When Is a Set of Lines in Space Convex?

Jacob E. Goodman

The Definition of a Convex Set

In $\mathbb{R}^d$, a set S of points is *convex* if the line segment joining any two points of S lies completely within S (Figure 1). The purpose of this article is to describe a recent extension of this concept of convexity to the Grassmannian and to discuss its connection with some other ideas in geometry. More specifically, the extension is to the so-called "affine Grassmannian" $G'_{k,d}$, the open manifold that parametrizes all the k-dimensional *flats* (translates of linear subspaces) in $\mathbb{R}^d$. In other words, rather than convex sets of points, we will be talking about convex sets of lines, for example, or of planes. Much of the material that this article deals with is based on joint work of the author's with Richard Pollack [7, 8], as well as with Rephael Wenger and others [3, 10].

Which properties of convex point sets would we expect to hold also for convex sets of k-flats?

The basic setup for point sets is this. To any set S of points is associated a set $\operatorname{conv} S$ containing S, called its convex hull (Figure 2), that satisfies:

1. monotonicity: $S \subset T \implies \operatorname{conv} S \subset \operatorname{conv} T$.
2. idempotence: $\operatorname{conv}(\operatorname{conv} S) = \operatorname{conv} S$.
3. antiexchange: $\operatorname{conv} S = S,\ x, y \notin S,\ x \neq y,\ y \in \operatorname{conv}(S \cup \{x\}) \implies x \notin \operatorname{conv}(S \cup \{y\})$ (Figure 3).

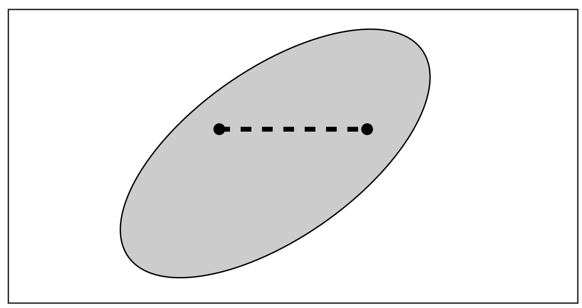

Figure 1. A convex point set.

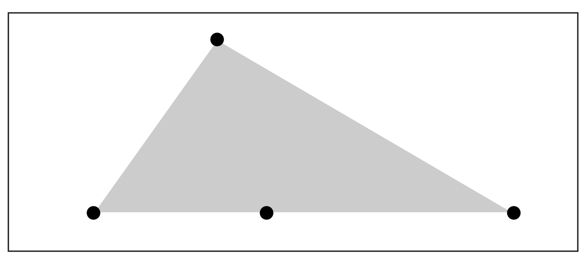

Figure 2. The convex hull of a set of four points.

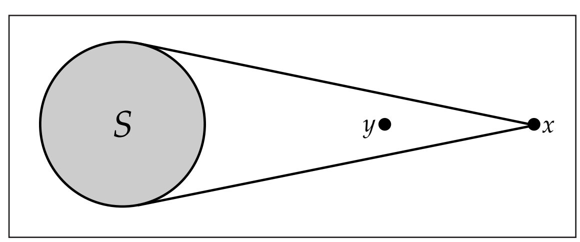

Figure 3. The antiexchange property.

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—J. E. Goodman

(Property 3 says essentially that conv induces a partial ordering on the complement of any convex set, which can be thought of intuitively as representing how "far away" points are from the set.)

4. nonsingular affine invariance: conv commutes with the action of the affine group. (If $\mathbf{A}(d, \mathbb{R})$ is the group of nonsingular affine transformations—nonsingular linear transformations plus translations—

$$\mathbf{A}(d, \mathbb{R}) = \left\{ \begin{bmatrix} L & \alpha \\ 0 & 1 \end{bmatrix} \,\middle|\, L \in \mathrm{GL}(d, \mathbb{R}),\ \alpha \in \mathbb{R}^d \right\},$$

and if $\sigma \in \mathbf{A}(d, \mathbb{R})$, then conv $(\sigma S) = \sigma(\text{conv } S)$ for $S \subset \mathbb{R}^d$; see Figure 4.) Then once we know what the convex hull of a set is, a set S is said to be *convex* if conv $S = S$.

The first three of these are usually taken to be the defining properties of what is known as an abstract "convex hull" operation, i.e., of a convexity structure, and the last condition means that this convexity structure is a natural one in the affine space $\mathbb{R}^d$.

But if we want to define a similar convexity structure on $G'_{k,d}$, for example on the set of lines in 3-space, we immediately run up against the following apparent obstacle. One of the characteristic features of a convex set of points is that it is connected. Yet we can show:

Theorem 1 [8]. There is no notion of convexity for lines or higher-dimensional flats that is nonsingular-affine-invariant, that satisfies the antiexchange property, and in which all convex sets are connected.

This is not difficult to prove. It means that we have to give up something. It turns out that if one is willing to give up the connectedness itself, to drop one's insistence that a convex set of flats should always be connected, then a rich theory emerges, one that extends many of the properties of convexity for point sets and that links up to a number of problems that have recently been studied under the heading of what is sometimes called "geometric transversal theory".

There are two ways to describe what the convex hull of a set of k-flats is, both of which extend characterizations of the convex hull of a point set. First let us go back to points for a moment.

Definition 1. Let us say that a point x in a flat F is *surrounded by* a set of points S in F if any *hyperplane* (flat of codimension 1) H in F passing through x lies strictly between two parallel hyperplanes H_1 and H_2 in F, each containing a point of S; in other words, any such hyperplane is "trapped" by points of S if it tries to escape by continuous translation to infinity (Figure 5).

Then it is easy to see that

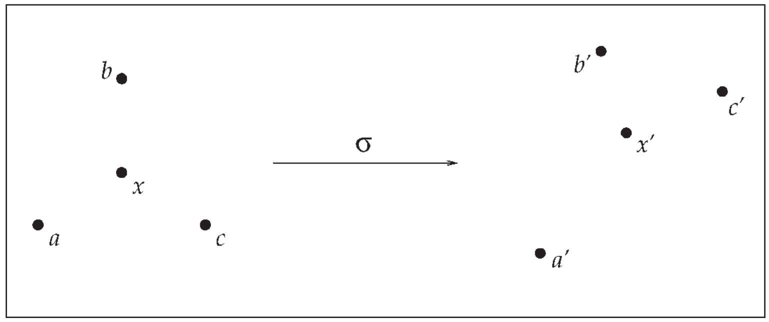

Figure 4. $x \in \text{conv}\{a, b, c\} \Longrightarrow x' \in \text{conv}\{a', b', c'\}$.

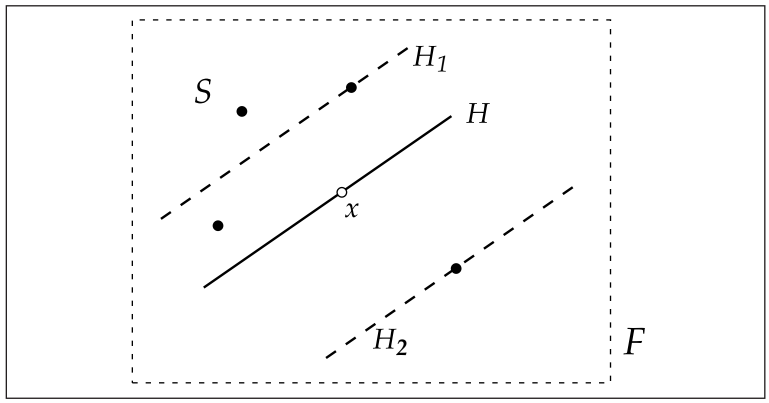

Figure 5. Point x is surrounded by set S.

