Book Review

Polyhedra

Reviewed by Bill Casselman

Polyhedra
Peter R. Cromwell
Cambridge University Press, 1997
460 pages
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This is an unusual book, one hard to classify, but certainly valuable and a labor of love. It is a book on what might be called the classical theory of polyhedra, as opposed to the modern theory of polytopes. It is concerned mostly with three-dimensional geometry and indeed mostly the geometry of previous centuries, although it does discuss recent and nonnegligible contributions to the main topic. It is probably not quite suitable as a text for an undergraduate geometry course, but it would prove invaluable as a reference book in such a course. The cost of the book is not excessive if the book is priced by size (!), but a paperback version would perhaps be more affordable for an undergraduate.

This is not a book on the history of geometry, but it includes many historical digressions in a somewhat informal but entertaining style. As a proper geometry book ought to, it contains many useful geometrical illustrations and some extremely impressive, even beautiful, figures excerpted from classical works of mathematics. The quality of the author's illustrations in the book might be slightly better, but the sheer quantity of figures is overwhelming, and the linkage between illustrations and discussion in the text is on the whole extremely good. It is perhaps unfortunate that almost the first illustration accompanying a proof, that of the incommensurability of the side and diagonal of a regular pentagon, is poorly conceived.

The major virtue of the book is that the author has managed to find many interesting paths in mathematics not often traveled. One pleasant feature is that many, if not most, arguments in the book seem to have been taken from original sources, but with useful and enlightening modifications.

The topics covered in Chapter 1 include a few brief remarks on early geometry among the Egyptians and Babylonians and a short discussion of the Greek discovery of irrationality. There is not much originality in these sections, which make up perhaps the weakest part of the book. It then discusses the volume of prisms, including some less well-known material from a classic Chinese work with some illuminating accompanying figures. It then discusses Hilbert's third problem, the one that originated with Hilbert's theory of equidecomposability and led to Dehn's discovery of the scissors equivalence classes of three-dimensional polyhedra.

Chapter 2 is on more or less regular polyhedra. The author describes the explicit construction of the five Platonic solids, dealing nicely with the different possible notions of regularity. (He takes up this account again later on in the first part of Chapter 6.)

Chapter 3 is an interesting tour of the history of geometry from the Hellenistic era through the

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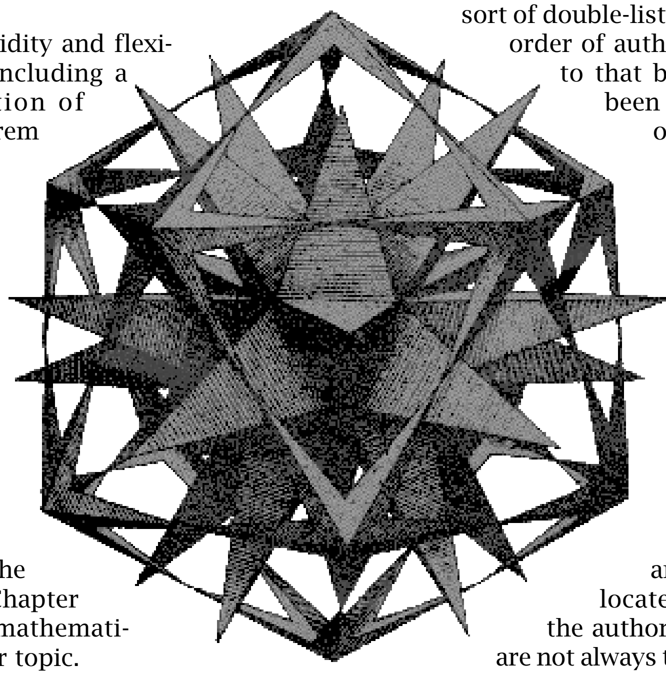
Renaissance. Again, this is not so much a history as a sequence of historically based digressions, but these digressions are often fascinating.

Chapter 4 is on relatively obscure but nonetheless intriguing geometrical work of Kepler.

Chapter 5 is about Euler's formula for what we call the Euler characteristic of polyhedra. It includes discussion of contributions of Descartes, Legendre, Cauchy, Möbius, and Poinsot, as well as less well-known mathematicians such as L'Huilier and Hessel.

Chapter 6 is on rigidity and flexibility of polyhedra, including a fairly clear exposition of Cauchy's rigidity theorem for convex polyhedra.

