The Mathematics of F. J. Almgren Jr.

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Frederick Justin Almgren Jr., one of the world's leading geometric analysts and a pioneer in the geometric calculus of variations, began his graduate work at Brown in 1958. It was a very exciting place and time for geometric measure theory. Wendell Fleming had just arrived and begun his collaboration with Herbert Federer, leading to their seminal paper "Normal and Integral Currents" in 1960 [19]. Among the major results in "Normal and Integral Currents" was a compactness theorem which implied existence of k-dimensional rectifiable area minimizing varieties with prescribed boundaries in $\mathbf{R}^n$. (Similar existence theorems were proved independently by Reifenberg and, in case $n = k + 1$, by De Giorgi.) Shortly afterward, Fleming (using earlier work of Reifenberg) proved that if $k = 2$ and $n = 3$, then the varieties are in fact smooth surfaces. Meanwhile De Giorgi (and subsequently, by a different argument, Reifenberg) did work implying that the varieties were smooth almost everywhere when $n = k + 1$, the case of hypersurfaces.

Fred came to Brown with an unusual background. He had just spent three years as a Navy pilot, and before that his undergraduate degree at Princeton had been not in mathematics but in engineering. In fact, as an undergraduate he took only three math courses: two semesters of honors calculus and one semester of differential equations and infinite series. Later he would jokingly accuse various mathematicians at Brown of calling him "the most ignorant person" that they had ever met.

"It was clear that he had great raw talent and good intuition," says Federer, "but indeed he knew very little mathematics then. There were even basic things in group theory, for example, that he had never heard of. When he asked me to be his advisor, I suggested the problem I did because he didn't know enough analysis for most problems in geometric measure theory."

Federer suggested a problem that was as much topological as measure theoretic. Four years earlier Dold and Thom had shown that there was a natural isomorphism between the homology groups of a compact manifold M and the homotopy groups of their associated symmetric product spaces. Federer realized that this could be interpreted as a statement about the homotopy groups of the space of 0-dimensional integral cycles of M. He conjectured a generalization to k-dimensional cycles, namely, that the mth homotopy group of the space of integral k-cycles in M is naturally isomorphic to the $(m+k)$th homology group of M. Almgren's thesis, published in the first volume of the journal *Topology* [2], proved that this was indeed the case.

Oddly enough, its publication caused trouble. Almgren was supposed to sign the copyright of his thesis over to Brown University, but could not since he had already signed it over to *Topology*. To the dean that meant Almgren could not graduate.

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This article is excerpted from a more complete article, to be published in the Journal of Geometric Analysis.

Federer actually had to go before the graduate council to have the dean overruled.

After his Ph.D., Almgren took a position at Princeton University. There he remained for the rest of his life, even taking most of his leaves of absence at the Princeton Institute for Advanced Study, though he attended conferences and made summer visits to institutes throughout the world.

After Almgren finished his thesis Federer suggested that he develop a Morse theory for minimal varieties, analogous to the well-developed theory of closed geodesics. In particular, although the Federer-Fleming paper proved existence of area-minimizing varieties (integral currents) in homology classes of compact Riemannian manifolds, there were no theorems asserting the existence of unstable minimal varieties.

Almgren's *Topology* result was a step in that direction, but it was not sufficient. Although the integral currents introduced by Federer and Fleming had proved to be ideally suited to the problem of minimizing area, they were not so suitable for problems in which the solutions are unstable critical points of the area functional. This is, in part, because the area functional is not continuous, but merely lower-semicontinuous on the space of integral currents. To prove the existence of a minimum, lower-semicontinuity suffices. But if one wishes to prove existence of a critical point by a mountain pass lemma, having only lower semicontinuity poses difficulties.

