Reports of AMS Association Resource Group

The AMSARG is a subcommittee of the AMS Committee on Education, chaired by Roger Howe and charged with representing the AMS to the National Council of Teachers of Mathematics (NCTM) in its revision of the NCTM Standards. Starting in 1989, the NCTM issued three sets of Standards for K–12 mathematics: one set on mathematics curricula, one on assessment, and one on the mathematics teaching profession. The revision will involve updating and refining the existing Standards and blending the three sets into a single document. More background about the work of the AMSARG may be found in the feature article by Roger Howe in this issue of the *Notices*. The questions that the AMSARG was to answer are listed in a letter from Mary M. Lindquist to Roger Howe that is reproduced in a box with his article. Briefly, these questions are: (1) Do the current statements of the Standards adequately communicate your view of the discipline? (2) Do the statements of the current curriculum Standards convey a sense of consistency and growth in content themes as the student moves across the grade levels? (3) Do the statements of the content Standards adequately reflect the mathematical understanding expected of a student graduating in the twenty-first century? (4) What suggestions could you make as to the most effective ways of blending the ideas of content, teaching, and assessment?

—*Roger Howe*

American Mathematical Society NCTM2000 Association Resource Group First Report January 1997

Introduction

We are very pleased to have been invited to participate in this consultative role in the revision of the NCTM Standards. We recognize that there are a variety of communities that have important stakes and interests in this process, and we share your conviction that the inclusion of as many of these communities as possible in each step of the process will result in a stronger document. We discussed a little among ourselves the difficulty of communicating with these various groups, since they have not only such different views of their own roles but also very fundamental questions such as the purposes of education and even the nature of mathematics. We recognize the challenges that face you and hope that we can be helpful.

The interest of the AMSARG in education is evidenced by the intense discussion that has been going on (via e-mail) for the past two months (one transcript runs to 140 printed pages). While a lot of this discussion has been simply the attempts of people who have not worked together and don't know each other personally to come closer, a number of points of agreement, and also of disagreement, have come up consistently. The following is an attempt to summarize these points.

In our discussions many specific observations about individual points in the curriculum Standards were made. We hope to organize these and transmit them as time allows, but in this initial report we discuss mainly what appeared to be the larger issues.

A. The Level of Ambiguity of the Standards

There was considerable agreement that the Standards need to be more explicit and less ambiguous. We considered some examples of that ambiguity that appeared from varying interpretations of the Standards by practitioners and others that arose from our own variations. We agreed that while vignettes may be helpful to many readers, it is important to make the Standards as clear and specific as reasonable and not to rely on the vignettes or other stories to *implicitly* develop standards. That said, vignettes are helpful to many and essential to some readers, especially in clarifying the meaning of terminology. Given this importance, vignettes should be written with great care and should carry serious mathematical content.

A.1. "Decreased Attention" and "Increased Attention" Tables

Several examples of ambiguity appear to have grown out of the "decreased attention" and "increased attention" tables in the Standards. The topics listed in the former have often been misinterpreted to mean almost no (or even absolutely no) attention. This often conflicts with not only the common-sense meaning of "decreased" but also the importance of related topics. An important example is the role of paper and pencil calculation. Another example concerns the apparent conflict between emphasizing the connections among various branches of mathematics—especially algebra and geometry, in our view—and the call for decreased emphasis on conic sections—which provide a beautiful illustration of a deep connection between these two topics. More of our discussion on these two topics is summarized below. There was considerable sentiment, though not consensus, for removing the "Increased-Decreased" tables from the Standards. If this is not done, effort should be made to avoid tendentiousness in them, and they should be supplemented with cautionary language emphasizing the need for judgment and balance in implementing proposed changes.

A.2. The Role of the Teacher

Other examples of ambiguity deal with pedagogy and teacher preparation. The Standards have led some teachers and leaders of teacher in-service development activities to call for a dramatic reduction to the point of almost total elimination of direct instruction. Most deep constructivists, in contrast to naive ones, appear to recognize that it is compatible with the view that individuals construct their own understanding to give the teacher a key role in the classroom, including directing the classroom dialogue, setting goals for student understanding, judging when that understanding has been achieved, and facilitating mathematical closure. We cannot tell whether the first editions of the Standards contain a deliberate overemphasis on reducing direct instruction in order to encourage more active student participation in class activities, but experience makes clear the need to be quite careful in documents like these to say what kinds of balance are needed and not to allow rhetorical excesses. Another example is the confusion—at least in our reading of the Standards—between calling for more mathematics in the preparation of teachers (which all of us enthusiastically support) and calling for changes in the nature of the mathematics taught to teachers (which some of us support to some extent, but about which others have deep concerns).

