

Doubling and Flatness: Geometry of Measures

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The goal of this article is to explore the relationship between the geometry of a domain Ω in Euclidean space and the properties of a canonical measure supported on $\partial\Omega$, which arises in potential theory. This measure is called the harmonic measure ω . In some instances the ω -measure of a set $E \subset \partial\Omega$ can be understood as the probability that the Brownian motion starting at a fixed point inside the domain will first hit the boundary at a point of E . Our project is twofold. Initially we study how the geometry of the boundary of a domain determines the regularity of the harmonic measure (see the section “Boundary Regularity” and [12]). Then we look at the converse problem, namely we establish how the regularity of the harmonic measure prescribes the geometry of the boundary (see the section “A Free Boundary Regularity Problem” and [13]).

The first problem is that of boundary regularity for the solution of a partial differential equation. This question has been extensively analyzed from two different points of view. One point of view is that the oscillation of the unit normal vector to the boundary controls the behavior of the “derivative” of harmonic measure (see [4, 10, 11, 18]). Almost all of these results are based on the assumption that the boundary of the domain is,

locally, the graph of a function which is required to be at least Lipschitz continuous. The other point of view is suitable in the more general setting where the boundary is only assumed to be non-tangentially accessible (i.e., points in the boundary can be reached by twisted cones from inside and outside the domain, which satisfies some additional connectivity hypothesis). In this case the harmonic measure is *doubling* (see [9] and also [5, 23] for related results). A measure is doubling if the measure of the ball of radius r controls the measure of the ball of radius $2r$. Our results partially reconcile the large disparity that exists between these two points of view.

The converse problem is a free boundary regularity problem. The regularity of the boundary of the set $\Omega = \{v > 0\}$, where v is a solution of the Laplace equation, is deduced from some information about the “regularity of the normal derivative” of v on $\partial\Omega$ (see “A Free Boundary Regularity Problem”). The main idea behind most results in this direction is that the oscillation (in a Hölder norm) of the “derivative” of harmonic measure controls the oscillation of the unit normal vector (also in the Hölder norm) (see [1, 2, 3]).

In the first two sections we introduce the notions of flatness and doubling. Our notion of flatness generalizes properties of C^1 hypersurfaces to much more general sets. Roughly speaking, a set is flat if it is well approximated by hyperplanes at every point and at every scale. In our work the doubling character of a measure and the flatness of a set replace conventional notions of regularity. In the context of the problems sketched above, the regularity of the harmonic measure is most often

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This article is based on a talk given at the Julia Robinson Celebration at MSRI, July 1996. The results discussed here are joint work with Carlos Kenig.

described in terms of its behavior with respect to the surface measure of the boundary of the domain. The notion of doubling enables us to talk about the regularity of the harmonic measure for domains whose boundary is so rough that the surface measure is not well defined. Similarly the regularity of the domain has almost always been described in terms of the oscillation of the unit normal vector to the boundary. Unfortunately this unit normal vector does not exist for many interesting domains (e.g., domains with fractal boundary or pseudo-spheres). The notion of flatness allows us to speak about the regularity of some of these domains.

For the sake of exposition we restrict our discussion to the case of unbounded domains. Thus Ω is always assumed to be an open connected unbounded domain in \mathbf{R}^{n+1} whose boundary separates \mathbf{R}^{n+1} . Namely, $\mathbf{R}^{n+1} \setminus \partial\Omega$ has exactly two non-empty connected components Ω and $\text{int } \Omega^c$. The canonical example to keep in mind is the upper half space \mathbf{R}_+^{n+1} . Similar results to the ones stated below hold for bounded domains satisfying appropriate separation and connectivity conditions.