A) $x \in$ conv S iff there is a flat F containing x within which x is surrounded by the points of S lying in F. (One cannot simply say "iff x is surrounded by the points of S", since S may be a lower-dimensional set.)

And of course we also have

B) $x \in$ conv S iff every convex point set meeting every point of S also meets x. (This is trivially true: it just amounts to saying that conv S is the intersection of all the convex point sets containing S.)

It turns out that "surrounded by" still makes perfectly good sense when the basic objects are flats of some fixed dimension k from 1 to $d - 1$ rather than simply points, and that (A) and (B) are still equivalent in that setting. And it also turns out that they imply the four basic properties of the convex hull operator in $\mathbb{R}^d$: monotonicity, idempotence, antiexchange, and invariance with respect to the affine group, and, happily, many other properties as well.

So here is our definition of the convex hull of a set of k-flats in d-space:

Definition 2. A k-flat l' belongs to the *convex hull* conv $\mathcal{L}$ of a set $\mathcal{L}$ of k-flats in $\mathbb{R}^d$ if it satisfies either of the following two conditions:

A) There is a flat F containing l' within which l' is surrounded by the flats of $\mathcal{L}$ lying in F;

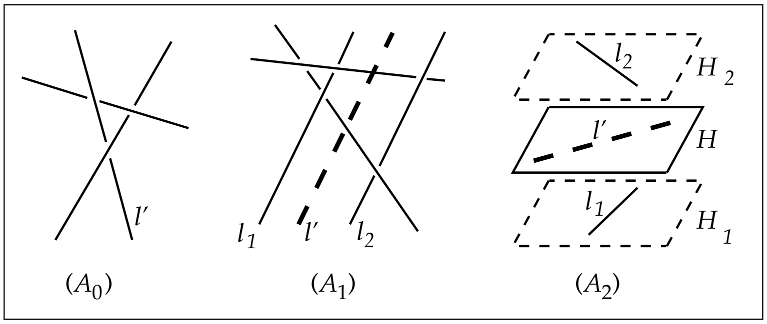

Figure 6. Line l' is surrounded by other lines.

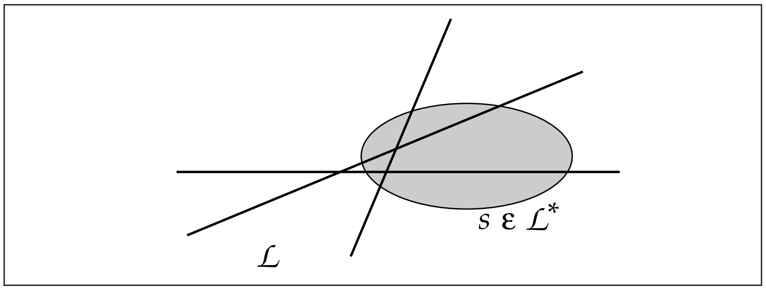

Figure 7. The dual of a set of lines.

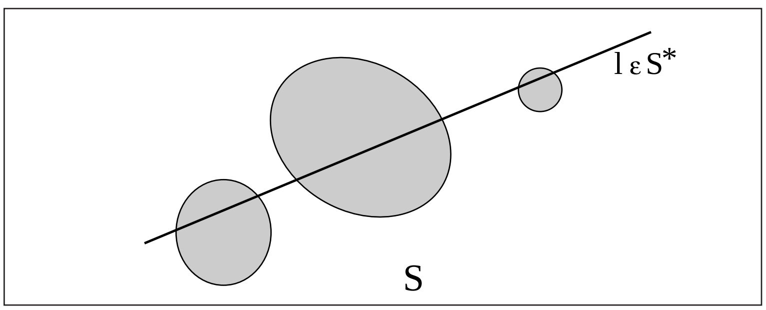

Figure 8. The dual of a set of convex point sets.

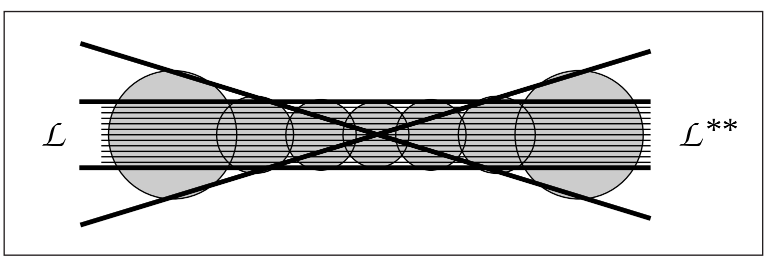

Figure 9. The convex hull of a set of lines.

B) Every convex point set meeting all the members of $\mathcal{L}$ also meets l'.

In (A) "surrounded by" has the same meaning as before: l' is *surrounded by* a set $\mathcal{L}$ of k-flats in a flat F if any hyperplane $H \subset F$ containing l' is sandwiched strictly between two other hyperplanes H_1 and H_2, each containing members of $\mathcal{L}$; in other words, every hyperplane $H \subset F$ containing l' is trapped by members of $\mathcal{L}$.

So, for example, just looking at the special case of lines in $\mathbb{R}^3$, we have:

If $\mathcal{L} \subset G'_{1,3}$, a line l' is in conv $\mathcal{L}$ if and only if (Figure 6)

A_0) $l' \in \mathcal{L}$, or

A_1) l' lies between two parallel lines $l_1, l_2 \in \mathcal{L}$, or

A_2) any plane Π passing through l' can be moved parallel to itself a positive distance in either direction until it contains lines of $\mathcal{L}$.

(A_2) describes the "generic" way in which a line is surrounded by other lines, while (A_0) and (A_1) describe the degenerate ways.

And then we have

Theorem 2 [8]. (A) and (B) in Definition 2 are equivalent.

Theorem 3. The operator conv on $G'_{k,d}$ satisfies

1. monotonicity: $\mathcal{L}_1 \subset \mathcal{L}_2 \implies \text{conv}\, \mathcal{L}_1 \subset \text{conv}\, \mathcal{L}_2$.

2. idempotence: $\text{conv}\,(\text{conv}\, \mathcal{L}) = \text{conv}\, \mathcal{L}$.

3. antiexchange: $\text{conv}\, \mathcal{L} = \mathcal{L},\ l_1, l_2 \notin \mathcal{L},\ l_1 \neq l_2,$ $l_2 \in \text{conv}\,(\mathcal{L} \cup \{l_1\}) \implies l_1 \notin \text{conv}\,(\mathcal{L} \cup \{l_2\})$.

4. nonsingular affine invariance: conv commutes with the action of the affine group. (If $\sigma \in \mathbf{A}(d, \mathbb{R})$ induces $\sigma_k : G'_{k,d} \to G'_{k,d}$, then $\text{conv}\,(\sigma_k \mathcal{L}) = \sigma_k(\text{conv}\, \mathcal{L})$ for $\mathcal{L} \subset G'_{k,d}$.)

There is a nice way to look at condition (B) in the definition. Given a set $\mathcal{L}$ of k-flats in $\mathbb{R}^d$, let us define its *dual*, $\mathcal{L}^*$, to be the set of all convex point sets S, each of which meets all the flats of $\mathcal{L}$ (Figure 7). And given a family $\mathcal{S}$ of convex point sets, define its k*-dual* $\mathcal{S}^*$ (actually $\mathcal{S}^{*_k}$, but simply $\mathcal{S}^*$ if we fix k) to be the set of all k-flats l meeting all the members of $\mathcal{S}$, the so-called k*-transversals* of the family $\mathcal{S}$ (Figure 8). Then condition (B) just amounts to saying that the convex hull of a set $\mathcal{L}$ of flats is its double dual $\mathcal{L}^{**}$. So if $k = 1$, say, start with a set $\mathcal{L}$ of four lines as in Figure 9, look at all the convex point sets meeting them ($\mathcal{L}^*$), and then look at *all* the lines meeting those convex sets ($\mathcal{L}^{**}$). They constitute the convex hull.