A.3. But How Specific?

While there was consensus that the Standards should be made more specific, there was enormous disagreement as to how thoroughly they should delineate the curriculum. This was the subject of the most heated discussion in the group.

On the one hand, a number of members argued persuasively for the advantages of a national curriculum. The examples of the Virginia state curriculum; the Japanese, Russian; and Dutch systems, and E. D. Hirsch's Core Knowledge program were adduced. The idea was brought forward that a national curriculum gives a basis for national discussion, so that the entire teaching profession can receive the benefit of research by small groups of teachers into specific classroom techniques. (The Japanese system was seen as the best example of this.) Also, a specific year-by-year curriculum would prevent misjudgment on the part of individual teachers or school districts about the importance of particular topics.

On the other hand, other members argued against a national curriculum. Examples were given where specific curriculum items led to a mechanical or rote mastery of the topics and encouraged assessment procedures which searched only for surface-level understanding of what mathematics looks like. The argument was made that a rigid national curriculum, dictated from above, would erode teacher and local autonomy, would not be accepted politically by school districts, and would place the classroom teacher in the position of a low-level "deliverer" of curriculum rather than a responsible member of an active profession.

We did agree that a call in the Standards for the development of detailed curricula at appropriate levels—whether school, district, state, or national—

would be very helpful, since such curricula could allow teachers to work together more effectively on common issues and could indicate clearly to all the expectations of the schools for student learning. One role of the new Standards could be to guide the development of such detailed curricula.

B. Algorithms

Two unrelated discussions took place with regard to this word. The first was about how arithmetic computation should be taught and particularly the role of the calculator in elementary school. The second was about the nature of algorithms in their more general mathematical context.

With respect to arithmetic computation, there was consensus that the use of calculators should support, but not supplant, other methods of computation, including paper-and-pencil algorithms. Other methods we contrasted with calculators included doing arithmetic mentally and using manipulatives to represent computations.

There was no consensus about how this might translate into classroom practice. Some members voiced concern about using the calculator at all in the early grades. Others pointed out that perhaps this decision should be left to teachers to work out.

The second discussion, about algorithms in general, was a bit more diffuse. The following summary seemed to fit everyone's ideas:

- Kids need to learn certain algorithms.
- They need to do this for three reasons:
  a. efficiency,
  b. mathematical understanding,
  c. the notion of algorithm itself.
- In an age of calculators and spreadsheets the notion of algorithm becomes even more important.

Conventional algorithms for basic arithmetic—addition, subtraction, multiplication, and division—were felt to be worth teaching for reasons (a), (b), (c) above and in particular for their preparatory value for the algebra of polynomials. Beyond that there was little consensus. There will be further comments on algorithms in the point-by-point comments to be submitted later.

There was a clear consensus that the use of algorithms, on whatever level, must be accompanied by understanding of how the algorithm works, not just what it accomplishes, and by discussion (wherever appropriate) of how algorithms can provide additional insight, not just specific answers.

These comments lead to the recommendation that the notion of algorithm in the new Standards be clarified and separated from the notion of arithmetic computation. Further, the status of algorithms and computation, especially in the "Increased Attention, Decreased Attention" lists, should be rethought.

C. Proof

The discussion on this topic was not nearly so thorough as the previous one, probably because of the limited time available. However, there seemed to be a consensus that the sudden appearance of "mathematical structure" in grades 9–12 should be rethought, expressed by one of us as follows: "…[I]t bothers me that there seems to be some very heavy line drawn at proof. I would hope, first of all, that if we really do a good job at developing mathematical reasoning skills in K–8, then proof would seem a natural next step in 9–12. In fact, I would put it almost as a litmus test of success in this area."

While the notion of logical deduction is not completely lacking in the description of K–8 education given in the Standards, the ARG discussion suggests that this strand could be made more prominent and more coherent. In particular, there is a need, once filled by the standard geometry class, for students to learn basic syllogistic logic, including notions such as converse, inverse, and contrapositive.

