that if T is a rational multiple of π, then the eigenvalues are either zero or else bounded away from zero by some fixed distance, and so $\Box^{-1}$ is bounded on the orthogonal complement of the null space of $\Box$. (In fact, it is compact on appropriate Sobolev spaces.) On the other hand, if T is a typical irrational multiple of π, then the eigenvalues of $\Box$ will approach arbitrarily close to zero,[1] so that $\Box^{-1}$ is unbounded and Rabinowitz's method no longer applies.

Rabinowitz's work inspired a great deal of additional research into the existence of periodic solutions for nonlinear partial differential equations, much of which is reviewed by H. Brezis in [4]. These investigations showed that while the hypotheses Rabinowitz made about the nonlinear term $f(u)$ could be weakened, the restriction to rational period seemed to be intrinsic to the variational method of constructing solutions. On the other hand, the construction of periodic solutions of (3) and (4) that I described above led to solutions of both rational and irrational period, and I want

[1]*For certain irrational periods T the eigenvalues of $\Box$ will again be bounded away from zero. The existence of periodic solutions for this set of frequencies is discussed (albeit with methods different from those of Rabinowitz) in [18]. However, there are "few" irrational periods with this property (they form a set of measure zero).*

next to describe how that Lyapunov-Schmidt approach allows one to circumvent this restriction that arises in the variational approach.

Consider a linear wave equation to which we add a term $g(x,u)$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + g(x,u)\ , 0 < x < \pi. \tag{14}$$

Expand $g(x,u)$ in a Taylor series to obtain $g(x,u) = g(x,0) + g_u(x,0)u + \mathcal{O}(u^2)$. Assume that $g(x,0) = 0$ so that $u \equiv 0$ is a solution of (14). Renaming $g_u(x,0) = \nu(x)$ and assuming for simplicity that the $\mathcal{O}(u^2)$ terms in the expansion are simply u^3, we are led to study:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \nu u + u^3\ , 0 < x < \pi. \tag{15}$$

The methods below generalize to a much wider class of equations (see [10, 2]) and also allow one to choose quite general boundary conditions at $x = 0$ and $x = \pi$, but for simplicity I will assume that $u(0,t) = u(\pi,t) = 0$, as in Rabinowitz's case. If $\{\phi_j(x)\}_{j=1}^{\infty}$ and $\{\omega_j^2\}_{j=1}^{\infty}$ are the eigenfunctions and eigenvalues of the Sturm-Liouville operator $L = -\frac{d^2}{dx^2} - \nu(x)$, one can find (infinitely) many periodic solutions to the linearized approximation to (15) of the form $(A\cos(\omega_j t) + B\sin(\omega_j t))\phi_j(x)$. My goal is to seek solutions of the full nonlinear equations which are "close" to these simple solutions. Assume, as we did before for the ordinary differential equation (3)–(4), that there exists a periodic solution of (15) with frequency Ω, and write it as

$$u(x,t) = \sum_{j=1}^{\infty}\sum_{n\in\mathbf{Z}} \hat{u}(n,j)e^{in\Omega t}\phi_j(x). \tag{16}$$

Inserting (16) into (15), one finds that the expansion coefficients $\{\hat{u}(n,j)\}$ must satisfy the (infinite) system of equations

$$(\omega_j^2 - n^2\Omega^2)\hat{u}(n,j) = \hat{V}(\hat{u})(n,j)\ , n \in \mathbf{Z}\ , \quad j = 1,2,3,\ldots, \tag{17}$$

where $\hat{V}(\hat{u})$ is the function which results from inserting the expansion (16) for u into the nonlinear term u^3 in (15) and then expanding that expression in terms of the eigenfunctions of the Sturm-Liouville operator in x and the exponentials in t.