Chapter 7 includes a very clear discussion of stellated polyhedra, and Chapter 8 is on symmetry. Chapters 9 and 10 are on various combinatorial questions about polyhedra, including various ways to color them and a very useful discussion of the role of computers in the proof by Appel and Haken on the four-color theorem. Chapter 10 will be for most mathematicians the least familiar topic.



This book covers a lot of territory. It covers a small number of topics in depth and a huge number of topics overall. Among my favorite discussions are the ones on Cauchy's rigidity theorem, including a very pleasant historical account of the notion of regularity, but I also liked very much the whole chapter on stellations. The number of little- known historical works of mathematics referred to is impressive and stimulating. The bibliography fills more than twenty pages and runs from items by Plutarch, Alberti, Dürer, Pacioli, Vasari, Descartes, Bonnet, and dozens of authors whose names I had never seen before, through Ernst Haeckel, Joseph Needham, Otto Neugebauer, and J. V. Field, and onto the more usual fare of Heath, Coxeter, Hilbert, Grünbaum, and Senechal. The history told in the book is rarely deep, but the breadth of topics covered more than makes up for that minor flaw. It seems likely that the author has actually examined most of the items, both well known and obscure, that he refers to.

I have a few minor complaints. The reference list is huge, and it is likely that most, if not all, of the items in the list are referred to in some way or another sooner or later. The author gives the sources for quotations, but no precise references for topics covered—no footnotes or numbered references, for example—and the task of tracking down particular topics among the references will often be daunting. My own ideal in this regard, in a book not unlike this one in many ways, is Neugebauer's *The Exact Sciences in Antiquity*, where at the end of each chapter references are threaded together beautifully. In *Polyhedra*, however, one is left on one's own. Better organization of the references would have been especially valuable considering the magnitude of the reference list. Even some sort of double-listing, one in alphabetical order of author's names in addition to that by chapter would have been nice. Another problem of a similar nature is that there are several figures extracted from old, even rare, editions of classical works. These are often extremely intriguing figures which one might like to see, so to speak, in the flesh. But in order to find them one would have to know the exact source—in which edition, for example, the figures were located and perhaps where the author's copy was found. We are not always told these things. My interpretation of this sort of problem is that the author has underestimated the interest his book will arouse and probably feels also that the book should not be weighed down with scholarly baggage. This is a legitmate concern, but surely some compromise was possible.

The author tells us that his choice of topics is personal and that the emphasis is on three-dimensional geometry throughout, but even so it would have been be useful to have some mention of a larger context for some topics. He discusses symmetries of three-dimensional polyhedra, for example, but there is no explicit mention of Coxeter groups (generated by reflections) despite the special role they play even in three dimensions. The proof of the existence of the regular polyhedra is essentially the classical proof, which seems to modern mathematicians a bit disjointed. In the section on Cauchy's enumeration of star polyhedra (Chapter 7) the author comes close to the proof of existence, appealing directly to symmetry that was apparently first discovered by Tits (explained in the one great book by Bourbaki), but doesn't quite get to it. Of course, this may be one place where the author feels the topic is so well covered elsewhere that it is not necessary to cover it here. Also missing is a discussion of the relationship between Poinsot's regular polyhedra and Riemann surfaces of higher genus, discovered by Du Val and Threlfall, which many find the most intriguing properties of these unusual geometrical objects.

The discussion of Cauchy's rigidity theorem is useful and original, and Cromwell's treatment of the famous lacuna in Cauchy's proof I found to be especially interesting, but it is still difficult to follow if one wants a complete proof, and most readers will want to look elsewhere for additional light. (My own favorite discussion is the one in Heinz Hopf's Springer Lecture Notes.) Perhaps because it is so well known, his discussion of the area formula for spherical triangles is brief, accompanied by no pictures. (Cromwell seems not to be aware of Thomas Harriot's unpublished but well-known proof from the year 1603.)

The figures Cromwell includes from historical documents are fine, but his own graphical work is occasionally lacking in panache. Even in the age of computers, drawing complicated figures in three dimensions is not easy, and in view of the sheer quantity of figures he does include this should not be counted a major sin. It is tempting to contemplate the nature of technological progress by contrasting the extraordinary detail in the figures reproduced from a sixteenth-century work by Wenzeln Jamnitzer with the rather spare illustrations made by the author himself. The revolution in mathematical typography brought about by $\TeX$ has not yet been extended to mathematical graphics. On the other hand, there is an insert of several color photographs of very fine quality.

There are a fair number of spelling and typographical errors in the book. An errata list (including a few replacement pages in PostScript) and other interesting things can be found at the author's Web page:

`http://www.liv.ac.uk/~spmr02/book/`.

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