Flatness

If $\Omega \subset \mathbf{R}^{n+1}$ is a smooth domain (i.e., $\partial\Omega$ is locally representable as the graph of a smooth function), its boundary is well approximated by n -dimensional affine spaces. For $Q \in \mathbf{R}^{n+1}$ and $r > 0$, $B(r, Q)$ denotes the $(n+1)$ -dimensional ball of radius r and center Q . The fact that $\partial\Omega$ can be well approximated by n -dimensional planes can be expressed as follows: for a point $Q \in \partial\Omega$ and a radius $r > 0$ sufficiently small there exists a number $\theta(r, Q) > 0$ such that the intersection of the boundary with the ball of radius r and center Q , $\partial\Omega \cap B(r, Q)$, is contained in a tubular neighborhood of width $r\theta(r, Q)$ about the tangent plane of $\partial\Omega$ at Q , $T_Q\partial\Omega$. Furthermore, the intersection of the tangent plane with the ball of radius r and center Q , $T_Q\partial\Omega \cap B(r, Q)$, is also included in a tubular neighborhood of width $r\theta(r, Q)$ about $\partial\Omega \cap B(r, Q)$. Since Ω is smooth, $\theta(r, Q)$ tends to 0 as r tends to 0. This suggests a notion of distance between the two sets. This distance is called the Hausdorff distance. We now formalize the concepts introduced above:

The Hausdorff distance D between two sets $A, B \subset \mathbf{R}^{n+1}$ is defined by:

$$D[A, B] = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}.$$

It follows that $D[A, B] \leq \delta$ if A is included in a δ -neighborhood of B and B is included in a δ -neighborhood of A .

For $Q \in \partial\Omega$ and $r > 0$ the deviation of $\partial\Omega$ from being an n -dimensional affine space at scale $r > 0$ is measured by the infimum of the scaled Haus-

dorff distances between the boundary and n -planes through Q , namely, by

$$\theta(r, Q) = \inf_L \left\{ \frac{1}{r} D[\partial\Omega \cap B(r, Q), L \cap B(r, Q)] \right\},$$

where the infimum is taken over all n -planes L containing Q .

Our work requires uniform control of several quantities on compact sets, thus for each compact set $K \subset \mathbf{R}^{n+1}$ we define

$$\theta_K(r) = \sup_{Q \in \partial\Omega \cap K} \theta(r, Q) \quad \text{and} \\ \theta(r) = \sup_{Q \in \partial\Omega} \theta(r, Q),$$

The quantity $\theta_K(r)$ provides a uniform measurement over K of how far $\partial\Omega$ is from being an affine plane at scale $r > 0$. It also gives an upper bound for the oscillation of the approximating affine spaces at scale r . In this sense it is a good replacement for the oscillation of the unit normal to the boundary, which measures the oscillation of the tangent planes.

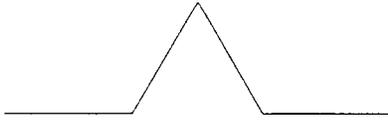
Definition 1.1. Let $\Omega \subset \mathbf{R}^{n+1}$. We say that Ω is a Reifenberg flat domain if there exists $\delta \in (0, 1/8)$ so that for each compact set $K \subset \mathbf{R}^{n+1}$ there exists $R > 0$ such that

$$(1) \quad \sup_{0 < r \leq R} \theta_K(r) \leq \delta, \quad \text{and} \quad \sup_{r > 0} \theta(r) \leq \frac{1}{8}.$$

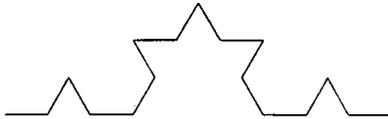
In the definition of a Reifenberg flat domain the parameter δ could have been chosen to be any positive number. On the other hand (1) only provides significant information for δ small. The choice of $1/8$ as an upper bound for δ is slightly arbitrary, but it is small enough to rule out some nasty examples (see Remark 2 below). The condition $\sup_{r > 0} \theta(r) \leq 1/8$ is not absolutely necessary but simplifies the exposition. It encodes the idea of flatness at infinity. In order to illustrate how varied the class of Reifenberg flat domains is we present two contrasting examples. Consider $\Omega = \{(x, t) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n, t > \varphi(x)\}$ where φ is a Lipschitz function with Lipschitz constant less than $1/8$. Ω is a Reifenberg flat domain. Moreover, since Lipschitz functions are differentiable almost everywhere, the unit normal vector to $\partial\Omega$ and the surface measure of $\partial\Omega$ are well defined notions.