Now try to mimic the previous approach to construct a solution of (15) which is "close" to the solution $u^0(x,t) = \epsilon\sin(\omega_1 t)\phi_j(x)$ of the linear wave equation. Recall that we saw in the case of ordinary differential equations that as the amplitude of the periodic solution varies, so does its frequency. The parameter ϵ is inserted in u^0 to allow us to easily vary the amplitude. If the frequency Ω of the periodic solution (16) is close to ω_1, then the linear operator on the left-hand side of (17) will have an eigenvalue very close to zero when $j = 1$ and $n = \pm 1$ and exactly equal to zero if $\Omega = \omega_1$. Thus as above define projection operators P as the projection onto the two-dimensional space spanned by $(\hat{u}(\pm 1, 1))$, and let $Q = P^{\perp}$. Applying these projection operators to (17), one obtains two equations very similar to (10) and (11). The only significant difference arises in the "Q-equation", which becomes

$$(\omega_j^2 - n^2\Omega^2)Q\hat{u}(j,n) = Q\hat{V}(\hat{u})(j,n)\ , \quad n \in \mathbf{Z}\backslash\{\pm 1\}\ , j = 2,3,4,\ldots. \tag{18}$$

In contrast to the case of ordinary differential equations, in which the linear part of (11) was obviously invertible so long as we avoided the resonant situation, one cannot expect the linear part of (18) to have bounded inverse. Sturm-Liouville theory implies that $\omega_j^2 \approx j^2 + c$, for some constant c; so in order to invert the linear part of (18), one must deal with expressions like $1/(j^2 - \Omega^2 n^2 + c)$, and for "typical" choices of Ω^2 there will be a sequence of pairs of integers $\{(n_\ell, j_\ell)\}_{\ell=1}^{\infty}$ for which $j_\ell^2 - \Omega^2 n_\ell^2 + c \to 0$ as $\ell \to \infty$. Trying to follow the construction above, these "small denominators" will frustrate any naive attempt to bound the inverse of the linear part of this equation. Thus, in marked contrast to the situation with ordinary differential equations, one encounters small denominators already in the construction of periodic solutions for partial differential equations, whereas for systems of ordinary differential equations they appear only in the construction of quasi-periodic solutions. But the analogy to quasi-periodic solutions of ordinary differential equations also suggests a method of circumventing this difficulty. The Kolmogorov-Arnold-Moser (KAM) theory was developed precisely to overcome small denominator problems in celestial mechanics, and since equation (15) is a Hamiltonian system, the classical KAM approach can be modified to deal with this problem (see [8, 15, 16, 23]). The KAM method starts from the Hamilton-Jacobi approach to integrating the equations of classical mechanics; one looks for a change of coordinates that preserves the Hamiltonian form of the equations of motion but such that after this change of variables the resulting system of differential equations is integrable. With rare exceptions it is impossible to find an explicit form for such a transformation, and the KAM theory uses Newton's method to construct better and better approximations to this change of variables and then shows that at least some of the solutions of these transformed systems converge to yield quasi-periodic solutions of the original equation.

The approach I will describe here, like the KAM theory, is based on Newton's method. However, it seems to offer certain advantages in searching for solutions of these partial differential equations.[2]

Consider (18) again, and assume that we know an approximate solution $\hat{u}^0$. In the present case, $\hat{u}^0$ will be the Fourier coefficients of $u^0(x,t) = \epsilon \sin(\Omega t)\phi_1(x)$, the solution of the linear wave equation, which we hope will approximate the periodic solution of the full nonlinear equation (15). Rewriting (18) as

$$F(\hat{u})(n,j) = -Q\hat{V}(\hat{u})(n,j) + (\omega_j^2 - n^2\Omega^2)\hat{u}(n,j) = 0\,, \tag{19}$$

one can attempt to improve the approximate solution $\hat{u}^0$ by writing $\hat{u} = \hat{u}^0 + \hat{v}$, linearizing (19) about $\hat{u}^0$ and then solving for the (presumably small) correction $\hat{v}$. This leads to a formula for $\hat{v}$ of the form

$$\hat{v} = -(D_{\hat{u}^0}F)^{-1}F(\hat{u}^0). \tag{20}$$

Estimating the size of $F(\hat{u}^0)$ is not difficult, since $\hat{u}^0$ is an approximate solution of (19). As is usual with Newton's method, the difficulty lies in estimating the inverse of the linear operator $D_{\hat{u}^0}F$. This is a particular problem in the present instance, since it is this factor that contains the small denominators.