D. Connection between Algebra and Geometry

There was a consensus that the connections between geometry and algebra, the two salient aspects of precollege mathematics, should be made frequently and early. Existing statements about this contained in the Standards should be strengthened, and more ways to make this connection should be ferreted out. Examples are: the number line, modelling addition with lengths and multiplication with areas, statistical graphics, conic sections, matrices.

While "conic sections" appears under algebra in the column "Decreased Attention" for grades 9–12, the importance of conic sections as a way of showing the connection between algebra and geometry should be reexamined.

E. The Organization of the Standards Documents

The ARG did not discuss directly the idea of "blending" the three Standards documents; discussion centered on the curriculum and evaluation Standards. In regard to these there was a consensus that the first four of these Standards are qualitatively different from the others. This may have been clear also to the original authors, since they are repeated on each of the three instructional levels. There was agreement that perhaps a tighter organization of the document might be achieved if these Standards were separated from the Standards concerned with description of content and discussed under a different and more descriptive title with less differentiation into grade levels.

American Mathematical Society NCTM2000 Association Resource Group Second Report June 1997

Responses to questions posed in the letter of 1 April 1997 from Joan Ferrini-Mundy and Mary Lindquist.

Question 1.

(a) What is meant by "algorithmic thinking"? **(b)** How should the Standards address the nature of algorithms in their more general mathematical context? **(c)** How should the Standards address the matter of invented and standard algorithms for arithmetic computations? (d) What is it about the nature of algorithms that might be important for children to learn?

Response to Question 1.

(a) We do not know a useful reply to this question in the context of K–12 mathematics. "Algorithmic thinking" conjures up no ready images or category of ideas for us. We feel that in some sense the question is not productive. An important feature of algorithms is that they are automatic and so do not require thought once mastered. Thus learning algorithms frees up the brain to struggle with higher-level tasks. On the other hand, algorithms frequently embody significant ideas, and understanding of these ideas is a source of mathematical power. We feel it should be a goal that children should understand why and how the algorithms they use work. Our predilection is that this understanding be achieved as soon as possible—ideally, at the time of introduction of the algorithm. However, we recognize that in some cases operational mastery of an algorithm can support the conceptual understanding, which might be more difficult without such mastery. Thus sometimes it can be sound pedagogy to teach an automatic procedure first and discuss the reasons for its success later. However, we strongly support the principle that such conceptual understanding be a firm goal.

(b) We believe that the notion of an algorithm, as a guaranteed method to solve a problem, can be presented in the elementary grades. This would involve at least the following four aspects:

(1) Presentation of the idea of an algorithm as a procedure *guaranteed* to solve a type of problem, accomplish a class of computation, or some other desired goal. (Examples would not even have to be limited to mathematics; thus, in language, verb conjugation, case formation, plural formation, etc., are (sometimes strictly, sometimes less so) algorithmic.)

(2) Experience with some specific algorithms. We believe that these should include standard algorithms for the four basic operations of arithmetic. (By "standard" we do not mean to imply that there is a unique "standard" algorithm for each arithmetic operation; however, the possibilities for "standard" algorithms for arithmetical operations will necessarily be highly constrained.)

(3) The standard algorithms of arithmetic should be seen as examples in a much broader class of things called algorithms. The fact that computer programs, even the computer games the kids play, are embodiments of algorithms could be mentioned to illustrate what a many-splendored class algorithms form. It would probably be well to cover in detail other algorithms beyond those for the basic arithmetic operations to underscore the fact that "algorithm" does not simply mean "rule for doing arithmetic". The Euclidean algorithm for finding the GCD of two integers is directly relevant to ideas of elementary arithmetic, undergirds some important theoretical facts, and is well suited to calculator implementation.

(4) An algorithm is not the same as what it does. Thus the addition algorithm is to be distinguished from the idea of addition.

Algebra presents a natural context for considering algorithms at a higher level. Some essential ideas are:

(1) The fact that mathematical procedures can be algorithmized is a key to the usefulness of mathematics and the reason that it can be automated: algorithms are the source of the power of computers.

(2) That the guarantee of validity of algorithms is accomplished by *proof* and that this is a fundamental feature of mathematics. (However, the role of proof extends far beyond guaranteeing the correctness of algorithms.)

