Conferences

Short Course on Nonlinear Control

San Antonio, Texas, January 11–12, 1999

The American Mathematical Society, in conjunction with its 105th Annual Meeting, will present a two-day Short Course on *Nonlinear Control* on Monday and Tuesday, January 11 and 12, 1999, at the Hilton Palacio Del Rio, San Antonio. The program is under the direction of **Hector J. Sussman**, Rutgers University, and **Keven Grasse**, University of Oklahoma.

Synopses and accompanying reading lists follow. Lecture notes will be available to those who register. Advance registration fees: $80 ($35 student/unemployed/emeritus); on-site registration fees $95 ($45 student/unemployed/emeritus). Registration and housing information can be found in this issue of the *Notices*; see the section "Registering in Advance and Hotel Accommodations" in the announcement for the meetings in San Antonio.

Dynamic Programming and Viscosity Solutions

H. Mete Soner, Princeton University

Synopsis

Consider the classical calculus of variations problem to minimize

$$\int_0^T L(\xi(s), \dot{\xi}(s), s)\, ds,$$

over all Lipschitz continuous functions $\xi : [0, T] \to \mathcal{R}^d$.

In the Hamilton-Jacobi approach, one embeds this problem into a family of similar problems

$$v(x,t) := \inf \left\{ \int_0^T L(\xi(t), \dot{\xi}(t), t)\, dt \right\},$$

parametrized by the initial data $\xi(t) = x$. Then, for every $h > 0$, the value function v satisfies

$$v(x,t) = \inf \left\{ \int_t^{t+h} L(\xi(t), \dot{\xi}(t), t) dt + v(\xi(t+h), t+h) | \xi : [t, t+h] :\to \mathcal{R}^d \right\}.$$

This minimization principle, called *dynamic programming*, yields a differential equation for the value function. Indeed, if we assume that the value function is continuously differentiable,

$$-\frac{\partial}{\partial t} v(x,t) + H(x, t, \nabla v(x,t)) = 0, \quad (x,t) \in \mathcal{R}^d \times (0,T)$$

where

$$H(x,t,p) := \sup \left\{ -\alpha \cdot p - L(x,t,\alpha) \quad | \quad \alpha \in \mathcal{R}^d \right\}.$$

Moreover, the minimizer ξ^* satisfies the differential equation

$$\dot{\xi}^*(t) = \nabla v(\xi^*(t), t).$$

In the sixties R. Bellman formulated a general stochastic optimal control problem and the corresponding dynamic programming principle. As a corollary, one obtains a nonlinear partial differential equation satisfied by the value function provided that the value function is sufficiently smooth. We will call this equation the *dynamic programming equation.*

This approach provides a very general way of calculating the value function and the minimizing trajectories. However, the value function is generally not smooth enough to justify these calculations. In their celebrated paper, M.
Crandall and P.-L. Lions overcame this difficulty by introducing a weak notion of a solution for the dynamic programming equation called *viscosity solutions*. The main idea is to define super- and subsolutions and then define the solution as a function which is both a sub- and a supersolution. In addition to this definition, Crandall and Lions also proved a uniqueness result. This is particularly important, as one generally calculates the value function using a numerical method. Then the uniqueness guarantees that the solution calculated by the computer well approximates the value function.

In this talk I will first explain the Crandall and Lions theory for dynamic programming. Then I will outline several applications to financial mathematics. In particular, I will use the viscosity solutions to calculate the superreplication cost of a contingent claim in the presence of portfolio constraints. This problem is related to backward-forward stochastic differential equations.

Reading List

[1] G. Barles, *Nonlinear control systems,* 3rd ed., Springer-Verlag, New York, 1995 (see esp. Chaps. 1, 2).

[2] M. Crandall and P.-L. Lions, Trans. Amer. Math. Soc.

[3] M. Crandall, L. C. Evans, and P.-L. Lions, Trans. Amer. Math. Soc.

[4] M. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.

[5] W. H. Fleming and H. M. Soner, *Viscosity solutions and controlled Markov processes*, Springer-Verlag, New York, 1991.