Surprisingly, the hint as to how one should control this inverse comes from quantum mechanics! To see why, look a little closer at the form of this operator. It acts on functions defined on the (n,j) lattice, and so its action can be described by its matrix elements, which are:

$$(D_{\hat{u}^0}F)(n,j;n'j') = -\,D_{\hat{u}^0}Q\hat{V}(n,j;n',j') + \delta_{n,n'}\delta_{j,j'}(\omega_j^2 - n^2\Omega^2). \tag{21}$$

Denote the diagonal piece by

$$\mathcal{V}(\Omega)(n,j) = (\omega_j^2 - n^2\Omega^2). \tag{22}$$

The small denominators arise from $\mathcal{V}(\Omega)$. The off-diagonal piece $D_{\hat{u}^0}Q\hat{V}$ is more problematic. At first sight it looks as if it has no structure at all. In order to better understand what happens, consider the special case in which $v(x)$ (in (15)) is zero. In that case, $\phi_j(x) = \sin(jx)$, and $\hat{u}^0 = \frac{\epsilon}{2}\delta_{j,0}(\delta_{n,1} - \delta_{n,-1})$. This allows one to compute the nonlinear term in (19) explicitly, and one finds that $D_{\hat{u}^0}Q\hat{V}(n,j;n',j')$:

- is $\mathcal{O}(\epsilon^2)$,
- vanishes if $|n-n'| + |j-j'| > 2$.

Another operator with properties similar to $D_{\hat{u}^0}Q\hat{V}$ is the finite difference Laplacian $\epsilon^2\Delta$, defined by

$$\epsilon^2(\Delta u)(n,j) = -4\epsilon^2 u(n,j) + \epsilon^2(u(n+1,j) + u(n-1,j) + u(n,j+1) + u(n,j-1))\,. \tag{23}$$

Note that the matrix elements of $\epsilon^2\Delta$ are $\mathcal{O}(\epsilon^2)$ and vanish if $|n-n'| + |j-j'| > 2$, just like those of $D_{\hat{u}^0}Q\hat{V}$. Therefore, as a model to try to understand the behavior of $D_{\hat{u}^0}F$ in (21), consider

$$H = -\epsilon^2\Delta + \mathcal{V}(\Omega). \tag{24}$$

Note that H is just the Hamiltonian operator of quantum mechanics and that mathematical physicists have developed a host of techniques to study its inverse. W. Faris [11] has surveyed a number of techniques and results related to the inverse of such operators, and one can adapt some of the methods he described there to the present problem. In particular, the techniques developed by Fröhlich and Spencer [12, 19], to invert operators like H are particularly relevant. Identify the points (n,j) in the two-dimensional lattice at which $\mathcal{V}(\Omega)(n,j)$ is particularly small as "singular sites". More precisely, define the singular sites as the sites (n,j) at which $|\mathcal{V}(\Omega)(n,j)| < 1/10$. Let $S = \{\text{set of all singular sites}\}$. For ϵ sufficiently small, we can invert H on the complement of the singular sites by a Neumann series, and we see that

$$(H|_{S^c})^{-1} = \sum_{n=0}^{\infty} \frac{1}{\mathcal{V}(\Omega)}(\epsilon^2\Delta\frac{1}{\mathcal{V}(\Omega)})^n. \tag{25}$$

The convergence of this series follows from the fact that on S^c, $|1/\mathcal{V}(\Omega)| \le 10$, so for ϵ sufficiently small, $\epsilon^2\Delta\frac{1}{\mathcal{V}(\Omega)}$ will have norm less than 1. A more careful analysis of the sum in (25) shows that not only does it converge but the matrix elements of $(H|_{S^c})^{-1}$ decay exponentially with separation, i.e.,

$$(H|_{S^c})^{-1}(n,j;n',j') \approx \mathcal{O}(\epsilon^{2(|n-n'|+|j-j'|)})\,. \tag{26}$$

In order to estimate the inverse of H on the entire lattice, Fröhlich and Spencer incorporate the singular sites inductively. Begin by defining a subset S_1 of S as those sites (n,j) at which $1/10 \ge |\mathcal{V}(\Omega)| \ge \epsilon$; these are the "not too singular sites", if you like. Writing

$$H|_{S^c\cup S_1} = H|_{S^c} \oplus H|_{S_1} \oplus \Gamma\,, \tag{27}$$

where Γ describes the matrix elements of H which connect sites in S^c to S_1, one can expand