It is trivial to see that this concept of duality satisfies the usual conditions:

1. $\mathcal{L}_1 \subset \mathcal{L}_2 \implies \mathcal{L}_1^* \supset \mathcal{L}_2^*$; $\mathcal{S}_1 \subset \mathcal{S}_2 \implies \mathcal{S}_1^* \supset \mathcal{S}_2^*$;

2. $\mathcal{L}_1 \subset \mathcal{L}_1^{**}$; $\mathcal{S}_1 \subset \mathcal{S}_1^{**}$.

These imply immediately that $\mathcal{L}_1^{***} = \mathcal{L}_1^*$ and $\mathcal{S}_1^{***} = \mathcal{S}_1^*$, and it follows that a set $\mathcal{L}$ of flats is convex if and only if it is self-dual ($\mathcal{L}^{**} = \mathcal{L}$). Equivalently,

A set $\mathcal{L}$ of k-flats is convex iff $\mathcal{L}$ is the set of all common k-flat transversals of some family $\mathcal{S}$ of convex point sets.

If $\mathcal{S}$ can be taken to consist of a finite family, we say that $\mathcal{L}$ is *finitely presented*; if $\mathcal{S}$ consists of a *single* convex point set, we call $\mathcal{L}$ *principal*.

This, incidentally, is one of the main reasons why we are interested in convex sets of flats: because of their connection to what is known as "geometric transversal theory". Before discussing this connection, though, let us look at some examples of convex sets, say for lines in $\mathbb{R}^3$.

Some Examples of Convex Sets of Lines

Recall that there are two equivalent ways to think of a set of lines being convex: one is that every line surrounded by the set must belong to the set; the other is that it is the set of all common line transversals to some family of convex point sets.

1. First, an example of a convex set of lines that we find useful in guiding our intuition: Consider the 1-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ (Figure 10). On it there is a set of rulings: the line $\{x = 1, z = y\}$ and all of the lines obtained by rotating it about the z-axis, as well as a second such set generated by $\{x = 1, z = -y\}$. Consider the first of these sets; call it $\mathcal{L}$. What is $\operatorname{conv} \mathcal{L}$?

It turns out that it consists precisely of all the lines "inside" the hyperboloid, together with the members of $\mathcal{L}$ themselves.

It is not hard to see this, using the "surrounding" criterion. For example, the z-axis belongs to $\operatorname{conv} \mathcal{L}$ because any plane containing the z-axis, when translated (in either direction perpendicular to itself) a unit distance, passes through one of the rulings in our set. The same thing holds (with modification) for *any* line inside the hyperboloid. On the other hand, if l' is not inside the hyperboloid, it is not hard to find a plane through l' that can be translated *away* from the origin so as never to pass through one of our rulings.

2. Here is a second example. Let $\mathcal{L}$ be a (necessarily discontinuous) section of the tangent bundle to a unit sphere in $\mathbb{R}^3$; i.e., choose a tangent line at each point (Figure 11).

Then it is not hard to see that $l' \in \operatorname{conv} \mathcal{L}$ if and only if either l' meets the interior of the sphere or $l' \in \mathcal{L}$.

3. A third example consists of a set $\mathcal{L}$ of parallel lines, such as the three extreme lines in Figure 12. Take any plane Π cutting them, and take the convex hull $\overline{S}$ of their trace S in that plane. Then the convex hull of $\mathcal{L}$ consists precisely of the lines through $\overline{S}$ parallel to the original set $\mathcal{L}$.

4. What happens, though, if we take, say, a *finite* set of lines in $\mathbb{R}^3$, no two parallel, as in Figure 13? It is easy to see, from the "surrounding" criterion, that such a set is already convex!— there is no line outside the set that is surrounded by the set. So this is an example, perhaps an extreme one, of a convex set of lines that is disconnected, as predicted by Theorem 1.

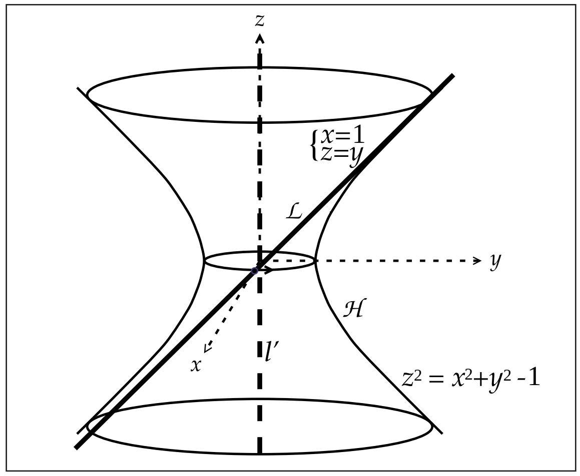

Figure 10. Rulings on a hyperboloid.

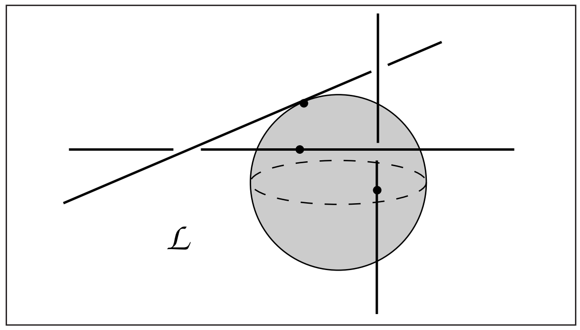

Figure 11. Lines tangent to a sphere.

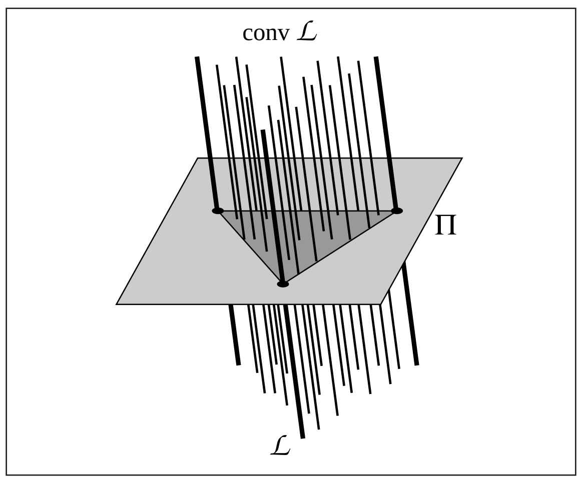

Figure 12. A set of parallel lines.

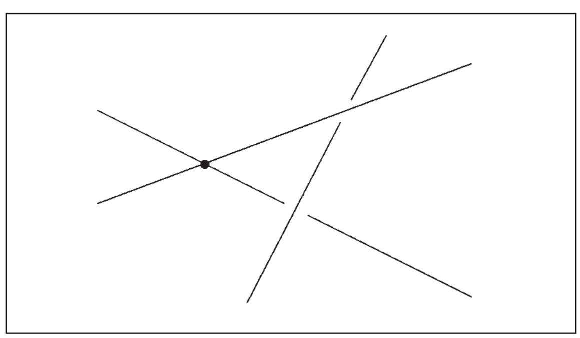

Figure 13. A finite set of lines without parallels.

