Wolfgang Heinrich Johannes Fuchs (1915–1997)

J. Milne Anderson, David Drasin, and Linda R. Sons

Biographical Sketch

Wolfgang Heinrich Johannes Fuchs was born in Munich on May 19, 1915. He joined the faculty of Cornell University in 1950, where he remained through his retirement in 1985 until his death on February 24, 1997. His life and career were characterized by an unrelentingly positive and supportive attitude. He read avidly (in many languages), travelled widely, and was devoted to intellectual dignity and the international mathematical community. He wrote two important monographs and more than sixty-five papers in complex function theory and related areas. These achievements were recognized by the award of three fellowships: Guggenheim (1955), Fulbright-Hays (1973), and Humboldt (1978).

Wolfgang graduated in 1933 from Johannes Gymnasium in Breslau (Wrocław), where his teacher, Hermann Kober (remembered for his *Dictionary of Conformal Representations*), convinced him to become a mathematician. Wolfgang's obituary of Kober, published in 1975, contains warm memories of those times. Outside school hours he studied Russian and Chinese.

Graduation occurred shortly after Hitler assumed power in Germany. Wolfgang's parents were classified as Jews, and they recognized at once that a normal life would be impossible in Germany. They arranged for Wolfgang to enter St. John's College, Cambridge, in the fall term of 1933 and were able to join him before war erupted in 1939.

At this time the predominant figures in analysis at Cambridge were G. H. Hardy and J. E. Littlewood, and Wolfgang soon came under their spell. He received a Ph.D. in 1941 under the direction of A. E. Ingham [10]; we discuss this work below.

In 1938 Wolfgang received a fellowship to Aberdeen, where he was fortunate to have W. W. Rogosinski, another refugee, as a colleague. They had known each other from Cambridge days and shared an interest in summability. Their collaboration intensified in the summer of 1940, when both were interred on the Isle of Man as "enemy aliens". He would later describe this period as a "beautiful summer vacation": there was a rich mathematical environment, and, since a chef of Buckingham Palace was another detainee, the food was rather good.

In 1943, while still at Aberdeen, he met and married Dorothee Rausch von Traubenberg, another refugee from Germany, who was a student at the university. "His relationship with her was the center of his being" [W. K. Hayman]. Dorothee came from a distinguished academic family. Her father was dismissed from a professorship at Kiel in 1937 for his pacifist beliefs, but, above all, for

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being married to a Jewish woman. He continued his research in atomic physics without a proper position, assisted only by his wife. However, his sudden death in 1944 left her unprotected, and Dorothee's mother was soon sent to Theresienstadt, where she was ordered to document this research. In this way her life was spared until the camp was liberated by the Red Army in 1945.

In time, Wolfgang's work (in particular, [12]) attracted the attention of R. P. Agnew, who was chairman of the Cornell mathematics department. Agnew invited Wolfgang to Cornell for 1948–49, where he accepted a permanent appointment.

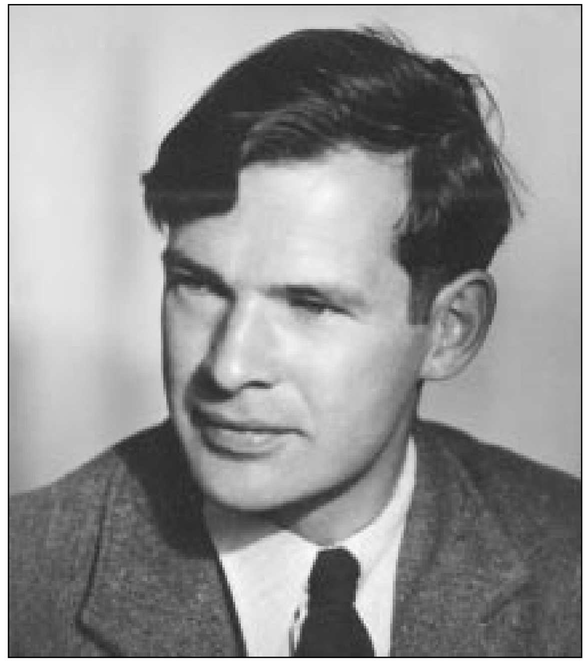
At Cornell Wolfgang carried out his research on Nevanlinna theory (value-distribution theory). This was the most sustained output of his career and is discussed below. While Nevanlinna theory was already well known among complex analysts, it was usually viewed as a tool rather than as a subject of its own. It had attracted almost no research in the U.S. before the early 1950s.