Consider now the domain $\Omega \subset \mathbf{R}^2$ to one side or the other of the set $\{(x, t) \in \mathbf{R}^2 : t = 0, |x| \geq 1\} \cup S_\beta$, where S_β is a self-similar snowflake of angle β with $0 < \sin \beta < 1/8$. S_β is obtained as the limit of a sequence of curves built from the generating curve C_0 by an iteration of similitudes (i.e., contractions and rotations) analogous to the one used to construct the Koch

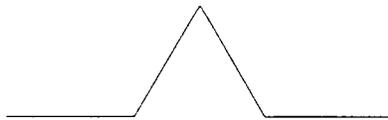
snowflake, $S_{\pi/3}$. Recall that the Koch snowflake is constructed from the following generating curve C_0



by iteration of the similitude that maps C_0 onto C_1 .



After 9 iterations we have:



S_β is a flat version of $S_{\pi/3}$, where the angle of the spike with respect to the horizontal is β instead of $\pi/3$. S_β is a fractal set, which admits a Hölder continuous parameterization that is nowhere differentiable. This implies that both the unit normal vector and the surface measure of $S_\beta \subset \partial\Omega$ are not well defined. On the other hand, it is not difficult to see that since $0 < \sin \beta < 1/8$, Ω is a Reifenberg flat domain. In fact, $\sup_{r>0} \theta(r) = \sin \beta$, where $\theta(r) = \sup_{Q \in \partial\Omega} \theta(r, Q)$.

Reifenberg flat domains should not be seen as exotic objects; they exhibit minimal geometric conditions necessary for some natural properties in analysis and potential theory to hold. As we shall see there is a correspondence between the flatness of a domain and the doubling properties of the harmonic measure. We now introduce Reifenberg vanishing domains which are the appropriate generalization of smooth domains in this context. Note that if $\Omega \subset \mathbf{R}^{n+1}$ is smooth, then the approximation of $\partial\Omega$ by affine spaces improves as r tends to 0. This translates into the following statement: for each compact set $K \subset \mathbf{R}^{n+1}$ we have

$$\lim_{r \rightarrow 0} \theta_K(r) = 0.$$

Definition 1.2. Let $\Omega \subset \mathbf{R}^{n+1}$. We say that Ω is a Reifenberg vanishing domain if Ω is a Reifenberg flat domain and if for each compact set $K \subset \mathbf{R}^{n+1}$

$$\lim_{r \rightarrow 0} \theta_K(r) = 0.$$

Summarizing, the fact that Ω is a Reifenberg flat domain guarantees that at small scales $\partial\Omega$ can be approximated by n -planes. This approximation is uniform on compact sets. The deviation of $\partial\Omega$ from being an n -dimensional affine space only depends on the parameter $\delta \in (0, 1/8)$. If Ω is a Reifenberg vanishing domain, the approximation improves as the scale diminishes. On the other hand, one should not be misled to believe that if Ω is a Reifenberg vanishing domain, then $\partial\Omega$ admits tangent planes. In fact, let $\varphi : \mathbf{R} \rightarrow \mathbf{R}$ be defined by

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k \sqrt{k}}.$$

The function φ can be shown to belong to λ_* , the little-o Zygmund class (see [27], page 47). This implies that φ is well approximated by affine functions whose graphs are affine spaces. Using this information, one can show without difficulty that $\Omega = \{(x, t) \in \mathbf{R}^2 : t > \varphi(x)\}$ is a Reifenberg vanishing domain. On the other hand, φ is a continuous function which is nowhere differentiable (in particular it is a variant of the Weierstrass function). Thus $\partial\Omega$ is not rectifiable (for a precise definition of rectifiability see [24]), which in particular implies that $\partial\Omega$ does not have a tangent line almost everywhere; i.e., there is not even a weak notion of the unit normal vector to $\partial\Omega$. Furthermore, the surface measure to $\partial\Omega$ is not well defined (see [26]).