[2] *In particular, the methods described below seem to be able to handle a greater variety of boundary conditions than the classical KAM techniques, and they permit extension to higher-dimensional spatial domains [1, 2, 9]. Recent work of J. Bourgain [3] indicates that these ideas may even offer advantages over the classical KAM methods in the construction of quasi-periodic solutions of systems of Hamiltonian* ordinary *differential equations.*

$$(28)\quad (H|_{S^c\cup S_1})^{-1} = (H|_{S^c} \oplus H|_{S_1})^{-1} + (H|_{S^c} \oplus H|_{S_1})^{-1} \times \Gamma(H|_{S^c} \oplus H|_{S_1})^{-1} + \dots .$$

The terms in this expansion are estimated with the help of the following remarks: $(H_{S^c})^{-1}$ has been estimated above; the operator $(H|_{S_1})^{-1}$ has norm bounded by $\mathcal{O}(\epsilon)$, since we know that $|1/\mathcal{V}(\Omega)| \leq 1/\epsilon$ at sites in S_1; and finally, since the only off-diagonal terms in H are those coming from $\epsilon^2\Delta$, we know that the norm of Γ is bounded by $C\epsilon^2$. These observations suffice to prove that $(H_{S^c\cup S_1})^{-1}$ is bounded and has matrix elements that decay exponentially, and one proceeds inductively to incorporate more and more singular sites into the region on which one can control H^{-1}.

A potential problem arises when one considers very singular sites, for example, sites S_n at which $\mathcal{V}(\Omega) \approx \mathcal{O}(\epsilon^n)$, with $n \geq 2$. In this case the factors of $(H_{S_n})^{-1}$ which appear in the analogue of (28) have norms that are $\mathcal{O}(\epsilon^{-n})$. In order to prove the convergence of the series in this case, Fröhlich and Spencer developed a clever alternating expansion in which a factor of $(H|_{S_n})^{-1}$ is preceded by a factor of $(H|_{S^c\cup S_1\cup \dots S_{n-1}})^{-1}$, evaluated at two widely separated lattice sites. The exponential decay of $(H|_{S^c\cup S_1\cup \dots S_{n-1}})^{-1}$ is then used to offset the large factors of $(H|_{S_n})^{-1}$, and the inductive procedure gives estimates of H^{-1} on larger and larger subsets of the lattice.

In order to be able to separate the factors of $(H|_{S_n})^{-1}$ by factors of $(H|_{S^c\cup S_1\cup \dots S_{n-1}})^{-1}$ evaluated at widely separated lattice sites, the different components of S_n must themselves be far apart. Therefore, to apply the Fröhlich and Spencer expansion to estimate $(D_{\hat{u}^0}F)^{-1}$, one must insure that the singular sites of $\mathcal{V}(\Omega)$ are widely separated. Since $V(\Omega)(n,j) = \omega_j^2 - n^2\Omega^2$ and since the asymptotics of eigenvalues of Sturm-Liouville operators imply that $\omega_j^2 \approx j^2 + c$, the singular sites (n, j) are those at which

$$(29)\qquad j^2 - n^2\Omega^2 + c \approx 0 \, .$$

For (29) to hold, one must have j approximately equal to $\pm\sqrt{n^2\Omega^2 + c} = \pm n\Omega + \mathrm{O}\,(1/n)$; this means that the singular sites must lie near the straight lines $j = \pm n\Omega$. If, in addition, Ω is poorly approximated by rational numbers,[3] one can show that the singular sites occur only at widely separated locations along these lines. This is more than sufficient information to apply the Fröhlich-Spencer method, and one finds that the operator $(D_{\hat{u}^0}F)^{-1}$ in (20) is bounded and decays exponentially away from the diagonal, giving one a good estimate of the corrections $\hat{v}$ that arise in Newton's method.

Before leaving this point let me remark that this method does not allow one to control the inverse of $(D_{\hat{u}^0}F)^{-1}$ for all choices of the frequency Ω, but only for *almost* all choices. This restriction arises because the singular sites of $\mathcal{V}(\Omega)$ may not be widely separated if Ω is well approximated by rational numbers, or $\mathcal{V}(\Omega)(n,j)$ may vanish for certain choices of Ω and (n, j). In either case the methods of Fröhlich and Spencer no longer apply. In particular, frequencies which are rationally related to some ω_j must be excluded.