Wolfgang was drawn to the subject by Albert Edrei, of Syracuse University. Edrei had been trained by G. Pólya and M. Plancherel in Zürich and thus had a strong function-theoretic background. In 1950 Edrei, then at the University of Saskatchewan, heard I. J. Schoenberg lecture about his conjecture on the characterization of generating functions of "totally positive sequences". Within a few months Edrei applied Nevanlinna theory to settle this problem completely. This convinced him that many important and beautiful results were ripe for harvest, and in 1955, at a mathematics picnic at Fall Creek Park in Ithaca, Wolfgang agreed to join him in the chase.

Nevanlinna theory remained the main focus of both Edrei and Fuchs for the remainder of their careers. They, their students and co-authors, and other groups developing in China, England, Germany, and parts of the former Soviet Union entered a long golden age of one-variable value-distribution theory. Almost all subsequent work that dealt with functions of finite order used the formalisms and computational techniques that Edrei and Fuchs introduced. When Rolf Nevanlinna died, Wolfgang was the obvious choice to deliver the address devoted to Nevanlinna's theory at the memorial conference in 1981.

Edrei and Fuchs remained close friends until Wolfgang's death, and Edrei died the following year, on April 29, 1998.

Wolfgang was also anxious to study new ideas from others, and his monographs influenced a generation of complex analysts. Extremal length? In [20] P. Koosis writes, "The most accessible introduction is in W. Fuchs' little book" [16]. This "little book" also includes Mergelyan's solution to the weighted approximation problem and an attractive selection of ideas from potential theory.



Photograph courtesy of Dorothee Fuchs.

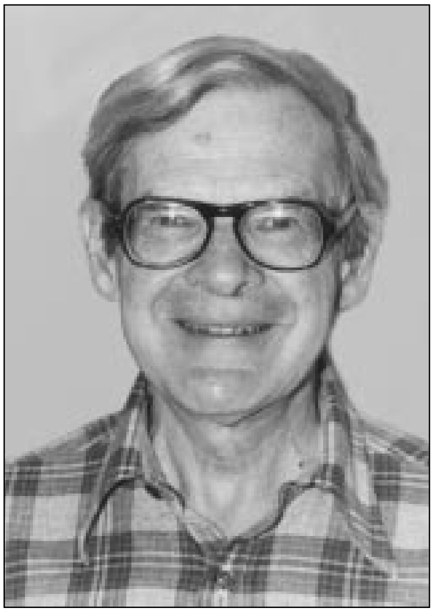
Wolfgang Fuchs, 1949.

He read widely and kept detailed notebooks with his own derivations and impressions from his reading. Frequently he would contribute these expositions to the original authors, and often this became what was published. This made him a valuable editor of several mathematical journals, notably the *Proceedings of the AMS*.

"My recollection is that [Wolfgang] always felt that one-variable value-distribution theory …was too narrow and needed infusion of the kind of geometric ideas advocated by Ahlfors. …[The] essence of his message was just to broaden the subject. At first I thought he was merely making a philosophical statement, but twice he wrote me complimenting me on specific things I said in my geometrical work. Although I believe he was unduly impressed with something rather trivial with which he happened not to be familiar, I came to understand that this advocacy was more than an empty gesture" [H. Wu].

"What impressed me is that he was still very eager to learn what was happening …, so he would take new papers and really work through them in all details" [W. Bergweiler, who was at Cornell in 1987–89 under Wolfgang's aegis].

Two of his many foreign contacts warrant special mention. In fall 1964 he participated in an official exchange with the USSR Academy of Sciences and attended an international conference in Erevan the following summer. There he intensified his contacts with several mathematicians in the outstanding schools of value-distribution theory and approximation theory. Soon after, he worked out Arakelyan's construction of functions of finite order with infinitely many Nevanlinna defects (available at that time only in a short *Doklady* note), and his 1967 Montreal lectures [18], with all details, became the standard reference for this work. Work by the Soviet mathematicians Keldysh, Mergelyan, and Goldberg, in addition to Arakelyan, occupy the major portion of [18]. His efforts to bring important work to the attention of Western mathematicians continued all his life. This orientation also led him to purchase gift memberships in the American Mathematical Society for several colleagues from abroad.


Fuchs in Berlin, October 1978.

"Mathematicians of the world can admire how in the years of the cold war he was building bridges between the divided spheres of mathematicians. Today it is hard to imagine …what was the common situation of fifteen years ago. [In those times] personal contacts were the privilege of only a very narrow circle. W. Fuchs understood that separation and mutual mistrust would only be detrimental to the science of mathematics. Many people understood this fact, but only a few took an active part in popularizing the achievements of Soviet mathematicians in the West" [A. A. Goldberg].