A way to understand the pathologies present in Reifenberg vanishing domains is to think about them as domains which admit $C^{0,\alpha}$ parameterizations for every $\alpha \in (0, 1)$ but which might not admit $C^{0,1}$ parameterizations. As a matter of fact, Reifenberg's theorem guarantees that the boundary of a Reifenberg vanishing domain is locally representable as the image (not the graph!) via a homeomorphism of an open subset of \mathbf{R}^n (see [17, 20]). Moreover, the proof also shows that this homeomorphism yields a local Hölder parameterization of the boundary. On the other hand, the example above shows that the boundary of a Reifenberg vanishing domain might not contain any Lipschitz piece. Some refinements of the Reifenberg vanishing condition guarantee the existence of $C^{0,1}$ (i.e., bilipschitz) parameterizations for the corresponding class of domains (see [25, 26]).

Remark. Reifenberg introduced this notion of flatness in 1960 (see [20]). He was interested in the existence and regularity of solutions for the Plateau problem in higher dimensions. The result mentioned above enabled him to show that an n -dimensional minimal surface with prescribed boundary is a topological manifold except for a set of n -dimensional Hausdorff measure zero.

Doubling

A measure ω , supported in a subset Σ of \mathbf{R}^{n+1} , is doubling if the ω -measure of the ball of radius $2r$ and center $Q \in \Sigma$ can be controlled by the ω -measure of the ball of radius r and center Q for $r > 0$ small enough. If Ω is a smooth domain, the surface measure of its boundary is a doubling measure. On the other hand, since the surface measure of the boundary of a general Reifenberg flat domain is not well defined, we are compelled to use a different measure which makes sense in a more general context. We motivate the definition of this measure by looking at smooth domains. Let $\Omega \subset \mathbf{R}^{n+1}$ be a smooth (unbounded) domain which is flat at infinity in the sense that $\sup_{r>0} \theta(r) \leq 1/8$; then there exists a harmonic function v defined in Ω satisfying

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Delta = \sum_{i=1}^{n+1} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian. The function v is uniquely determined up to multiplication by a positive constant and is called the Green's function of Ω with pole at infinity (see [13]). In general the Green's function of a domain Ω with pole at $X_0 \in \Omega$ is a positive harmonic function in $\Omega \setminus \{X_0\}$ which vanishes on $\partial\Omega$. The function v can be constructed as the limit of scaled Green's functions whose poles converge to infinity. If $\phi \in C_c^\infty(\mathbf{R}^{n+1})$, integration by parts gives us Green's formula

$$\int_{\Omega} (v\Delta\phi - \phi\Delta v)dx = \int_{\partial\Omega} (v\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial v}{\partial\nu})d\sigma,$$

where σ denotes the surface measure and $\frac{\partial}{\partial\nu}$ denotes the normal derivative at the boundary (i.e., $\frac{\partial}{\partial\nu} = \nu \cdot \nabla$ where ν denotes the outward unit normal and ∇ denotes the gradient). Since v is harmonic in Ω and vanishes on the boundary, the integration by parts formula above becomes

$$(2) \quad \int_{\Omega} v\Delta\phi dx = - \int_{\partial\Omega} \phi \frac{\partial v}{\partial\nu} d\sigma.$$

By analogy with the conventional Poisson kernel, $-\frac{\partial v}{\partial\nu}$ is called the Poisson kernel of Ω with pole at infinity. The measure ω supported in $\partial\Omega$ and defined by

$$\omega(A) = \int_{\partial\Omega \cap A} h d\sigma, \quad \text{where } h = -\frac{\partial v}{\partial\nu},$$

for any Borel set $A \subset \mathbf{R}^{n+1}$, is called the harmonic measure of Ω with pole at infinity. Both the Poisson kernel and the harmonic measure are determined up to multiplication by a positive constant.