The fact that this method results in periodic solutions whose frequencies are "poorly approximated" by rational numbers may seem counterintuitive at first sight. One might expect that periodic orbits would correspond to rational frequencies, or at least nearly rational frequencies. However, requiring Ω to be poorly approximated by rational numbers really implies that no multiple of Ω is too nearly commensurate with any of the other natural frequencies of the nonlinear wave equation (15), i.e., with any of the other $\omega_j \approx j$. Physically, the occurrence of such a resonance or near resonance can lead to an exchange of energy between the periodic solution we are trying to construct and other modes of the system, and this loss of energy can destroy the periodic motion we seek. Indeed, the Kirkwood gaps in the asteroid belt occur for exactly this reason. Resonances between the periods of the orbits of asteroids that would fill these gaps and the period of the orbit of Jupiter prevent periodic orbits from forming in these regions.

One can construct an inductive argument based on the ideas above and show that it converges to a solution of the Q-equation (18) as one iterates this procedure. Proceeding then as we did for ordinary differential equations, one defines $\hat{y} = P\hat{u}$ and $\hat{z} = Q\hat{u}$. The iterative argument just described allows one to construct a smooth function $\hat{z} = \hat{y}(\Omega, \hat{y})$ which for a set of frequencies Ω of large measure solves (18). Inserting this solution into the equation which results when one applies the projection operator P to (17), one again obtains a pair of equations for the two remaining coefficients $\hat{y}$, which one solves "by hand", and one finally obtains [9]:

Theorem 1. If the potential $v(x)$ satisfies a finite number of explicit conditions, then there is a smooth curve $\Omega = \Omega(r)$ such that for r in a Cantor set of positive measure, the nonlinear wave equation (15) has a periodic solution with frequency $\Omega(r)$ and amplitude r.

Remark 1. The conditions imposed on $v(x)$ are satisfied generically and can be checked for specific

[3] *By "poorly approximated" I mean that there exist positive constants c and τ such that $|j - n\Omega| \geq cn^{-\tau}$, for all $(n, j) \neq (0, 0)$. It is easy to show that almost every irrational number Ω satisfies such an estimate for some choice of c and τ.*

cases. For example, if $v(x) = m^2$, the Klein-Gordon equation, these conditions are satisfied for almost every choice of m (but not for $m = 0$).

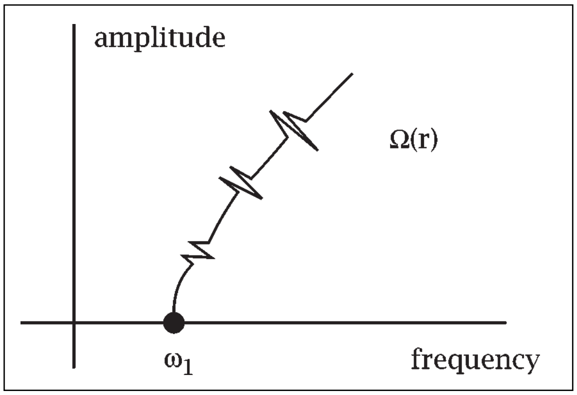

Figure 2. The bifurcation diagram for a periodic orbit of a nonlinear partial differential equation. The broken parts of the curve represent regions in which points on the graph of $\Omega(r)$ do not correspond to solutions of the nonlinerar wave equation (15).

This theorem is perhaps best illustrated by Figure 2, which highlights both the similarities and differences between the case of partial differential equations and the analogous results for ordinary differential equations illustrated in Figure 1. Just as in Figure 1, one has a smooth curve relating the amplitude of the solution to its period. However, in the present case not every point on the curve corresponds to a solution, but only those which lie in a Cantor set. The points that are excluded are those frequencies which were "too well" approximated by rational numbers in the process of inverting the linear operator to solve the "Q-equation" in the Lyapunov-Schmidt procedure. Note, however, that since the Cantor set of frequencies for which solutions are known to exist has positive measure, there must exist at least some periodic solutions with irrational period. Indeed, these methods are in some sense complementary to the variational techniques pioneered by Rabinowitz, since they fail for the rational periods for which the variational approach is well suited.