(3) There can be different algorithms to accomplish the same task, and one algorithm might be better in one context and worse in another. A good example of this could be the comparison of Cramer's Rule with elimination for the solution of linear systems of equations. Cramer's Rule gives an explicit formula for the solution of a system of linear equations in terms of determinants. Elimination does not provide a formula, but instead describes a procedure that will lead to the answer. One's initial predilection might be to prefer the formula; but, in fact, the formula for determinants involves so much computation and is so vulnerable to round-off error that Cramer's Rule is impractical for large systems, and determinants, when needed, are usually computed using elimination. Nevertheless, Cramer's Rule is important in some cases and for conceptual purposes. Such compar-

isons should be encouraged throughout the curriculum.

(4) There is a strong connection between algebraic formulas and algorithms. In particular, an algebraic expression is a sort of "loose algorithm": it is a recipe for producing some quantity from others by means of algebraic operations. Thus "$y = 3x + 2$" can be translated: "to get y from x, multiply x by three and then add two". However, an algebraic formula is not quite an algorithm, because algebraic notation has built into it ambiguities that are known not to matter to the final outcome. Thus, in computing a sum of terms an algorithm would specify which pair of terms to add first, which further term to add to the result, and so on; but an algebraic expression does not specify an order of addition, because the associative law for addition tells us that the order of addition does not affect the final outcome. This algorithmic viewpoint can be usefully applied to the understanding of identities, which are seen as a statement that two (rough) algorithms are equivalent, in the sense that they yield equal results. For example, the standard identity $a^2 - b^2 = (a + b)(a - b)$ says that the two procedures are:

(i) Take two numbers, square the first, square the second, subtract.

(ii) Take two numbers, add them, subtract the second from the first, multiply the two results. Both yield the same final result.

(5) Algorithms have a recursive structure, and this recursive structure is a source of power: once one problem has been solved, the solution can be applied to further problems. The quadratic formula provides an example here: taking the algorithmic point of view toward formulas, one sees that the quadratic formula gives a procedure for finding the roots of an arbitrary quadratic equation in one unknown. It does so by expressing the solution in terms of the standard arithmetic operations and the operation of taking a square root. Thus it presupposes the ability to perform these operations.

(6) The recursive nature of algorithms is analogous to the recursive nature of mathematics itself. This recursive structure is a prime feature of logical deduction and of axiomatic systems, in which you can use either the basic postulates, or previous theorems, in proving a new result.

Some Further Comments on Question 1 (b)

It probably would be valuable to revisit the algorithms of arithmetic from the higher perspective of algebra.

Geometric constructions are effectively algorithms. (We do not mean that devising a construction is algorithmic, but that a completed construction can be read as a set of instructions to do various basic operations to produce the desired geometric object.)

It is natural to discuss algorithms in relation to computers. Getting a computer reliably to do a multistep calculation, perhaps via a spreadsheet program, rather than formal programming would provide excellent hands-on training in algorithms. A simple programming environment like Logo offers experience in the recursive aspect of algorithms.

Related to machine computation, it is an interesting issue what algorithms calculators actually implement to compute the functions they offer. We do not suggest that this should be in the school curriculum, but it would be desirable for secondary teachers to know this so they could discuss it with their more advanced and interested students. The CORDIC algorithm uses very strongly the structure of elementary functions, especially the addition laws for trig functions, so it illustrates the power of such structural facts.

(c) We are aware of some suggestive studies (for example, by C. Kamii), as well as the practice of some foreign countries (e.g., Switzerland, Japan) which do well on TIMMS, that support the idea that extensive practice with mental computation helps develop strong number sense. Since the standard algorithms tend to be optimized for pencil-and-paper computation and not for mental computation, practice in mental arithmetic will probably lead to alternative algorithms. In particular, in practical problems involving addition or multiplication, estimation usually is a consideration, and for purposes of estimation the natural way to add is to combine like digits from left to right rather than from right to left, as in the standard pencil-and-paper algorithm, which is concerned instead with minimizing the amount of rewriting. We can believe that investigating and comparing the methods that arise may well help understanding of arithmetic. More generally, we find plausible the idea that devising personal ways to deal with arithmetic problems can promote number sense. On the other hand, we suspect it is impractical to ask all children personally to devise an accurate, efficient, and general method for dealing with addition of any numbers—even more so with the other operations. Therefore, we hope that experimental periods during which private algorithms may be developed would be brought to closure with the presentation of and practice with standard algorithms. Also, we hope care would be taken to ensure that time spent developing and testing private algorithms will not significantly slow overall progress. We believe that neither pure rote mastery of algorithms nor purely privately invented algorithms can optimize learning of arithmetic. Finding a good balance between the two is a delicate business and a matter for much practice and study. Guidance here (and elsewhere) might be found by examination of curricular materials from high-ranking TIMMS countries.