In particular, if $\Omega = \mathbf{R}_+^{n+1}$, $v(x_1, \dots, x_{n+1}) = x_{n+1}$, then the Poisson kernel of \mathbf{R}_+^{n+1} with pole at infinity is identically 1 and the harmonic measure of \mathbf{R}_+^{n+1} with pole at infinity is the Lebesgue measure of \mathbf{R}^n . Thus (2) becomes

$$\int_{\mathbf{R}_+^{n+1}} x_{n+1} \Delta\phi dx_1 \dots dx_{n+1} = \int_{\mathbf{R}^n} \phi dx_1 \dots dx_n.$$

The Hopf boundary lemma combined with classical boundary regularity results for the solution of the Laplace equation on a smooth domain guarantees that the harmonic measure with pole at infinity ω is *asymptotically optimal doubling*. This means that ω is a Radon measure; i.e., the ω -measure of compact sets is finite, and for each compact set $K \subset \mathbf{R}^{n+1}$ such that $K \cap \partial\Omega \neq \emptyset$ and for each $\tau > 0$,

$$(3) \quad \begin{aligned} \liminf_{r \rightarrow 0} \inf_{Q \in K \cap \partial\Omega} \frac{\omega(B(r\tau, Q))}{\omega(B(r, Q))} &= \tau^n \\ \limsup_{r \rightarrow 0} \sup_{Q \in K \cap \partial\Omega} \frac{\omega(B(r\tau, Q))}{\omega(B(r, Q))} &= \tau^n. \end{aligned}$$

On the one hand, (3) states that ω is a doubling measure. On the other hand, it claims that as $r \rightarrow 0$ the ratio $\frac{\omega(B(2r, Q))}{\omega(B(r, Q))}$ behaves more and more like the corresponding ratio for the Lebesgue measure (resp. Hausdorff measure) in n -dimensional Euclidean space (resp. n -dimensional smooth hypersurface). Nevertheless, (3) does not imply anything about the behavior of the ratio $\frac{\omega(B(r, Q))}{r^n}$ for $Q \in \partial\Omega$ as r tends to 0. In fact, if $\Omega = \{(x, t) \in \mathbf{R}^2 : t > \varphi(x)\}$ for $\varphi(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k \sqrt{k}}$, then ω is asymptotically optimally doubling, but for each compact set $K \subset \mathbf{R}^2$, $\sup_{K \cap \partial\Omega} \frac{\omega(B(r, Q))}{r}$ tends to infinity as r tends to 0.

In summary, a smooth domain is Reifenberg vanishing and its harmonic measure with pole at infinity is asymptotically optimal doubling. We now show that these notions of flatness and doubling are deeply intertwined and are independent of the smoothness assumption, as suggested by the example above. They provide some weak notions of regularity which suffice to answer several questions in analysis and geometric measure theory. Before stating any results we need to guarantee that it makes sense to talk about the harmonic measure with pole at infinity for a general Reifenberg flat domain. This is the content of the next proposition.

Proposition [13]. Let $\Omega \subset \mathbf{R}^{n+1}$ be a Reifenberg flat domain; then there exist a unique (up to a positive constant multiple) function v , continuous on the closure of Ω , and a unique (up to a positive constant multiple) doubling Radon measure ω such that

$$\int_{\Omega} v \Delta \phi dx = \int_{\partial \Omega} \phi d\omega \quad \forall \phi \in C_c^\infty(\mathbf{R}^{n+1})$$

where

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial \Omega. \end{cases}$$

Here ω denotes the harmonic measure of Ω with pole at infinity, and v denotes the Green's function of Ω with pole at infinity.

Boundary Regularity

In this section we discuss how the regularity of the boundary of $\Omega \subset \mathbf{R}^{n+1}$ (a domain as in the proposition above) determines the regularity of its harmonic measure ω . If at each scale $\partial \Omega$ has big Lipschitz pieces—more precisely, if $\partial \Omega$ is uniformly rectifiable (see [6] for a precise definition)—then the surface measure σ , given by the restriction of the n -dimensional Hausdorff measure to $\partial \Omega$, i.e. $\sigma = \mathcal{H}^n \llcorner \partial \Omega$, is a Radon measure. Under the appropriate geometric conditions for Ω , results in [4, 5], and [23] guarantee that ω is a doubling measure and that σ and ω are mutually absolutely continuous (i.e., a subset of $\partial \Omega$ has σ -measure zero if and only if it has ω -measure zero). In this case the Radon-Nikodym theorem insures that the Poisson kernel $h = \frac{d\omega}{d\sigma} = -\frac{\partial v}{\partial \nu}$ (the Radon-Nikodym derivative of ω with respect to σ) exists. The Poisson kernel is the density of ω with respect to σ . In this setting the regularity of ω is described in terms of the behavior of h . If $\partial \Omega$ is not uniformly rectifiable, we concentrate on the doubling properties of ω rather than on properties of the Poisson kernel which might not even exist. The proofs of all the results discussed in this section mainly use techniques from partial differential equations.