The applications of this method have been greatly extended recently, primarily through the work of J. Bourgain [1, 2]. Bourgain has shown in particular that one can use this approach to construct periodic solutions for equations on spatial domains of arbitrary dimension as well as quasi-periodic solutions for equations on one- and two-dimensional domains. There is still no proof of the existence of quasi-periodic solutions on three- (or higher) dimensional spatial domains due to difficulties associated with solving the analogue of the "Q-equation" (18). Note that a key observation used in solving that equation was that the sites (n, j) at which the quantity $\omega_j^2 - n^2\Omega^2$ was small were widely separated in the (n, j) lattice. Since $\omega_j^2 \approx j^2 + c$, this is essentially a question about the distribution of lattice sites at which the quadratic form $j^2 - n^2\Omega^2$ takes on small values, a question that is easy to answer in terms of how well Ω is approximated by rational numbers. In order to construct quasi-periodic solutions, one must analyze the distribution of lattice sites at which quadratic forms in larger and larger numbers of variables become small, and for the quadratic forms that arise in three or more dimensions this remains an unsolved problem.

References

[1] J. Bourgain, *Construction of quasiperiodic solutions for Hamiltonian perturbations of linear equations and application to nonlinear pde*, Internat. Math. Res. Notices **11** (1994), 475 ff.

[2] ____, *Construction of periodic solutions of nonlinear wave equations in higher dimension*, Geom. Funct. Anal. **5** (1995), 105–140.

[3] ____, *On Melnikov's persistency problem*, preprint, 1997.

[4] H. Brézis, *Periodic solutions of nonlinear vibrating strings and duality principles*, Bull. Amer. Math. Soc. (N.S.) **8** (1983), 409.

[5] S. N. Chow and J. Hale, *Methods of bifurcation theory*, Grundlehren der mathematischen Wissenschaften, vol. 251, Springer-Verlag, New York, 1982.

[6] P. Concus, *Standing capillary-gravity waves of finite amplitude*, J. Fluid Mech. **14** (1962), 568–576.

[7] ____, *Standing capillary-gravity waves of finite amplitude: Corrigendum*, J. Fluid Mech. **19** (1964), 264–266.

[8] W. Craig, *Kam theory in infinite dimensions*, Dynamical Systems and Probablistic Methods in Partial Differential Equations (David Levermore, Percy Deift, and C. Eugene Wayne, editors), Lectures in Appl. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1996, pp. 31–46.

[9] W. Craig and C. E. Wayne, *Newton's method and periodic solutions of nonlinear wave equations*, Comm. Pure and Appl. Math. **46** (1993), 1409–1498.

[10] ____, *Periodic solutions of nonlinear Schrödinger equations and the Nash-Moser method*, Hamiltonian Mechanics: Integrability and Chaotic Behavior (John Seimenis, editor), NATO Adv. Sci. Inst. Ser., Plenum Press, New York, 1994, pp. 103–122.

[11] W. G. Faris, *Random waves and localization*, Notices Amer. Math. Soc. **42** (1995), 848–853.

[12] J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Comm. Math. Phys. **88** (1983), 151–184.

[13] O. Vejvoda, with L. Herrmann et al., *Partial differential equations: Time periodic solutions*, Marttinus Nijhoff Publishers, The Hague, Boston, and London, 1982.

[14] J. B. KELLER and L. TING, *Periodic vibrations of systems governed by nonlinear partial differential equations*, Comm. Pure Appl. Math. (1966), 371–420.

[15] S. B. KUSKIN, *Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum*, Funct. Anal. Appl. **21** (1987), 192–205.

[16] ——, *Nearly integrable infinite-dimensional Hamiltonian systems*, Lecture Notes in Math., vol. 1556, Springer-Verlag, New York, 1993.

[17] A. M. LIAPUNOV, *Problème général de la stabilité du mouvement*, Ann. of Math. Stud., no. 17, Princeton Univ. Press, 1947.

[18] B. V. LIDSKIJ and E. I. SCHULMAN, *Periodic solutions of the equation* $\ddot{u} - u_{xx} + u^3 = 0$, Funct. Anal. Appl. **22** (1988), 332–333.

[19] J. PÖSCHEL, *On Fröhlich Spencer estimates of Green's function*, Manuscripta Math. **70** (1990), 27–37.

[20] P. RABINOWITZ, *Free vibrations for a semilinear wave equation*, Comm. Pure Appl. Math. **30** (1977), 31–68.

[21] I. TADJBAKHSH and J. B. KELLER, *Standing surface waves of finite amplitude*, J. Fluid Mech. **8** (1960), 442–451.

[22] V. A. VITT, *Distributed self-oscillating systems* (Russian), Zurn. Teoret. Fiz. **4** (1934), 144–157.

[23] C. E. WAYNE, *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Comm. Math. Phys. **127** (1990), 479–528.