To handle these difficulties, Almgren turned to the class of surfaces that he called integral varifolds.[1] The area functional is continuous (with respect to weak convergence) on the space of integral varifolds, and he was able to carry out a Morse theory with them. In particular, he proved that every compact m-manifold contains stationary integral varifolds of each dimension less than m. He also proved a compactness theorem for integral varifolds. (The Banach-Alaoglu theorem trivially implies a compactness theorem for all varifolds, but the compactness for integral varifolds is quite subtle.) He also proved a striking isoperimetric inequality for stationary integral varifolds. All this was done in Almgren's 1965 *Theory of Varifolds*, mimeographed lecture notes that amazingly were never published. The work, extended and streamlined by Allard, was—with Almgren's encouragement—eventually published together with Allard's celebrated regularity theorem for integral stationary varifolds in [1].

[1] *Essentially the same class of surfaces had been introduced in 1951 by L. C. Young under the name "generalized surfaces" [30]. That term is perhaps confusing since there are many other generalizations of the familiar notion of surface: integral currents, integral flat chains, normal currents, real flat chains, flat chains mod p, and so on. Indeed, Young himself had first used the term "generalized surface" in a somewhat different context [29].*

So far the tremendous advances that had been made in the higher-dimensional calculus of variations had been limited to the area functional. In his 1968 paper [4] Almgren introduced the class of parametric elliptic functionals and (extending the techniques pioneered by De Giorgi) proved the fundamental regularity theorem: if a minimizing surface is weakly close to a multiplicity one disk, then it is smooth near the center of the disk. This implies that minimizing surfaces are smooth almost everywhere (if multiplicity is not counted) or on an open dense set (if multiplicity is counted).

Robert Hardt proved the analog of Almgren's theorem for boundary points [21]. Many years later, Bombieri [12] and Schoen and Simon [24] gave different proofs of Almgren's theorem, replacing some of Almgren's barehanded geometric constructions by more standard PDE techniques. However, the later proofs are limited to oriented surfaces, whereas Almgren's is not. Almgren's regularity theorem is used in almost all subsequent work on parametric elliptic functionals.

Soap-Film-Like Surfaces

Almgren's next major work on regularity was his 1975 monograph, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints* [5]. There were two major new features.

The first was that it developed a regularity theory for sets rather than currents. The theory of integral currents, as well as the classical theory of Douglas and Rado, can be regarded as providing solutions to Plateau's problem. But neither theory very accurately models soap films and soap bubbles, which were what Plateau actually studied. For example, the classical Douglas-Rado solutions are sometimes immersed, which real soap films never are. Conversely, soap films display types of singularities that are present neither in the Douglas-Rado minimal surfaces nor in mass-minimizing integral currents. (Reifenberg's theory modeled soap films rather well, but Almgren wanted to handle soap bubbles as well as films and arbitrary elliptic functionals, not just the area functional.) The difficulty with currents, as Almgren saw it, was that they had structure that the physical surfaces did not. "In this paper," he wrote, "we do not wish to assume the existence of a boundary operator…because in many of the geometric, physical, and biological phenomena to which the results and methods [of this paper] are applicable there seems to be no natural notion of such a boundary operator."

The lack of a boundary operator meant that comparison surfaces could no longer be constructed by cut-and-paste operations. Instead, all comparison surfaces had to be obtained by (not necessarily one-to-one) Lipschitz deformations of the original surface. This limitation meant that proofs were rather cumbersome compared to those in Almgren's earlier work on elliptic regularity.

The second major new feature of the paper is the extremely useful idea of surfaces that "almost" minimize a functional. Given an elliptic functional F, in addition to the basic problem of finding a surface that minimizes F, there are many closely related problems of interest: minimize F among surfaces enclosing a specified volume, or among surfaces in a Riemannian manifold (rather than in $\mathbf{R}^N$), or among surfaces that avoid a specified smooth obstacle, etc. Almgren realized that a solution surface S to any one of these problems must be what he called "(F, ϵ, δ)-minimizing". Roughly speaking, the portion of S in a small ball has F-energy close to the smallest possible F-energy (ignoring volume constraints, obstacles, etc.); the smaller the ball, the closer the F-energy to the minimum possible.