A classical boundary regularity result states that if the oscillation of the unit normal vector to the boundary is small in some Hölder norm, then so is the oscillation of the Poisson kernel and its inverse. Controlling the oscillation of h and of $1/h$ amounts to controlling the oscillation of $\log h$. More precisely, if Ω is a $C^{1,\alpha}$ domain for $\alpha \in (0, 1)$ (i.e., $\partial \Omega$ is locally representable as the graph of a function whose gradient is Hölder continuous with exponent α), then $\log h \in C^{0,\alpha}$. In particular h and $1/h$ are Hölder continuous (see [11]). Jerison and Kenig proved that if the oscillation of the unit normal vector to the boundary is small in the C^0 norm, then $\log h$ has small oscillation in an integral sense. Namely, the L^2 averages on balls of $\log h$ minus its average (i.e. the mean oscillation) converge to zero as the radius of the balls tends to zero. In this case $\log h$ is said to have vanishing mean oscillation, which we denote by $\log h \in VMO(d\sigma)$. Explicitly, if Ω is a C^1 domain (i.e., $\partial \Omega$ is locally representable as the graph of a

C^1 function), then $\log h \in VMO(d\sigma)$ (see [10]). This insures that $\log h$ can be well approximated by uniformly continuous functions in an integral sense (namely, in the mean oscillation norm). Nevertheless, it does not guarantee that $\log h$ is continuous. In fact, it is easy to construct examples of C^1 domains for which the logarithm of the Poisson kernel is not continuous.

Along these lines we prove that if the unit normal vector ν to the boundary of Ω has small integral oscillation, then so does the logarithm of the Poisson kernel. Our results concern domains where a suitable version of the divergence theorem holds, i.e., sets of locally finite perimeter (see [7]) whose boundary is Ahlfors regular. $\partial \Omega$ is Ahlfors regular if there exists a constant $C > 1$ such that, for $Q \in \partial \Omega$ and $r > 0$,

$$C^{-1} \leq \frac{\mathcal{H}^n(\partial \Omega \cap B(r, Q))}{r^n} = \frac{\sigma(B(r, Q))}{r^n} \leq C.$$

We show that if Ω is a set of locally finite perimeter whose boundary is Ahlfors regular and $\nu \in VMO(d\sigma)$, then $\log h \in VMO(d\sigma)$ (see [12]). A domain satisfying these hypotheses is called a chord arc domain with vanishing constant. In particular this result applies to a domain Ω whose boundary is locally representable as the graph of a function whose gradient is in VMO . Note that we establish the same result as Jerison and Kenig under much weaker assumptions. In some sense this shows that chord arc domains with vanishing constant provide a good generalization of C^1 domains from a potential theoretic point of view.

The first results of this type for nonsmooth domains were proved by Lavrentiev ($n = 1$; see [15]) and Dahlberg ($n \geq 2$). In particular Dahlberg showed that if Ω is a Lipschitz domain, then σ and ω are mutually absolutely continuous. In this case, while the Radon-Nikodym theorem guarantees only that $h \in L^1_{\text{loc}}(d\sigma)$, he proved that in fact $h \in L^2_{\text{loc}}(d\sigma)$ (see [4]). This result has played a central role in the study of boundary regularity for solutions of elliptic partial differential equations in nonsmooth domains.

We saw in the example above that the surface measure of the boundary of a Reifenberg vanishing domain need not be well defined. In this setting the regularity of the harmonic measure needs to be expressed in terms of its doubling properties, not in terms of the regularity of its density with respect to surface measure. The appropriate regularity statement is given by the following result.

Theorem 1 [12]. The harmonic measure of a Reifenberg vanishing domain is asymptotically optimal doubling.