Introduction to Differential-Geometric Control

Kevin A. Grasse, University of Oklahoma

Synopsis

A control system can be viewed informally as a dynamical object (e.g., ordinary differential equation) containing a parameter (control) which can be manipulated to influence the evolution of the system's trajectories through its underlying "state space". In particular, from a fixed initial state it is usually possible to reach (or control to) an entire family of other states by varying the control. This lecture will establish the conceptual and notational framework in which a nonlinear control system can be viewed as a collection of vector fields on a finite-dimensional differentiable manifold. Such a collection of vector fields is sometimes referred to as a *dynamical polysystem*. The control systems considered here are always understood to be finite-dimensional, deterministic, continuous-time systems (thereby excluding other classes of potentially interesting systems, such as stochastic, distributed-parameter, and discrete-time systems). After R. E. Kalman's introduction of the notion of control systems evolving on a state space (circa 1960), it was discovered somewhat later (circa 1970) that linear control systems (whose state space is a finite-dimensional vector space) could be elegantly and effectively studied via "coordinate-free" linear algebra, while for nonlinear control systems the natural coordinate-free setting was a finite-dimensional, differentiable manifold. It must be stressed that the introduction of manifolds is not made simply for the sake of idle generalization. There are several compelling reasons for doing this, among which are the following:

1. In some cases the state of the system evolves on a manifold by virtue of the system's intrinsic nature (viz. a controlled rigid body where a portion of the state is its orientation, which is represented by a three-by-three orthogonal matrix).
2. Even in cases where the state of the system evolves in a euclidean space, the so-called *reachable set* of the system (i.e., the set of states to which the system can be controlled) may be a subset of an immersed submanifold of the ambient euclidean space.
3. When the reachable set of the system is contained in an immersed submanifold of positive codimension in the ambient euclidean space, the necessary conditions for optimal controls (i.e., the Pontryagin Maximum Principle, to be discussed in subsequent lectures) never yield useful information in their classical, euclidean-space formulation, whereas they may yield useful information when formulated in a coordinate-free manner on manifolds.

We assume only a rudimentary familiarity with the theory of differentiable manifolds, vector fields, and ordinary differential equations (specifically conditions for existence and uniqueness of integral curves of vector fields and continuous dependence of solutions on parameters; Chap. 1 of [8] is more than sufficient). After a brief introduction to the concept of a control system, we will develop the connection between control systems and systems of vector fields. The lecture will proceed to discuss the reachable set of a control system and its basic properties. The Lie bracket of vector fields will emerge in a natural way from this discussion and will play a fundamental role throughout. Questions of the structure of the reachable set are most completely answered for so-called "forward-backward" (FB) reachable sets; i.e., the states we can reach going both forward and backward in time. We will see that the collection of FB-reachable sets of a control system partitions the ambient state manifold into a foliation with singularities (so not all leaves may be of the same dimension). Indeed, we will explicate how this "orbit theorem" provides a striking generalization of the traditional theorem of Frobenius studied in differential geometry. Of course, going "backward in time" is not physically meaningful unless the control system exhibits a very strong form of symmetry, but forward (F)-reachable sets are always contained in the FB-reachable sets, so the orbit theorem can still convey useful information about the F-reachable sets. Questions of structure of the F-reachable sets themselves are considerably more delicate and will be touched on in the next lecture. We will also introduce the concepts of *controllability* (the ability to reach any one state from any other) and *accessibility* (the ability to reach an open subset of the state space from any given initial state) and survey basic results that pertain to these important notions.

We conclude with a few comments about the reading list. Any one of the references [1], [2], [3], [4], [6] provides a good overview of the material of this lecture. Reference [4] is also an excellent introduction to the theory of control systems, but perhaps places more emphasis on the system-theoretic considerations than on geometric considerations. Either of references [5] and [7] gives a fairly complete exposition of the above-mentioned orbit theorem.

Reading List

[1] A. Isidori, *Nonlinear control systems,* 3rd ed., Springer-Verlag, New York, 1995 (see esp. Chaps. 1, 2).

[2] V. Jurdjevic, *Geometric control theory*, Cambridge University Press, Cambridge, 1997 (see esp. Chaps. 1–4).

[3] H. Nijmeijer and A. J. van der Schaft, *Nonlinear dynamical control systems*, Springer-Verlag, New York, 1990 (see esp. Chaps. 1, 2).

[4] E. D. Sontag, *Mathematical control theory,* 2nd ed., Springer-Verlag, New York, 1998 (see esp. Chaps. 1–3 for a general introduction to mathematical control theory and Chap. 4 for the more geometric aspects).

[5] ____, *Integrability of certain distributions associated with actions on manifolds and applications to control problems*, Nonlinear Controllability and Optimal Control (H. J. Sussmann, ed.), Marcel Dekker, New York, 1990, pp. 81–131.