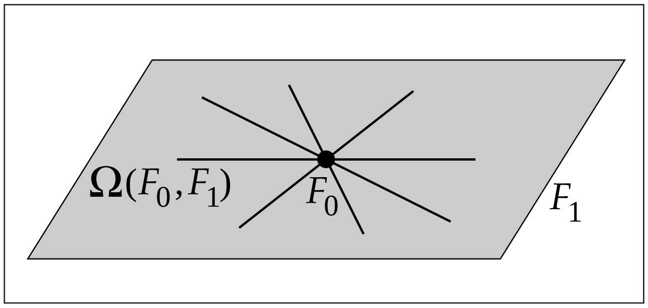

Figure 14. A partial flag of flats.

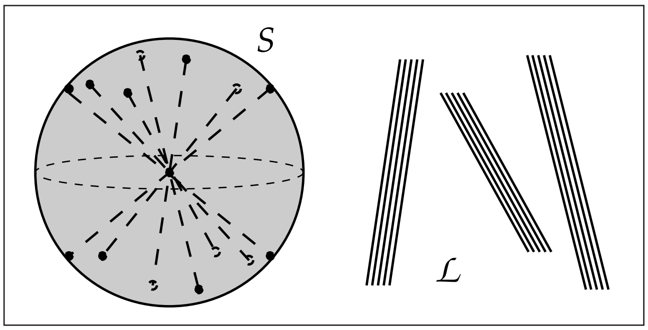

Figure 15. A parallel-closed set of lines.

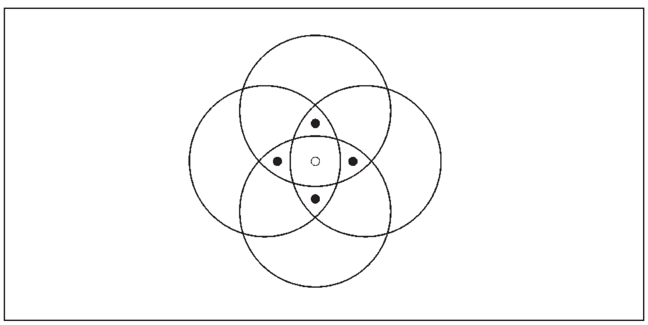

Figure 16. Helly's theorem in the plane.

Nothing like this happens, of course, with point sets, but for a very simple reason: any two points are "parallel" in the sense that each is a translate of the other. And we have seen that for a set of parallel lines the convex hull *is* connected. So this is a major difference between convexity for point sets and for sets of flats—the possibility that for flats a convex set may turn out to be disconnected or even discrete.

5. Certain Schubert varieties in an affine Grassmannian are convex. These are defined as follows: Let $\Phi : F_0 \subset F_1 \subset \cdots \subset F_k$, where the inclusions are strict, be a (partial) flag of flats in $\mathbb{R}^d$; an example is shown in Figure 14 for $d = 3$ and $k = 1$. The *Schubert variety* $\Omega(F_0, \ldots, F_k)$ determined by Φ, which consists of all the k-flats $l \in G'_{k,d}$ with $\dim(l \cap F_i) \geq i$ for all $i = 0, \ldots, k$, and the algebraic set $\Omega^0(F_0, \ldots, F_k)$ determined by the *strict* Schubert conditions coming from Φ, which consists of all the k-flats $l \in G'_{k,d}$ with $\dim(l \cap F_i) = i$ for all $i = 0, \ldots, k$, are convex in certain cases, for example if $d = 3$ and $k \leq 3$. We do not yet know all of the values of k and d for which this holds; in fact, this seems to be an interesting problem.

6. For a final example consider a centrally symmetric point set S on a 2-sphere in $\mathbb{R}^3$, and let $\mathcal{L}$ be the set of all lines parallel to the lines joining the center of the sphere to the points of S, a "parallel-closed set" (Figure 15). Then it turns out that $\mathcal{L}$ is convex if and only if the set S is the complement of a union of great circles.

These parallel-closed families, in fact, turn out to be rather interesting. They can be thought of as duals of families of "convex point sets at infinity", in an appropriate sense. We will say more about them below.

Geometric Transversal Theory

Let us look at convexity on the affine Grassmannian from the point of view of what is called "geometric transversal theory". Helly's theorem [12], one of the cornerstones of combinatorial geometry, says that for a family of at least $d+1$ compact convex sets in $\mathbb{R}^d$, if every $d+1$ are pierced by a point, then all the sets are pierced by a single point (Figure 16).

In 1935 Vincensini asked whether a similar theorem could be proved if "pierced by a point" was replaced by "met by a line", or by a plane, or a flat of dimension k in d-space. This question led to a series of papers by, among others, Santaló, Klee, Hadwiger, Debrunner, Grünbaum, Danzer, Valentine, and Eckhoff, giving various generalizations of Helly's theorem to line transversals and higher-dimensional transversals and to families of convex bodies of various kinds (parallelopipeds, translates of a fixed body, and so on). An excellent survey of work in geometric transversal theory up to 1962 is given in [4]. In the past two decades there has been a sort of explosion of interest, with work by Katchalski, Lewis, Liu, Zaks, Tverberg, Dol'nikov, Alon, Kalai, Kleitman, Montejano, and many other people. In recent years the thread has also been picked up by computer scientists, for example Agarwal, Aronov, Avis, Edelsbrunner, Pellegrini, Robert, Sharir, Shor, and Wenger, among others, who are interested in the combinatorial complexity of the space of transversals of a family of convex sets, both for lines and for higher-dimensional "stabbers", in connection with the complexity of algorithms in computational geometry. Up-to-date surveys include [5, 9, 17, 18].

The author's own interest in the subject dates to a paper with Pollack [7], in which we generalized a theorem of Hadwiger's on line transversals:

Theorem 4 (Hadwiger's Transversal Theorem) [11]. A family S of pairwise disjoint compact convex sets in the plane has a line transversal if and only if there is a linear ordering of S such that every three convex sets are met by a directed line consistently with that ordering.

Thus if there is some numbering of the sets such that any three line up in increasing order, as in Figure 17, then the family has a line transversal. Without the assumption about the sets having a consistent order the theorem would be false; in fact, there is no "pure" Helly-type theorem that says that if enough of the sets have a common line transversal, then all of them do: the example shown in Figure 18, consisting of a family of four line segments and a centrally located point, generalizes.

But if we are interested in the existence of, say, a *planar* transversal for a family of compact convex sets in 3-space, how can we generalize this notion of "consistent linear order"? The answer is provided by the concept of the *order type* of a set of points in the plane or in general in $\mathbb{R}^d$. This is a very simple idea that has proven useful in a great many situations where one wants to talk about higher-dimensional "order" properties of a set of points.

Definition 3. If S is a set of labeled points, $P_1 = (x_1^1, \dots, x_1^d), \dots, P_n = (x_n^1, \dots, x_n^d) \in \mathbb{R}^d$, the *order type* of S is the family of orientations of the $(d+1)$-tuples of points, i.e., the mapping that associates to each $(d+1)$-tuple $i_0 < \dots < i_d$ the sign of the determinant

$$\begin{vmatrix} 1 & x_{i_0}^1 & \dots & x_{i_0}^d \\ \vdots & & \ddots & \\ 1 & x_{i_d}^1 & \dots & x_{i_d}^d \end{vmatrix}.$$

So, for example, the sets S and T in Figure 19 have the same order type, labeled as they are, since each triple on the left is counterclockwise if and only if the corresponding triple on the right is, but the sets S and U do not, since P_1, P_2, P_3 and R_1, R_2, R_3 have opposite orientations. In fact, it is easy to see that there is *no* labeling in which their order types agree.