This theorem shows that if the boundary of a domain is well approximated by affine spaces in

the Hausdorff distance sense, then, from a doubling point of view, its harmonic measure behaves like the harmonic measure of the half spaces bounded by these affine spaces. Theorem 2, discussed in the next section, establishes the converse of this statement. This provides a complete characterization of Reifenberg vanishing domains in terms of the doubling properties of the harmonic measure. The main ingredients in the proof of Theorem 1 are the maximum principle, the comparison principle for nontangentially accessible (NTA) domains (of which Reifenberg flat domains are an example), and the boundary regularity for nonnegative harmonic functions on NTA domains (see [9, 12]).

A Free Boundary Regularity Problem

In this section we discuss the converse problem. Namely, we explain that either the regularity of the Poisson kernel of a domain or the doubling properties of its harmonic measure determine the regularity of its boundary. This problem should be understood as a free boundary regularity problem in the sense that we are trying to deduce the regularity of the boundary of the set $\Omega = \{v > 0\}$, where v denotes the Green's function with pole at infinity from some information about the "regularity of the normal derivative" of v on $\partial\Omega$. In the case where the boundary of the domain is rectifiable we have a unit normal vector to $\partial\Omega$. Moreover, the normal derivative of v at the boundary is a well defined function h , the Poisson kernel with pole at infinity. In the case where the boundary is not rectifiable, we do not have a unit normal vector to the boundary and therefore the "normal derivative" should be understood not as a function but as a measure, the harmonic measure with pole at infinity. This interpretation is validated by the integral equality that appears in the proposition at the end of "Doubling". The proofs of these results require some tools from geometric measure theory that are developed in [14] and [13], and are inspired by techniques discussed in [16] and [19].

Roughly speaking, Alt and Caffarelli (see [1, 2, 3]) showed that on a domain which is sufficiently flat the behavior of the logarithm of the Poisson kernel determines the regularity and the geometry of the boundary. The key idea behind their proof is the following: in a domain whose boundary is a priori well approximated by affine spaces the uniform continuity of the logarithm of the Poisson kernel insures that as the scale decreases the approximation by affine spaces improves.

Theorem [1]. Assume that:

1. $\Omega \subset \mathbf{R}^{n+1}$ is a set of locally finite perimeter whose boundary is Ahlfors regular,
2. $\Omega \subset \mathbf{R}^{n+1}$ is a Reifenberg flat domain and (1) holds for some $\delta > 0$ small enough,
3. $\log h \in C^{0,\beta}$ for some $\beta \in (0, 1)$;

then Ω is a $C^{1,\alpha}$ domain for some $\alpha \in (0, 1)$ which depends on β . Moreover, if h is identically equal to 1, then Ω is a half-space.

Note that these results combined with those mentioned above reinforce the idea that the regularity of the boundary of a domain (described in terms of the oscillation of the unit normal vector) and the regularity of its harmonic measure (described in terms of the regularity of the logarithm of its density) are "equivalent". Along these lines we prove that this "equivalence" prevails even when the notions of smoothness involved are weaker than the ones above. We show that on a domain which is flat enough, the behavior of the logarithm of the Poisson kernel together with the doubling properties of the harmonic measure determine the regularity and the geometry of the boundary. We say that Ω is a chord arc domain with small constant if Ω is a set of locally finite perimeter whose boundary is Ahlfors regular for which the unit normal vector ν to the boundary has small mean oscillation (i.e., the L^2 averages of ν minus its average are small). In [13] (see also [8]) we prove that if Ω is a chord arc domain with small constant, ω is asymptotically optimal doubling, and $\log h \in VMO(d\sigma)$, then $\nu \in VMO(d\sigma)$. This result should be contrasted with the results discussed in "Boundary Regularity" concerning chord arc domains with vanishing constant. This yields the following theorem: Ω is a chord arc domain with vanishing constant if and only if Ω is a chord arc domain with small constant whose harmonic measure ω is asymptotically optimal doubling and whose Poisson kernel h satisfies $\log h \in VMO(d\sigma)$.