[6] H. J. Sussmann, *Lie brackets, real-analyticity, and geometric control*, Differential Geometric Control Theory (R. W. Brockett et al., eds.), Birkhäuser, Boston, 1983, pp. 1–116 (see esp. §§1–3).

[7] ____, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–189.

[8] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Scott, Foreman and Co., Glenview, IL. 1971.

Motion Control of Mechanical Systems

Richard M. Murray, California Institute of Technology

Synopsis

Recent advances in geometric mechanics, motivated in large part by applications in control theory, have introduced new tools for understanding and utilizing the structure present in mechanical systems. In particular, the use of geometric methods for analyzing Lagrangian systems with both symmetries and nonintegrable (or nonholonomic) constraints has led to a unified formulation of the dynamics that has important implications for a wide class of mechanical control systems. In this lecture I will introduce some of the fundamental geometric concepts which are basic to nonlinear control of mechanical systems and illustrate their utility on a variety of engineering examples and applications. The focus of the lecture will be on motion control between equilibrium points of the system or along a reference trajectory.

The underlying mechanical structure which one exploits is based on the association of kinetic energy with a natural Riemannian metric for the system. This allows Lagrange's equations to be represented using covariant differentiation with respect to the Levi-Cevita connection given by the kinetic energy. We use this framework to define the symmetric product between vector fields and show how it arises in the characterization of configuration accessibility and equilibrium controllability of general mechanical systems. This allows one to characterize specialized controllability properties for mechanical systems that are required for motion control.

Using this framework, one can undertake a geometric approach to control of locomotion systems, such as mobile robots and highly articulated snake robots. A common feature of these systems is the role of constraints on the behavior of the system. Typically, these constraints force the instantaneous velocities of the system to lie in a restricted set of directions but do not actually restrict the reachable configurations of the system. A familiar example in which this geometric structure can be exploited is parallel parking of an automobile, where periodic motion in the driving speed and steering angle can be used to achieve a net sideways motion. By studying the geometric nature of velocity constraints in a more general setting, it is possible to synthesize gaits for snake-like robots, generate parking and docking maneuvers for automated vehicles, and study the effects of rolling contacts on multifingered robot hands. Simulations and videotape of experiments performed at Caltech will be used to illustrate the main ideas.

Finally, we consider the problem of real-time trajectory generation and tracking for mechanical systems. Using the paradigm of two degrees of freedom control, we illustrate the role of real-time trajectory generation in nonlinear control. Important systems to consider are differentially flat systems, for which the trajectory generation problem is conceptually simple and computationally tractable. We illustrate the use of flatness in two different flight control examples and provide some constructive conditions for checking flatness of mechanical systems. Experimental results using a flight control testbed at Caltech will be presented to demonstrate the performance of flatness-based controllers.

We assume only a rudimetary familiarity with the theory of differentiable manifolds and vector fields, at the level of [1]. A summary of the basic ideas presented in this lecture can be found in a recent review article [2] and the references therein.

Reading List

[1] W. M. Boothby, *An introduction to differentiable manifolds and Riemannian geometry,* 2nd ed., Academic Press, 1986.

[2] R. M. Murray, *Nonlinear control of mechanical systems: A Lagrangian perspective*, Ann. Rev. Control **21** (1997), 31–45.

The Maximum Principle and Reachable Sets—A Classical Perspective

Heinz Schättler, Washington University

Synopsis

We consider a control system of the form $\Sigma : \dot{x} = f(x,u), x \in M, u \in U$ where M is a C^∞ manifold and the control set U is arbitrary. Admissible controls are Lebesgue measurable functions u defined on a compact interval with values in a compact subset of U. A point q is reachable from p in time t if there exists an admissible control u such that the corresponding trajectory x satisfies $q = x(t)$. The set of all points which are reachable in time t is the time-t-reachable set denoted by $Reach_{\Sigma,t}(p)$, and $Reach_{\Sigma,\leq T}(p)$ denotes those points which are reachable in times t, $0 \leq t \leq T$. If T is small, we call this set the small-time reachable set.

The Pontryagin Maximum Principle [1] gives necessary conditions for the endpoint of a trajectory to lie in the boundary of the reachable set. These conditions are obtained by approximating the reachable set with a convex cone. We briefly outline this classical construction, emphasizing its geometric context under sufficiently strong regularity conditions, which allows us to ignore technical aspects. We also show how these constructions can be used to derive the classical necessary conditions for optimality in the optimal control problem. Connections with classical results from calculus of variations will be made [2].