The order type of a set of points is also known as a "realizable acyclic oriented matroid". Roughly speaking, an oriented matroid is an axiomatically defined structure that generalizes many of the incidence and order properties of a configuration of points. This notion will occur in Theorem 6 below. A more precise definition, which we do not need, may be found in [2].

Now the Hadwiger theorem for line transversals requires that the sets be met consistently by directed lines. What do we mean by the order in which a family of convex sets meets a directed line l? Simply choose any point from the intersection of each set with l, and look at their order along l (Figure 20). The pairwise disjointness hypothesis guarantees that for a given line transversal meeting any number of the sets, the order in which it meets them will not depend on which point we choose from each set.

How can we guarantee the same thing for, say, an oriented plane transversal? A moment's reflection shows that we should assume that no three of the sets have a line transversal (if they did, as in Figure 21, the choice of points x, y_i, z could turn out clockwise, counterclockwise, or degener-

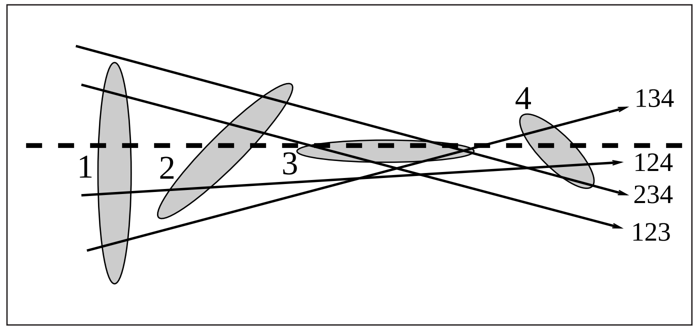

Figure 17. Hadwiger's transversal theorem.

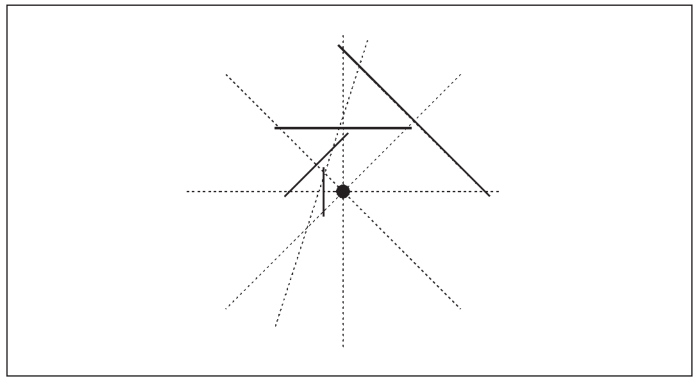

Figure 18. Any 4 sets have a common transversal, but all 5 do not.

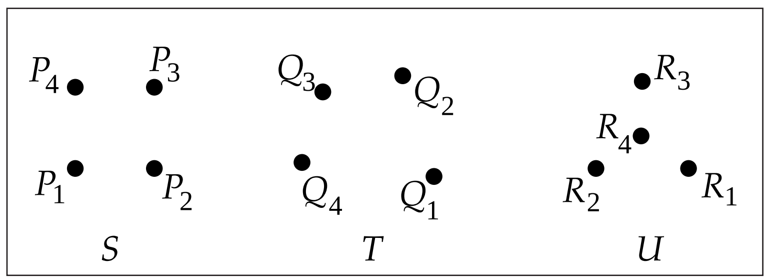

Figure 19. U **has a different order type from** S **and** T**.**

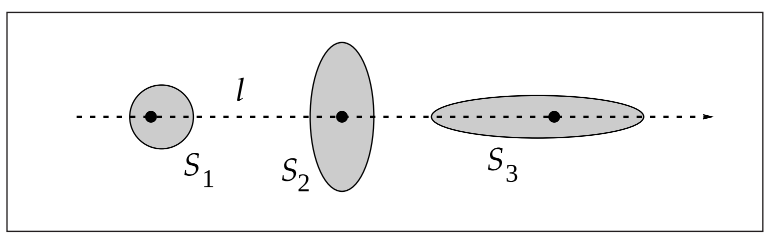

Figure 20. The order of the 3 sets is well-defined.

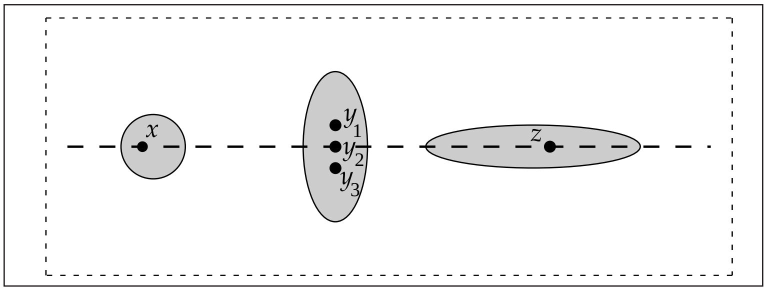

Figure 21. The orientation of the 3 sets is ambiguous.

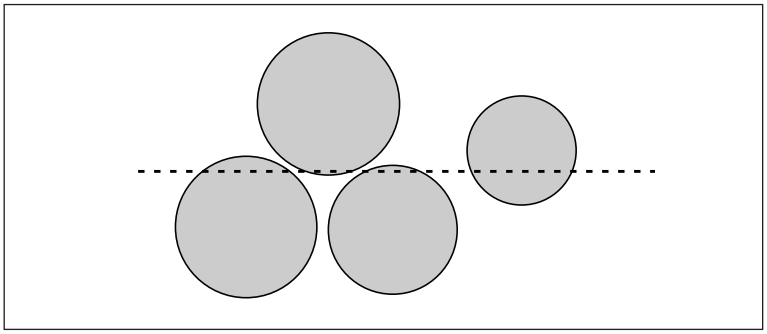

Figure 22. 0-separated sets in the plane.

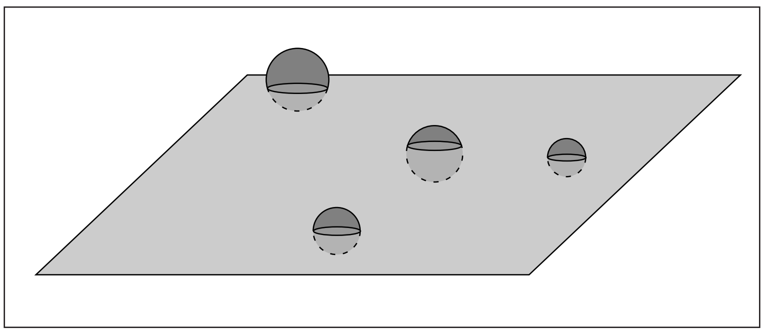

Figure 23. 1-separated sets in 3-space.

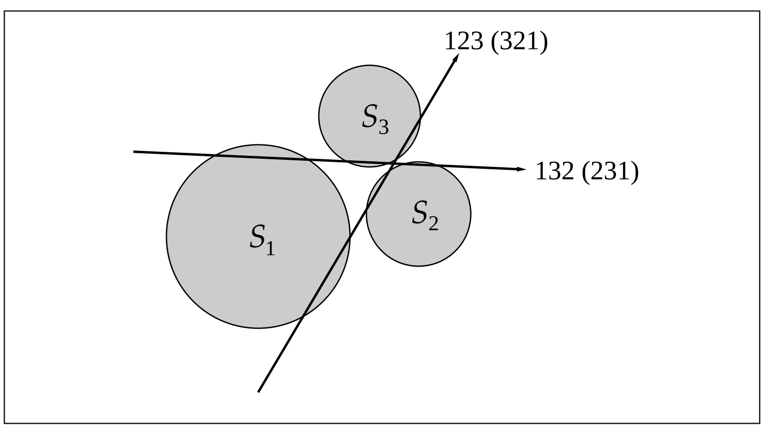

Figure 24. Two geometric permutations.

ate, depending on which side of the line we chose them). So we call a family of point sets *k-separated* if no $k+2$ of them have a common k-transversal; thus 0-separated means that no two have a point in common, which simply means that the sets are pairwise disjoint (Figure 22), as in the original Hadwiger theorem, and 1-separated means that no three line up (Figure 23).