Consider, for example, a circle and the length functional: a small arc of the circle comes close to minimizing length in that the ratio of its arc-length to the chord is only slightly greater than one, and indeed the ratio tends to one as the length tends to 0.

It turns out that the fundamental regularity theorem (a minimizing surface weakly near a multiplicity one disk is regular near the center of the disk) for minimizing surfaces is also true and only slightly more difficult to prove for (F, ϵ, δ)-minimizing surfaces. (Here regular means $C^{1,\alpha}$ for a suitable $\alpha > 0$.) Thus Almgren simultaneously handled the various problems mentioned above.

A special case of the results in Almgren's monograph is the existence and almost-everywhere regularity of surfaces in $\mathbf{R}^3$ that accurately model physical soap films and soap bubble clusters. Jean Taylor wrote a thesis [25] leading to her celebrated 1976 theorem [26] that Almgren's soap-bubble-like surfaces do indeed have exactly the structure described by Plateau: they consist of smooth surfaces which meet in threes along smooth curves, which in turn meet in fours at isolated points. Almgren and Taylor described their work in a beautiful *Scientific American* article [9] on the physics and mathematics of soap films.

Surfaces of Higher Codimension

In the early 1970s very little was known about regularity of mass-minimizing surfaces of codimension greater than one. The singular set of such an m-dimensional surface was known to be closed and nowhere dense, but it was not known to have m-dimensional measure 0. Around 1974 Almgren started on what would become his most massive project, culminating ten years later in a three-volume, 1,700-page proof that the singular set not only has m-dimensional measure 0 but in fact has dimension at most $(m - 2)$ [6]. This dimension is optimal, since by an earlier result of Federer there are many examples in which the dimension of the singular set is exactly $(m - 2)$. Sheldon Chang, Almgren's eleventh Ph.D. student, proved in his 1986 thesis that for the case $m = 2$ the singular set is not only 0-dimensional (which some cantor sets are) but is locally finite away from the boundary [15].

The reader may wonder why Almgren's theorem is so much more complicated than all the regularity results that preceded it. Perhaps the most fundamental difference is the following. For the various surfaces (e.g., mass-minimizing hypersurfaces, size-minimizing surfaces, mass-minimizing surfaces mod 2) in which almost everywhere regularity was already known, one can determine whether a point is a regular point just by examining its tangent cone: the point is regular if and only if it has a plane as a tangent cone.

For mass-minimizing surfaces of higher codimension, this is no longer true. Suppose, for example, that a two-dimensional mass-minimizing surface has as a tanget cone (at a certain point) a plane with multiplicity 2. The point could be a regular point, namely, part of a smooth multiplicity 2 surface, or it could be a singular branch point (such as the origin in the surface $\{(z^2, z^3) : z \in \mathbf{C}\} \subset \mathbf{C}^2 \cong \mathbf{R}^4$).

Thus the problem is multiplicity. If a tangent cone is a plane with multiplicity one, the standard regularity theory applies. In codimension 1, multiplicity is not a problem: an integral current weakly close to a multiplicity k disk can be decomposed into layers (from top to bottom), each of which is close to the same disk with multiplicity 1. But in codimension greater than 1, such decomposition is impossible because there is no appropriate analog of "top" and "bottom".

To get an idea of how subtle Almgren's theorem is, imagine an m-dimensional mass-minimizing surface consisting of two distinct smooth pieces which intersect and are tangent to each other along a set S. Almgren's theorem asserts in this case that the set S has dimension $\leq (m - 2)$, and thus it can be regarded as a delicate unique continuation theorem. Now imagine that the two pieces are actually joined by infinitely many little handles near S. The surface can no longer be locally represented by graphs of functions solving a PDE, but the same conclusion about S has to be reached.