The first-order necessary conditions for optimality of the Maximum Principle provide the fundamental mechanism for restricting the class of trajectories to a smaller family of candidates for optimality. In general, however, the structure of optimal trajectories is left wide open, and further reductions need to be achieved. In the remainder of the lecture we will describe two approaches intended to achieve such restrictions.

The first pursues a precise construction of the small-time reachable set. If it is possible to analyze the boundary of the reachable set directly, no information will be lost and better characterizations of boundary trajectories will be obtained. This idea will be developed for single-input systems in low dimensions which are linear in the control under general conditions that are expressed in terms of the Lie brackets of the control vector fields and their relations at a reference point. The guiding principle in these constructions is to proceed from the "general" to the "specific" [3]. The procedure has proven effective to solve optimal control problems for low-dimensional systems [4, 5].

The second approach which we describe is an adaptation of the method of characteristics. The necessary conditions for optimality of the Maximum Principle also give the characteristic equations for the corresponding Hamilton-Jacobi-Bellman equation. If the flow of extremals covers a region R in the state space diffeomorphically, then a smooth solution to the Hamilton-Jacobi-Bellman equation can be constructed on R, and, very much as in the classical calculus of variations, sufficient conditions for optimality follow. We briefly outline the classical construction but focus on geometric properties of the value function as fold or simple cusp singularities arise in the flow of extremals [7]. The corresponding syntheses of optimal solutions show very close connections with the structure of the reachable sets discussed earlier.

Reading List

[1] L. Pontryagin, V. Boltyansky, R. Gamkrelidze, and E. Mishchenko, *The mathematical theory of optimal processes*, Wiley-Interscience, New York, 1962.

[2] H. J. Sussmann and J. C. Willems, *300 years of optimal control: From the Brachystochrone to the Maximum Principle*, IEEE Control Systems (1997), 32–44.

[3] H. J. Sussmann, *The structure of time-optimal trajectories for single-input systems in the plane: The C^∞ nonsingular case*, SIAM J. Control Optim. **25** (1987), 433–465.

[4] B. Piccoli, *Classification of generic singularities for the planar time-optimal synthesis*, SIAM J. Control Optim. **34** (1996), 1914–1946.

[5] A. J. Krener and H. Schättler, *The structure of small time reachable sets in low dimension*, SIAM J. Control Optim. **27** (1989), 120–147.

[6] H. Schättler, *Extremal trajectories, small-time reachable sets and local feedback synthesis: A synopsis of the three-dimensional case*, Nonlinear Synthesis (C. I. Byrnes and A. Kurzhansky, eds.), Birkhäuser, Boston, 1991, pp. 258–269.

[7] M. Kiefer and H. Schättler, *Cut-loci and cusp singularities in parametrized families of extremals*, Optimal Control: Theory, Algorithms and Applications (W. W. Hager and P. M. Pardalos, eds.), Kluwer, 1998.

Feedback Stabilization

Eduardo D. Sontag, Rutgers University

Synopsis

This presentation will deal with the problem of stabilization of an equilibrium for finite-dimensional systems $\dot{x} = f(x, u)$ evolving in a Euclidean space. That is, the objective is to find a *feedback law* $u = k(x)$ rendering the origin of the *closed-loop system* $\dot{x} = f(x, (x))$ locally or globally asymptotically stable. The problem of stabilization of equilibria is one of the central issues in control, of interest in itself and also because many other objectives—such as tracking, disturbance rejection, or output feedback—involve stabilizing suitable quantities (such as an error signal). Furthermore, understanding this simpler problem is a first step in dealing with more complicated issues, such as the stabilization of periodic orbits or general invariant sets.

For *linear* control systems the theory of stabilization is well understood, cf. [11], and it will be reviewed in the talk. Examples will be given, and we will explain the implications for local stabilization of nonlinear systems.

For linear systems if stabilizability is at all possible, then there is a linear feedback that achieves the objective. For nonlinear systems it has been known since the late 1970s that, in general, there are topological obstructions to the existence of even continuous stabilizers, cf. [12, 7, 2, 11]. We will discuss these obstructions, stated in terms of degree theory, and also the use of techniques from nonsmooth analysis and differential games, cf. [3], to deal with discontinuous controllers.

On the other hand, it is also known that in those cases in which continuous stabilizers do exist it is then also possible to pick an infinitely differentiable (in the complement of zero) k. This fact follows from the rich theory of *control-Lyapunov functions* (clf) which constitute an extension of the classical concept of Lyapunov functions from dynamical system theory. We will discuss clf, as well as their use in recursive design and the formulation of "universal" formulas for feedback controls, cf. [1, 8, 5, 6, 11].