Then the generalized Hadwiger theorem turns out to be:

Theorem 5 [7]. A $(d-2)$-separated family S of compact convex sets in $\mathbb{R}^d$ has a hyperplane transversal if and only if there is a set S of points in $\mathbb{R}^{d-1}$ such that every $d+1$ convex sets are met by an oriented hyperplane consistently with the order type of S.

This result has since been generalized further by Pollack and Wenger [15] and more recently still further by Anderson and Wenger to the following, which eliminates entirely the separatedness condition in Hadwiger's original theorem, drops the assumption that the order type of the sets is realizable by points, and also subsumes an earlier theorem of Katchalski:

Theorem 6 [1]. Let S be a finite family of connected sets in $\mathbb{R}^d$. Then S has a hyperplane transversal if and only if, for some k with $0 \le k < d$, there is a rank $k+1$ acyclic oriented matroid on S such that every $k+2$ members of S are met by an oriented k-flat consistently with that oriented matroid.

The outstanding problem in this connection is to find a similar result, if one exists, for transversals of codimension greater than 1. Here something beyond the order type will be needed, just as the original Hadwiger theorem needed something beyond a Helly-type condition: For each n there is an example, due to Aronov et al., of a linearly ordered family of convex sets in 3-space such that every n are met by a line transversal consistently with the order, but no line transversal exists for the entire family (see [9], Theorem 2.9).

Geometric Permutations

A second thread in geometric transversal theory, also involving hyperplane transversals, has a more combinatorial flavor. It involves the idea of what is called a "geometric permutation". A family of convex sets in the plane may be met by directed lines in several different orders, even if the sets are disjoint, as in Figure 24. Each such order (together with its reverse) is called a *geometric permutation*. If there are n mutually disjoint sets, how many geometric permutations can they have?

It was shown by Katchalski, Lewis, and Zaks [14] that there can be as many as $2n-2$ (see Figure 25, where half of the permutations for $n = 5$ are indicated), and by Edelsbrunner and Sharir [6] that this is, in fact, the maximum number. What about the same problem for hyperplane transversals in $\mathbb{R}^d$?

Again, the notion of *order type* tells us what a "geometric permutation" should mean in this case: a pair consisting of a $(d-1)$-dimensional order type determined by intersecting a $(d-2)$-separated family of convex sets with a hyperplane transversal, together with its reverse, as shown in Figure 26. Then we have the following result, which generalizes, at least asymptotically, the upper bound in the plane due to Edelsbrunner and Sharir:

Theorem 7 [3]. Let S be a $(d-2)$-separated family of n compact convex sets in $\mathbb{R}^d$. Then S has $O(n^{d-1})$ geometric permutations induced by hyperplane transversals.

The key step in the proof turns out to involve looking at the space of oriented hyperplanes that are *common tangents* to a $(k-2)$-separated family of $k \leq d$ strictly convex sets in $\mathbb{R}^d$ and showing that it is homeomorphic to 2^k copies of the sphere S^{d-k}. For example, Figure 27 indicates one of the four circles of oriented planes that are common tangents to the two spheres shown.

Now it is clear that any two k-flat transversals inducing distinct geometric permutations on a $(k-1)$-separated family of compact convex sets must belong to distinct connected components of the entire space of transversals; for example, in the plane, if two lines meet a family of pairwise disjoint sets in different orders, then one line cannot be moved continuously to the other, keeping it transversal to all the sets, since that would violate the separatedness. The converse was shown by Wenger to hold in the case of hyperplane transversals:

Theorem 8 [16]. Let S be a $(d-2)$-separated family of compact convex sets in $\mathbb{R}^d$. Two hyperplane transversals induce the same geometric permutation on S if and only if they lie in the same connected component of the space of transversals to S.

So in terms of the convexity structure on the space $G'_{d-1,d}$ of hyperplanes, this last result gives a description of the various connected components of a convex set of hyperplane transversals if the presentation is given by a separated family: each connected component corresponds to a different geometric permutation; see Figure 28. (This shows again why convex sets of higher-dimensional flats cannot be expected to be connected, in general, as indicated in Example 4 above: because of the possibility that they may correspond to distinct geometric permutations.)

Summing up, one might say that geometric transversal theory involves the study of the space of k-flat transversals to a given family of convex point sets with regard to its topological properties, its combinatorial properties, and so on. The theory of convexity on the affine Grassmannian explores the *inverse* problem: given a set $\mathcal{L}$ of k-flats, under what circumstances is $\mathcal{L}$ the set of all k-transversals to some family of convex point sets? The answer, provided by Theorem 2 above, is: if

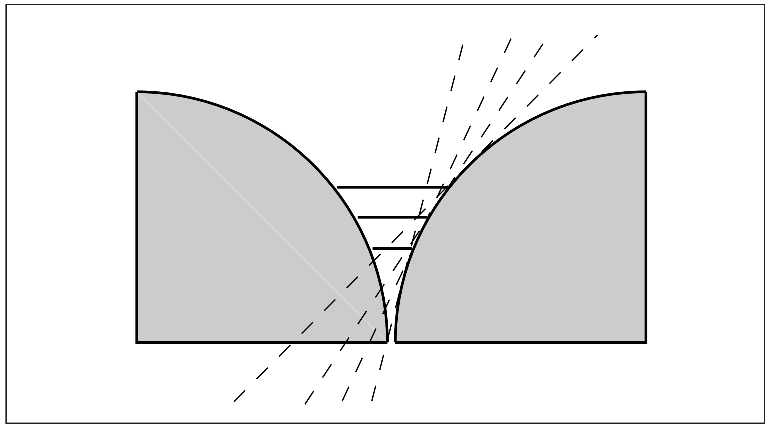

Figure 25. 5 sets with 8 geometric permutations.

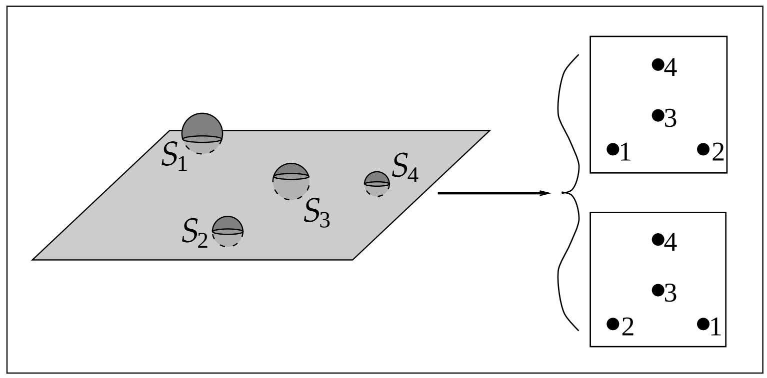

Figure 26. A planar geometric permutation.