The enormity of Almgren's paper has deterred all but a few people from reading very much of it. Nevertheless, some ideas from it have begun to have an impact. For a description of some of those ideas, see [28]. See [16] for a short overview of the proof.

The Isoperimetric Inequality

Almgren discovered several beautiful theorems relating to the classical isoperimetric inequality. The classical inequality states that if S is a closed

k-dimensional surface in $\mathbf{R}^{k+1}$ and if M is the region bounded by S, then

$$\mathcal{H}^{k+1}(M) \leq c_k \mathcal{H}^k(S)^{1+\frac{1}{k}} \tag{1}$$

with the constant c_k chosen so that equality holds if S is a sphere. (Here $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure: length if $k = 1$, area if $k = 2$, and so on.)

In their 1960 paper "Normal and Integral Currents" [19] Federer and Fleming proved a beautiful generalization to k-dimensional surfaces in $\mathbf{R}^n$. Of course, if $n > k + 1$, there will be many $(k + 1)$-dimensional surfaces bounded by S. Federer and Fleming proved that for any S there exists a surface M bounded by S for which essentially the same inequality still holds, though their proof did not give the best constant c_k. Of course, one may as well take M to be the least area surface with boundary S. Thus their isoperimetric inequality can be stated as follows: if M is a $(k + 1)$-dimensional mass-minimizing integral current in $\mathbf{R}^n$ with boundary S, then

$$\text{mass}\,(M) \leq c_{k,n}(\,\text{mass}\,(S))^{1+\frac{1}{k}}. \tag{2}$$

Almgren's first discovery along these lines was his beautiful 1965 proof ([3], [1], §7) that (2) is true (possibly with a worse constant) not only for minimizing surfaces M but also for surfaces that are merely stationary for the area functional. (To say that M is stationary means that it satisfies the first-derivative test for minima: if we deform M, then the initial derivative of area is 0.) In particular, it applies to any smooth surface M whose mean curvature is 0 at every point.

The value of the best constants in these isoperimetric inequalities remained open for sixteen years after the Federer-Fleming paper was published. Then in 1986 Almgren published a beautiful proof [7] that the best constant for the Federer-Fleming inequality (2) is the same as for the classical isoperimetric inequality. Thus among all $(k + 1)$-dimensional mass-minimizing integral currents of a given mass, a ball in $\mathbf{R}^{k+1}$ of the appropriate radius has the smallest possible boundary mass.

One ingredient of Almgren's proof is the following fact (which is quite interesting in its own right, even for smooth surfaces): if S is any closed k-dimensional surface with mean curvature everywhere bounded above by k, then the area of S is greater than or equal to the area of the unit k-sphere $\partial\mathbf{B}^{k+1}$, with equality if and only if S is isometric to $\partial\mathbf{B}^{k+1}$.

Almgren's proof of this fact is typical of his geometric ingenuity and ability to look at problems from an unusual point of view. First he takes the convex hull C of S. Then he thickens it a little to get a convex body C' with a well-defined unit normal at every boundary point. Let R be the set of points x in $\partial C'$ such that the point in C nearest to x belongs to S. Of course, the image of $\partial C'$ under the Gauss map is the unit sphere $\partial\mathbf{B}^n$. Now Almgren proves that the image of $\partial C' \setminus R$ has area 0, and he bounds the area of the image of R (by a straightforward calculation of the jacobian determinants) in terms of the area of S and the given bound on mean curvature of S. Putting these together gives the inequality.

(See [27] for a more detailed expository account of Almgren's proof.)

Almgren's optimal isoperimetric inequalities paper applies only to minimizing surfaces. Whether his inequality for stationary (but not necessarily minimizing) surfaces is true with the same constant is still not known (though there are some results in this direction: [17, 18, 22, 23]).