Finally, the talk will touch upon the existence and genericity of inputs for nonsingular controllability. The results to be covered are proved by an application of techniques from the theory of real-analytic sets. Examples will be given to illustrate the use of these results, which lead to (1) numerical methods for path planning and (2) constructions of time-varying periodic stabilizers $u = k(t, x)$. References for this part include [4, 10, 9].

Several of the main references, which should be consulted for more technical details, can be found at `http://www.math.rutgers.edu/~sontag/`.

Reading List

[1] Z. Artstein, *Stabilization with relaxed controls*, Nonlinear Anal., Theory, Methods Appl. **7** (1983), 1163–1173.

[2] R. W. Brockett, *Asymptotic stability and feedback stabilization*, Differential Geometric Control Theory (R. W. Brockett, R. S. Millman, and H. J. Sussmann, eds.), Birkhäuser, Boston, 1983, pp. 181–191.

[3] F. H. Clarke, Yu. S. Ledyaev, E. D. Sontag, and A. I. Subbotin, *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Automat. Control **42** (1997), 1394–1407.

[4] J.-M. Coron, *Global asymptotic stabilization for controllable systems without drift*, Math Control, Signals, and Systems **5** (1992), 295–312.

[5] A. Isidori, *Nonlinear control systems: An introduction*, 3rd ed., Springer-Verlag, Berlin, 1995.

[6] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and adaptive control design*, Wiley, New York, 1995.

[7] E. D. Sontag and H. J. Sussmann, *Remarks on continuous feedback*, Proc. IEEE Conf. Decision and Control (Albuquerque, 1980), pp. 916–921.

[8] E. D. Sontag, *A "universal" construction of Artstein's theorem on nonlinear stabilization*, Systems Control Lett. **13** (1989), 117–123.

[9] ____, *Control of systems without drift via generic loops*, IEEE Trans. Automat. Control **40** (1995), 1210–1219.

[10] ____, *Spaces of observables in nonlinear control*, Proc. Internat. Congr. Math., 1994, vol. 2, Birkhäuser-Verlag, Basel, 1995, pp. 1532–1545.

[11] ____, *Mathematical control theory: Deterministic finite dimensional systems*, 2nd ed., Springer-Verlag, New York, 1998.

[12] H. J. Sussmann, *Subanalytic sets and feedback control*, J. Differential Equations **31** (1979), 31–52.

Recent Results on the Maximum Principle of Optimal Control Theory

Hector J. Sussmann, Rutgers University

Synopsis

The Maximum Principle of Optimal Control Theory is a far-reaching generalization of the necessary conditions for optimality of a trajectory given by the classical Euler-Lagrange equation of the calculus of variations, which applies to many other problems that are not reducible to the calculus of variations setting. As in H. Schättler's lecture, we will deal only with finite-dimensional, deterministic control systems, thereby excluding other classes of potentially interesting problems arising from stochastic or distributed-parameter systems. (The class of problems to be considered will, however, include discrete-time as well as continuous-time systems.)

The classical version of the Principle was stated and proved in the 1962 book [7] (cf. also [1] and [6]). Since then, the theorem has been extended in several directions, leading to stronger results for wider classes of problems under weaker technical hypotheses. Until recently it was not clear whether these numerous extensions could be unified into a single general result proved by means of a single method. "Nonsmooth" versions (e.g., Clarke [2]), involving dynamical equations with a Lipschitz but not continuously differentiable right-hand side, were proved by successive penalization techniques, whereas classical smooth versions with stronger conclusions (e.g., high-order optimality conditions) were proved by adapting Weierstrass's idea of "needle variations". Moreover, versions applicable to some very nonsmooth (e.g., non-Lipschitz) problems could be derived by ad hoc methods. All this added up to a somewhat chaotic situation, where many different techniques and constructions were needed to prove results that were all clearly closely related. This was obviously undesirable for aesthetic reasons, and in addition had the unpleasant effect that the existing results did not add up to a version that would cover "hybrid" problems: If the dynamics satisfies different technical conditions in different regions of the state space, and an arc ξ visits more than one of these regions, then each of the existing versions would apply to a piece of ξ, but none of them would yield a conclusion valid for the whole arc.