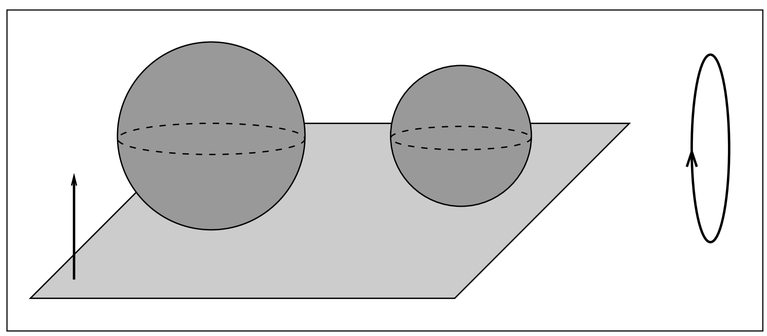

Figure 27. A circle of oriented tangent planes.

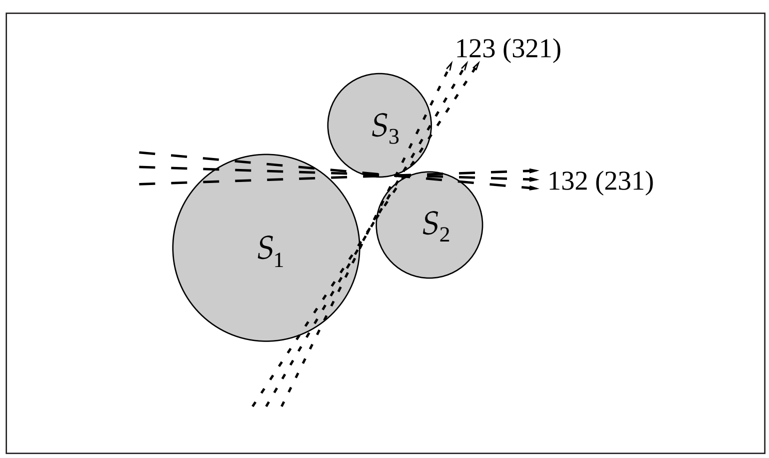

Figure 28. The two connected components of a convex set.

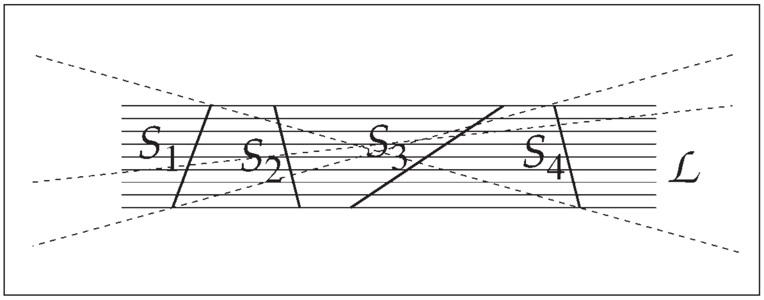

Figure 29. $\mathcal{L}$ is not finitely presented.

and only if every k-flat surrounded by $\mathcal{L}$ already belongs to $\mathcal{L}$.

Properties of Convex Sets: Some Results and Questions

Of the various properties of convex point sets that extend to convex sets on the affine Grassmannian, here are just a few.

We have already mentioned the fact that convex sets of k-flats are preserved under nonsingular affine transformations. Just as is true for convex point sets, convex sets of k-flats are also preserved under restriction to subspaces and under restriction to direction:

Theorem 9 [8]. Let $\mathcal{L} \subset G'_{k,d}$ be convex.

i) If F is an m-flat in $\mathbb{R}^d$ with $m \geq k$, the restriction $\mathcal{L}|_F$ of $\mathcal{L}$ to F, consisting of all the members of $\mathcal{L}$ that lie in F, is a convex subset of $G'_{k,m}$.

ii) If F is an m-flat in $\mathbb{R}^d$ with $m \leq k$, then the set $\mathcal{L}\|_F$ of all flats in $\mathcal{L}$ parallel to F is convex.

Another property that extends nicely from convex point sets to convex sets of flats is the Krein-Milman property. For point sets this says that every compact convex set of points is the convex hull of its extreme points. If we define an *extreme flat* of a convex set of flats in $\mathbb{R}^d$ as one that does not lie in the convex hull of the remaining flats of the set, then we have

Theorem 10. Every compact convex set of flats in $\mathbb{R}^d$ is the convex hull of its set of extreme flats.

So far the questions we have been dealing with have been largely ones that make sense for convex point sets; we have seen that many of the standard properties of convex point sets extend to convex sets of flats.

Now let us consider a different kind of question that does not come up for convex point sets, one that arises from the "double dual" characterization of the convex hull of a set of flats. This can be thought of as the question of *canonical representation*.

If a convex set $\mathcal{L}$ of k-flats is the set of transversals to a *finite* set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of convex point sets, recall that $\mathcal{L}$ is called *finitely presented*. It is easy to see that not all convex sets of k-flats are finitely presented, for example, the set of all lines in the plane lying between two parallel lines (Figure 29). Here one needs infinitely many convex point sets S_i to limit their common transversals to just the lines of $\mathcal{L}$.

But suppose $\mathcal{L}$ *is* finitely presented. Then we may be able to throw away convex point sets S_i that are not needed in the presentation to get an *irredundant* presentation, and we may be able to shrink each set S_i as much as possible to get a *minimal* presentation. More precisely,

Definition. If $\mathcal{L}$ is a convex set of k-flats in $\mathbb{R}^d$ and $\mathcal{S}$ a family of convex point sets, the presentation $\mathcal{L} = \mathcal{S}^*$ is *irredundant* if $\mathcal{L} \subsetneq \mathcal{S}_0^*$ for every proper subset $\mathcal{S}_0$ of $\mathcal{S}$. On the other hand, if $\mathcal{S}^*$ becomes strictly smaller whenever any $S \in \mathcal{S}$ is replaced by a proper subset, the presentation $\mathcal{L} = \mathcal{S}^*$ is called *minimal*.

Not every convex set of k-flats has a minimal, irredundant presentation, for example the set of lines between two parallels, but it is easy to prove

Theorem 11 [8]. If $\mathcal{L} \subset G'_{k,d}$ is finitely presented by a set $\mathcal{S}$ of compact convex point sets, then $\mathcal{S}$ can be refined to a family $\mathcal{S}_0$ of compact convex point sets that gives a minimal, irredundant presentation of $\mathcal{L}$.

In the special case where $\mathcal{L}$ is a finite set of hyperplanes in general position in $\mathbb{R}^d$ (such a set is easily seen to be convex by the surrounding criterion), we can prove more:

Theorem 12 [8]. If $\mathcal{L}$ is a finite subset of $G'_{d-1,d}$ consisting of n hyperplanes in general position, then $\mathcal{L}$ has a minimal, irredundant presentation by $2(d-1)(n-d)+2^d$ compact convex point sets.

Figure 30 shows such a presentation for the case of n lines in the plane ($d = 2$ in this case, so that $2(d-1)(n-d)+2^d = 2n$).

We have no such result for the case of k-flats with $1 \leq k \leq d-2$; it would be nice to know what happens there.

Another question that does not arise for convex point sets but does for convex sets of flats of positive dimension is the following: If a convex set of k-flats can have more than one component, is it true, at least, that each component by itself is convex? With no restriction at all on the convex set, the answer turns out to be no—there is a rather complicated example [10] of a set of lines in 3-space whose convex hull has two connected components, one of which is nonconvex. But if we restrict our convex sets in fairly natural ways, the answer seems to be yes, at least in certain dimensions.