Curvature Flows

In recent years Almgren was very interested in dynamic problems in which surfaces move through space with velocities related to their curvatures. His first major contribution to this field was not a paper but a graduate student: in 1975, Ken Brakke wrote a remarkable thesis on mean-curvature flow for varifolds [13]. However, the thesis was ahead of its time and had little impact for many years. After Richard Hamilton's stunning success with the Ricci curvature flow in 1982 [20], Gerhardt Huisken, Mike Gage, Matt Grayson, and Hamilton himself began obtaining striking theorems about mean curvature flow using methods of classical PDE and differential geometry rather than geometric measure theory. The field became a large and active one, as it continues to be. After a few years researchers in the field became aware of Brakke's thesis and were amazed by the wealth of insights it contained. Meanwhile, Brakke had also become interested in numerical work, and today he is even more widely known for his extremely powerful Surface Evolver software for simulating geometric flows [14].

Smooth surfaces evolving under curvature flows typically become singular at a finite time, after which it is not clear even how to define the flow with classical PDE and differential geometry. Thus, people have proposed various notions of weak or generalized solutions to geometric evolution equations.

These notions allow one to prove existence of solutions for all time. Typically it is relatively easy to prove that the generalized solutions agree with the classical ones as long as the surfaces are smooth. It is rather difficult to prove partial regularity results for solutions after singularities form. (Indeed, I believe that Brakke's regularity theorem is the only such result known. That is, the various partial regularity theorems all use Brakke's.)

Almgren and his collaborators Jean Taylor and Lihe Wang wanted a model that would apply to many kinds of geometric evolutions occurring in nature. Brakke's mean curvature flow models motion of grain interfaces in annealing metals, but mean-curvature flows in general are relevant only when the physical surface energies are isotropic. Almgren, Taylor, and Wang also wanted to model crystal growth, in which the surfaces energies are highly nonisotropic; indeed, for crystals the surface energy density typically depends in a nondifferentiable way on the unit normal.

They proposed such a model in [10]. The idea was to replace the parabolic problem by a series of minimization problems at discrete time steps. The flow is then the limit (as the time step size goes to 0) of the minimization problem. More precisely, at each time one chooses a surface that minimizes the sum of two quantities: the surface energy and a term that measures how far the surface is from where it was at the previous time step.

The main theorem of [10] asserts that a region bounded by the evolving surface is a Holder-continuous function of time. The authors also prove that for smooth initial surfaces and smooth elliptic surface energy density functions their flow agrees with the classical one until singularities appear.

In [11] Almgren and Wang incorporated temperature into the model so that one could, for example, model melting or freezing ice. Thus, in addition to a moving region (the "ice") there is a temperature function. Both in the region and in its complement, the temperature should satisfy a heat equation. Whereas mathematicians studying this problem routinely use the same heat equation in both regions, Almgren characteristically does not, since (as he points out elsewhere [8]) the heat capacity of water is twice that of ice. Furthermore, within the crystal the model allows heat to flow more easily in some directions than in others. Naively one might define the ice region to be the set of points at which the temperature is below freezing, but this neglects (among other things) the "Gibbs-Thomson effect", that is, the dependence of the local freezing temperature (along an interface) on curvature (of that interface). As Almgren explains, "Small crystals with high curvatures melt even though the temperature of the surrounding liquid is slightly below the freezing temperature given in handbooks." [8]

Almgren and Wang [11] define a kind of evolution that includes these many physical effects, they prove existence for all time, and they prove (among other results) that temperature depends Holder-continuously on time.

When illness struck in the summer of 1996, Almgren was busy with a number of projects. One of his main goals was to prove regularity theorems for the Almgren-Taylor and Almgren-Taylor-Wang curvature flows.

Other Works

Almgren wrote a great many other papers, including important work with Schoen and Simon on elliptic regularity, with Allard on uniqueness of tangent cones, with Lieb on liquid crystals and on spherical symmetrization of functions, and with Thurston on the "convex hull genus" of space curves. See [28] for descriptions of some of this work and for fuller accounts of some of the work described here.

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