We note that to use invented algorithms in teaching, as opposed to their private use by students, will require teachers to be quite expert about the alternative algorithms which are possible. We suspect that the range of algorithms that will arise and that survive a test of reasonable generality will not be huge, and it could be a beneficial research activity to investigate and classify these and incorporate the results into teachers' manuals so that teachers could be prepared to discuss invented algorithms profitably as they arise. We understand that Japanese teachers' manuals frequently discuss the ramifications of a given topic and survey possible student responses. Such manuals would be most desirable in the U.S. We hope that children who invent algorithms could usually be brought to understand the relation between their method and the standard algorithm.

Regarding the algorithms for arithmetic, an important point to be made is that our way of writing numbers, e.g., decimal notation, is an algorithm, a very sophisticated and powerful algorithm. It produces very high information density and is marvelously adapted to computation. Furthermore, it is the result of a lengthy process of development and was not essentially complete until late in the first millennium (and not generally adopted in Europe until the sixteenth century). It incorporates in its very structure all the basic operations of rational algebra—counting, addition, multiplication, and exponentiation. Finally, it conditions the other algorithms we use—for example, an addition algorithm is not something that tells us how to add—which is a primitive intuition; an addition algorithm is something that tells us how to express the sum of two numbers, each expressed in standard decimal form, in standard decimal form also. It is probably not appropriate to tell all of this to children, but some propaganda to help them appreciate what a marvelous machine they are operating (whether or not they are using a calculator!) might be useful. Ethnomathematics might help supplement and reinforce the comparisons available through the traditional study of roman numerals, which were also a decimal system, but less systematic than our Hindu-Arabic one. Teachers should be deeply aware of the algorithmic qualities of our decimal notation and of the reasons for its power. In particular, they should keenly appreciate that our decimal notation is a highly unnatural creation, one which took about four of our five millennia of civilization to produce, and that its efficiency and apparent simplicity are the result of the sophistication of its construction. From a practical point of view, we suspect that a sufficiently deep appreciation of the beauty, power, and sophistication of the decimal system could help teachers bridge the gap between standard arithmetic algorithms and the ones invented by their students.

Standard algorithms may be viewed analogously to spelling: to some degree they constitute a convention, and it is not essential that students operate with them from day one or even in their private thinking; but eventually, as a matter of mutual communication and understanding, it is highly desirable that everyone (that is, nearly everyone—we recognize that there are always exceptional cases) learn a standard way of doing the four basic arithmetic operations. (The standard algorithms need not be absolutely unique, just as there are variant spellings between, say, the U.S. and England, but too much variation leads to difficulties.) We do not think it is wise for students to be left with untested private algorithms for arithmetic operations—such algorithms may only be valid for some subclass of problems. The virtue of standard algorithms—that they are *guaranteed* to work for *all* problems of the type they deal with—deserves emphasis.

We would like to emphasize that the standard algorithms of arithmetic are more than just "ways to get the answer"—that is, they have theoretical as well as practical significance. For one thing, all the algorithms of arithmetic are preparatory for algebra, since there are (again, not by accident, but by virtue of the construction of the decimal system) strong analogies between arithmetic of ordinary numbers and arithmetic of polynomials. The division algorithm is also significant for later understanding of real numbers. For all its virtues, decimal notation suffers a significant drawback over, say, standard notation for fractions: decimal numbers (meaning decimal fractions with finitely many terms) do not allow division. This can be remedied at the cost of using infinite decimal expansions, but this is a big leap, and the general infinite decimal is not rational. To understand that rational numbers correspond to repeating decimals essentially means understanding the structure of division of decimals as embodied in the division algorithm. We do not see that naive use of calculators can be of much help here: the length of repeat of a decimal will typically be comparable to the size of the denominator, so that 7/23 or 5/29 will not reveal any repeating behavior on standard calculators.

(d) The most important thing is that they exist; there are uniform ways of solving an entire class of problems.

It is important to understand that in learning an algorithm you are confronting the essence of the phenomenon with which the algorithm deals, since it is guaranteed to accomplish its aim. Also, that, once learned, the algorithm gives you automatic mastery over the topic.