For example, we have:

Theorem 13 [10]. Let $\mathcal{S}$ be a finite family of pairwise disjoint compact convex point sets in $\mathbb{R}^3$, and let $\mathcal{S}^*$ be the convex set of lines dual to $\mathcal{S}$. If $\mathcal{L}$ is a connected component of $\mathcal{S}^*$ (see Figure 31), then $\mathcal{L}$ is convex. Moreover, $\mathcal{L}$ is itself the space of line transversals of some finite family of pairwise disjoint compact convex point sets.

We conjecture that a similar theorem holds in any dimension.

Finally, here is a combinatorial question that also has no counterpart for point sets; this has to do with partitioning the Grassmannian into proper convex subsets.

For point sets, $\mathbb{R}^d$ can be partitioned into two proper convex subsets, for example a closed half-space and its complement. For k-flats in $\mathbb{R}^d$, though, it turns out that this is impossible in general, even though there *is* always a partition into a finite number. We can prove, for example,

Theorem 14 [8]. The lines in $\mathbb{R}^3$ cannot be partitioned into two nonempty convex sets.

On the other hand, there *is* a partition of $G'_{1,3}$ into *three* nonempty convex sets. For example, here is one way to get it:

Take an octahedron centered at the origin in $\mathbb{R}^3$, and 3-color its boundary complex so that

i) opposite faces get the same color;

ii) the color of each vertex agrees with that of some incident edge;

iii) the color of each edge agrees with that of some incident face; and

iv) whenever a vertex and an incident face have the same color, then every edge incident to both has the same color as well.

The image on the cover of this issue of the *Notices* shows an example of such a coloring.

Then take for $\mathcal{L}'_1, \mathcal{L}'_2, \mathcal{L}'_3$ the sets of lines joining the origin to the points in each color class and for $\mathcal{L}_i$ the set of all lines in $\mathbb{R}^3$ parallel to the lines of $\mathcal{L}'_i$. Then it turns out that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ form a convex partition of the set of all lines in $\mathbb{R}^3$.

This is, in fact, an example of a partition into *parallel-closed* convex sets. For partitions of this kind we can prove:

Theorem 15 [8]. Let $G_0 \subset G_1 \subset \cdots \subset G_{d-1}$ be a flag of subspaces of $\mathbf{P}^{d-1}$, with $\dim G_i = i$, and set $G_{-1} = \emptyset$. Let S be the set of all subsets $\{i_0, \ldots, i_{k-1}\}$ with $i_0 < \ldots < i_{k-1}$ of the integers $0, \ldots, d-1$, and let S_{d-1} consist of all k-sets $i_0 < \ldots < i_{k-1}$ in S with $i_{k-1} = d-1$. For each element $\sigma = \{i_0, \ldots, i_{k-1}\}$ of S, put $i_{-1} = 1$ and $i_k = d$, and let Φ_σ be the Schubert cell defined by

$$\Phi_\sigma = \{\phi \in G_{k-1,d-1} \mid \dim \phi \cap G_i = j \quad \text{for } i_j \le i < i_{j+1},\ -1 \le j \le k-1\}.$$

Let $\mathcal{L}_\sigma = \delta^{-1}(\Phi_\sigma)$, where $\delta : \mathbb{R}^d \setminus \{O\} \to \mathbf{P}^{d-1}$ is the canonical map. Then each $\mathcal{L}_\sigma$ is a parallel-closed convex set of k-flats, as is the union

$$\mathcal{L}_{d-1} = \bigcup \mathcal{L}_{i_0,\ldots,i_{k-2},d-1},$$

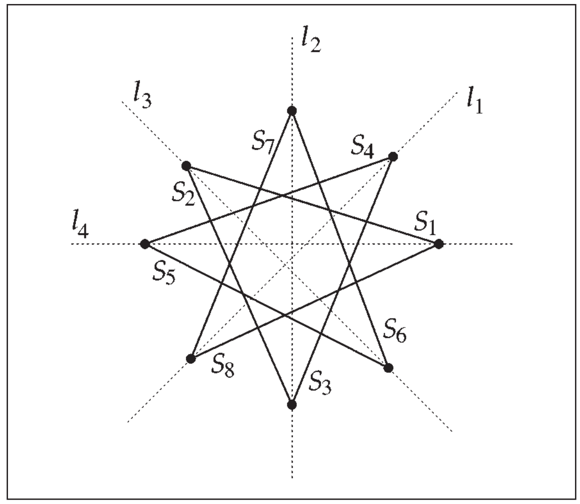

Figure 30. A minimal, irredundant presentation.

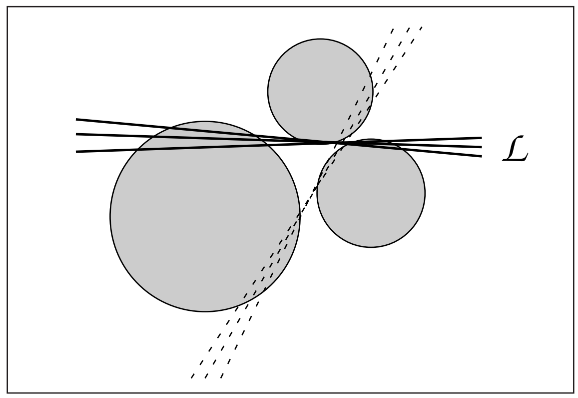

Figure 31. A connected component of a convex set of lines.

and

$$G'_{k,d} = \left(\bigcup_{\sigma \in S \setminus S_{d-1}} \mathcal{L}_\sigma \right) \cup \mathcal{L}_{d-1}$$

is a partition of $G'_{k,d}$ into $\binom{d-1}{k} + 1$ nonempty parallel-closed convex sets.

In particular, this gives an upper bound on the smallest number (> 1) of convex sets into which $G'_{k,d}$ can always be partitioned.

The only lower bound we have is for the case of parallel-closed convex sets:

Theorem 16 [8]. If $G'_{k,d} = \bigcup_{i=1}^n \mathcal{L}_i$ is a partition of $G'_{k,d}$ into $n > 1$ nonempty parallel-closed convex sets, then $n \ge d - k + 1$.

(For example, for lines in 3-space, at least three parallel-closed sets are needed.)

We suspect that the lower bound of $d - k + 1$ holds for *all* convex sets, not just parallel-closed sets, but at the moment we have no proof.

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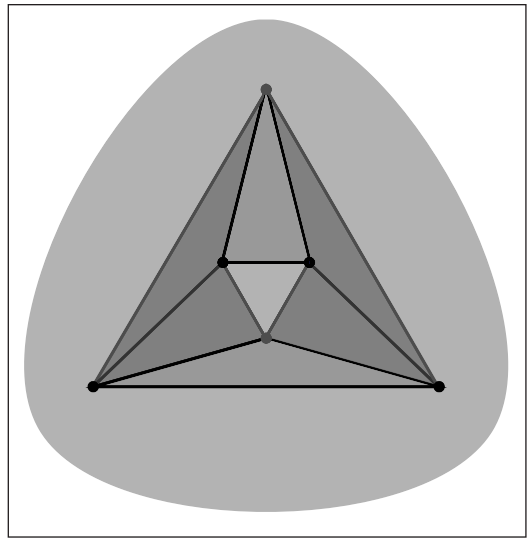

About the Cover

Convex Partition of a Grassmanian: The cover figure is a *Schlegel diagram* (a planar projection from a point close to an interior point of some facet) of the regular octahedron in $\mathbb{R}^3$. Its coloring partitions the Grassmannian $G'_{1,3}$ of lines in 3-dimensional space into three sets, as follows: Color each line through the center of the octahedron by the color of its points of intersection with the octahedron; this makes sense, since the face coloring is centrally symmetric. Then color all parallel translates of a given line by the same color. The set of lines in each color class turns out to be convex.