Note that all of these results convey the idea that the oscillation of the unit normal to the boundary of a domain and the oscillation of the logarithm of its Poisson kernel are "equivalent" quantities on flat domains. For chord arc domains in \mathbf{R}^2 this equivalence was explicitly proved by Pommerenke. He showed, using complex analytic methods, that the unit normal to the boundary belongs to $VMO(d\sigma)$ if and only if $\log h \in VMO(d\sigma)$ (see [18]). It is the subject of current investigation whether this result is also true in higher dimensions. More precisely, we would like to show that if $\Omega \subset \mathbf{R}^{n+1}$ is a chord arc domain with small constant and $\log h \in VMO(d\sigma)$, then ω is an asymptotically optimal doubling measure. This would allow us to apply the result stated above to conclude that Ω is a chord arc domain with vanishing constant, i.e., that $\nu \in VMO(d\sigma)$.

For general Reifenberg flat domains the notion of surface measure of the boundary is not well defined. Therefore the regularity of the free boundary in this case depends solely on the doubling properties of the harmonic measure.

Theorem 2 [13]. Let $\Omega \subset \mathbf{R}^{n+1}$ be a Reifenberg flat domain whose harmonic measure is asymptotically optimal doubling; then Ω is a Reifenberg vanishing domain.

Theorem 2 is a corollary of a more general result: *A Reifenberg flat set which supports an asymptotically optimal doubling measure is Reifenberg vanishing.* Theorems 1 and 2 provide a complete characterization of Reifenberg vanishing domains in terms of the doubling properties of their harmonic measure. The proof of Theorem 2 uses tools from [16, 19]. It relies heavily on the Kowalski-Preiss classification of n -uniform measures (see [14] and Remark 1 below) as well as on the notions of tangent and pseudo-tangent measure (see [13]).

Remarks

1. In Alt and Caffarelli's result as well as in Theorem 2 the assumption that Ω is a Reifenberg flat domain is crucial. An example by Kowalski and Preiss (see [14]) combined with a calculation of the harmonic measure carried out in [13] shows that in dimensions greater than 4 there exist unbounded domains whose Poisson kernel at infinity is identically 1, whose harmonic measure with pole at infinity is asymptotically optimal doubling, and which are very far from being Reifenberg vanishing. If $n \geq 3$, let

$$\Omega = \left\{ (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} : |x_4| < \sqrt{x_1^2 + x_2^2 + x_3^2} \right\}.$$

Ω is an unbounded nontangentially accessible domain whose harmonic measure ω with pole at infinity appropriately normalized satisfies

$$\omega = \sigma = \mathcal{H}^n \llcorner \partial\Omega \text{ which implies } h = \frac{d\omega}{d\sigma} = 1.$$

Moreover, ω is n -uniform; i.e., for $Q \in \partial\Omega$, $r > 0$, and $\tau > 0$

$$\begin{aligned} \omega(B(r, Q)) &= \mathcal{H}^n(\partial\Omega \cap B(r, Q)) = r^n \\ \text{which implies } \frac{\omega(B(r\tau, Q))}{\omega(B(r, Q))} &= \tau^n. \end{aligned}$$

On the other hand, it is easy to see that Ω is not Reifenberg vanishing. In fact, $0 \in \partial\Omega$ and for $r > 0$ the Hausdorff distance between $\partial\Omega \cap B(r, 0)$ and $L \cap B(r, 0)$ for any n -dimensional plane containing the origin is at least $r/\sqrt{2}$, i.e., $\theta(r, 0) \geq 1/\sqrt{2}$.

2. The previous characterization of Reifenberg vanishing domains is quite natural if looked at under the appropriate light. The Reifenberg flatness of a set and the doubling character of a measure extend some classical notions of regularity that require differentiability of the quantities in-

volved. In particular they are very well adapted to sets which are not rectifiable, whose Hausdorff dimension might be larger than their topological dimension, or which are not locally graphs. The notions of doubling and flatness presented here are powerful tools to study potential theory on continuous domains which lack differentiability.

Acknowledgments

I would like to thank Richard Pollack and Hugo Rossi for their comments on a preliminary version of this paper and Don Marshall who created the pictures that appear in "Flatness".

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