Recent developments have fundamentally changed this state of affairs and brought us very close to a complete unification of the various versions into one single result, proved using a uniform technique that applies in all cases and that works equally well for "hybrid" systems and for many other problems where the previous theorems either fail to apply or do apply but with conclusions that are too weak. The technique is that of "variations, flows, and generalized differentials", which combines (a) the classical needle-variations approach of Weierstrass, (b) the systematic use of a *reference flow* as opposed to a reference vector field, and (c) the use of suitable theories of *generalized differentials*.

The lecture will give a self-contained description of the technique and of the general results that can be proved with it. The main point of the approach involves realizing that the classical method used in [7] applies equally well if the classical notion of differential of a map is replaced by various other multivalued notions. (For example, the "generalized derivative" of the function $f(x) = |x|$ at 0 in the sense of F. Clarke's "generalized gradients" or that of J. Warga's "derivate containers" is the interval $[-1, 1]$.) It then turns out that the various previously existing versions of the Maximum Principle can all be obtained by the same method by just making an appropriate choice of a theory of multivalued differentials. Moreover, there exists a theory that includes all the others (the "multidifferentials", studied in [13]), and using this theory one gets the most general Maximum Principle, a version of which is stated in [14]. Finally, the approach based on using this theory is actually *simpler* than those using more restrictive theories and allows for a presentation with very few prerequisites other than basic calculus (on the level of the implicit and inverse functions) and standard real variables and functional analysis. (A few more sophisticated results, such as Sard's theorem, the Brouwer fixed-point theorem, and Rademacher's theorem on the a. e. differentiablity of Lipschitz functions will be used as well, but no detailed familiarity will be required.)

In the reading list provided below, the books [1], [6], and [7] contain versions of the classical maximum principle, and [2] contains F. Clarke's nonsmooth version. The books [5] and [8] are general introductions to control theory. A brief discussion of what is still missing to achieve complete unification will be given at the end of the lecture. The versions given in the papers [3] and [4] are examples of the kind of result *not* yet covered by the unified framework discussed in the lecture.

The papers by H. J. Sussmann in the reading list are available at the author's Web page: `http://www.math.rutgers.edu/~sussmann/currentpapers.html`.

Reading List

[1] L. D. Berkovitz, *Optimal control theory*, Springer-Verlag, New York, 1974.

[2] F. H. Clarke, *Optimization and nonsmooth analysis*, Wiley, New York, 1983.

[3] A. D. Ioffe and R. T. Rackafellar, *The Euler and Weierstrass conditions for nonsmooth variational problems*, Calc. Var. Partial Differential Equations **4** (1996), pp. 59–87.

[4] A. D. Ioffe, *Euler-Lagrange and Hamiltonian formalisms in dynamic optimization*, Trans. Amer. Math. Soc. **349** (1995), 801–817.

[5] V. Jurdjevic, *Geometric control theory*, Cambridge Univ. Press, 1997.

[6] E. B. Lee and L. Markus, *Foundations of optimal control theory*, Wiley, New York, 1990.

[7] L. Pontryagin, V. Boltianskii, R. Gamkrelidze, and E. Mishchenko, *The mathematical theory of optimal processes*, Wiley-Interscience, New York, 1962.

[8] E. D. Sontag, *Mathematical control theory*, Springer-Verlag, New York, 1990.

[9] H. J. Sussmann, *A strong version of the Lojasiewicz Maximum Principle*, Optimal Control of Differential Equations (N. H. Pavel, ed.), Dekker, New York, 1994.

[10] ____, *A strong version of the Maximum Principle under weak hypotheses*, Proc. 33rd IEEE Conf. Decision and Control (Orlando, FL, 1994), pp. 1950–1956.

[11] ____, *A strong maximum principle for systems of differential inclusions*, Proc. 35th IEEE Conference on Decision and Control (Kobe, Japan, 1996), IEEE Publications, 1996.

[12] H. J. Sussmann and J. C. Willems, *300 years of optimal control: From the Brachystochrone to the maximum principle*, IEEE Control Systems (1974), 32–44.

[13] H. J. Sussmann, *Multidifferential calculus: Chain rule, open mapping and transversal intersection theorems*, to appear in Optimal Control: Theory, Algorithms, and Applications (W. W. Hager and P. M. Pardalos, eds.), Kluwer, 1997.

[14] ____, *Geometry and optimal control*, to appear in Perspectives in Control, a book of essays in honor of Roger W. Brockett on the occasion of his 60th birthday (J. Baillieul and J. C. Willems, eds.), Springer-Verlag.