Thirdly, there is the demystification of machines: your calculator and your computer perform algorithms—in fact, that is all they do. Limitations on the algorithms limit the processes they can handle.

Again, it is important to distinguish between an algorithm and what it accomplishes.

Question 2.

(a) What mathematical reasoning skills should be emphasized across the grades? **(b)** How should the Standards address mathematical proof? Why? **(c)** How should the Standards address topics within mathematical structure?

(a) The most important thing to emphasize about mathematical reasoning is that it exists—more, that it is the heart of the subject, that mathematics is a coherent subject, and that mathematical reasoning is what makes it so. This need not be taught in so many words, in fact, probably should not be. Mathematics should simply be taught as a subject where things make sense and where you can figure out why they are the way they are. There are significant exceptions, of course: there are axioms, such as the field axioms in algebra. They should be introduced as summaries of what we know from our experience—intuitively acceptable general rules—and sold as firm principles we can rely on when we are in less familiar territory. Not everything need be justified ab initio—sometimes an important property or fact may take some getting used to and is best introduced first and justified later. In such cases it should be pointed out that the item can be justified and will be later, preferably at some identifiable time. If the idea that you can and should figure (most) things out (maybe with help from classmates or teachers or from the text) can be inculcated in elementary school, then illustrations of how you use axioms, like the field axioms, to extend your range—to go from arithmetic of numbers to arithmetic of polynomials—can be introduced in junior high. Also, in algebra derivations of important formulas should be given. Derivation as proof—as the justification for formulas and statements of algebra—should be pointed out. Also, there should be practice with proof in the sense of "local proof": statements should be given, to be justified on the basis of logic, together with simpler facts which are agreed to be taken for granted. Throughout, mathematical reasoning in the form of translating from concrete problem contexts to mathematics, then back again after mathematical processing, should also be developed. The complexity of the mathematical processing and of the translation step should be gradually increased with student age.

(b) We feel that if mathematical reasoning is handled well in elementary and junior high school, then students should be ready to see (fairly) formal proof in high school. Issues of formal logic—syllogisms, negation and/or statements, converses, contrapositives, inverses, and the beginnings of quantification—should receive serious attention. Also, the necessity of using language carefully should be discussed—the need to specify hypotheses and conclusions, to be clear on the difference between "if" and "if and only if" statements, the need for careful definitions rather than simply "intuition". Also, the principle that reasoning has to start somewhere and that the starting point is defined by the axioms should be explained clearly. The value of stating obvious facts as a basis from which to arrive at nonobvious ones should be demonstrated in toy examples as well as in mathematically significant ones. (This fits well with the notion of "semilocal" proof, mentioned below.) The history of non-Euclidean geometry might be discussed as an illustration of how rigor can sharpen and reform intuition. (The 2,000 year debate which led to non-Euclidean geometry could never have started if Euclid had not given his axiomatic foundations of geometry.) The traditional (meaning for students of our generation) place to learn formal proof was in geometry. With greater appreciation both of the difficulties and cumbersomeness of a full axiomatization of Euclidean geometry as well as of the importance of other geometries, we are hesitant to recommend a full axiomatic treatment of Euclidean plane geometry. Nevertheless, we feel that the opportunities for reasoning in this subject are very rich and that it has great intuitive appeal for many mathematically talented students. Thus we feel that reasoning should play a large role in geometry courses, perhaps in a kind of "semilocal" proof, where a few assumptions are used to justify fair-sized collections of theorems, of which some of the more elementary and "intuitively obvious" of the statements are accepted without proof (while other, similar statements are proved to illustrate the issues involved), and the less obvious results are proved or assigned as exercises. Neither would all proofs have to be synthetic; the derivation of the equation of an ellipse or hyperbola from their metric definitions is a nice combination of geometry and algebra.

(c) We are not enthusiastic about formal treatment of algebraic systems in K–12. It would be enough, and very beneficial, for some groundwork for understanding formalization to be laid by a deeper investigation of more concrete topics. This would include a good working understanding of the field axioms and their usefulness in algebra. It could include an understanding of a concrete but nonstandard algebraic system, such as modular arithmetic, and a deeper understanding of the complex numbers, and polynomials. Similarly, transformational geometry, linked to both synthetic geometry and to matrix algebra, could provide a rich intuitive background for the abstractions of algebra. Integrating these ideas into the curriculum presents challenges enough.