He was thrilled to make an official visit to China in 1980. The Cultural Revolution had ended only a few years earlier, and during that period no one in China could enter a scientific career. Wolfgang enthusiastically lectured and arranged for mathematics students to come to the U.S. to help restore mathematics in China. This was an important resource and contact for Chinese mathematicians, since function theory was one of the few active areas of mathematical research in China at that time (cf. [9]). "In simple words, the work done by Zhang [Guang-Hou] and me in the seventies was mainly based on the knowledge of French scholars [active before 1940] and the influence of Edrei and Fuchs's papers" [Yang Lo]. Wolfgang publicized the work of Yang, Zhang, and others and arranged for many mathematicians to visit the U.S. Thus he felt outrage at the massacre of June 1989 in Tiananmen Square and at once organized and arranged that the letter [1] be published in the *Notices*. "[It] was first suggested to me by [Wolfgang]. As a Chinese-American I would never have done it alone" [H. Wu]. His concern continued for the rest of his life: the letter [2] decried the reimprisonment of the dissident Wang Dan. Because of human rights considerations, he publicly declined later invitations to China and, on other occasions, to Israel.

Wolfgang was a charter member of the Ithaca chapter of Amnesty International and served as coordinator. "Our group had strong links with the international scientific community, and Wolfgang took an active role in establishing contacts with scientists in Eastern Europe, the Palestinians in the West Bank, and elsewhere, including such well-known dissidents as Andrei Sakharov and Yuri Orlov. He continued to attend meetings and support Amnesty activities long after the state of his health would have justified slowing down" [Peter Wetherbee].

He contributed a poem at the 1985 conference to celebrate de Branges's proof of the Bieberbach conjecture. It is the closing item in the published proceedings (1986), and the way he describes the history of the problem and its solution displays and preserves some of the charm his colleagues and friends long appreciated. "My first encounter with Wolfgang Fuchs changed my life. I visited Cornell in 1965 to consider an offer from the mathematics department. One evening in Wolfgang's home convinced me that it would be a privilege to live and work in the same community as this wonderfully wise, kind, and witty man" [Clifford Earle, 1997].

Some Mathematical Accomplishments

We describe some aspects of Wolfgang's research that display his breadth, insight, power, and influence in analysis. A full bibliography and discussion appear in a special issue of *Complex Variables* [3] dedicated to him and Edrei.

Thesis

Many theorems about entire and meromorphic functions are obtained by comparing growth rates of appropriate increasing real-valued functions. If f is entire, the most common such function associated to f is the *maximum modulus*

$$M(r) = M(r, f) \equiv \max_{\theta} |f(re^{i\theta})|,$$

but for given $p > 0$ we could as well consider $M_p(r, f)$, the L_p-mean of f on $\{|z| = r\}$. Wolfgang's thesis [10] confirmed a remarkable conjecture by Ingham: when $p \neq \infty$, M_p itself is almost an analytic function.

> Let f and g be analytic in $\{r_1 < |z| < r_2\}$ and suppose for a fixed $0 < p < \infty$ we have $M_p(r, f) = M_p(r, g)$ for a sequence of r with limit point in (r_1, r_2). Then $M_p(r, f) \equiv M_p(r, g)$ for $r_1 < r < r_2$.

This theorem completely fails when $p = \infty$. Hayman considers this and [8], written with Erdős, his favorites. The theorem is not hard to prove when f and g have no zeros, and "the proof of the analyticity [in r of the L_p mean] across the modulus of a zero is a brilliant and subtle piece of work" [Hayman].

Nevanlinna Theory

Wolfgang's greatest impact on American mathematics came from his work on Nevanlinna theory. Nevanlinna developed his theory in the 1920s as a potential-theoretic analysis of Picard's theorem (1879), which asserts that a nonconstant meromorphic function in the plane cannot omit three values. The obvious example $f(z) = e^z$ shows that the theorem is sharp. For the next fifty years, Borel, Valiron, and others attempted to find more insightful proofs. Not only was Nevanlinna's approach the most successful, but his techniques became standard in potential theory and the foundation for a subject of its own. Thus, let f be meromorphic in the plane. If $0 < r < \infty$ and $n(r,a)$ is the number of solutions to the equation $f(z) = a$, with $|z| < r$, account being taken of multiplicities, we set

$$N(r,a) = \int^r n(t,a)t^{-1}\,dt$$

(this is slightly modified when $f(0) = a$). Nevanlinna's *characteristic* $T(r) = T(r,f)$ can be defined as

$$T(r) = \int_{\hat{C}} N(r,a)\,d\mu(a),$$

where μ is the uniform distribution on the Riemann sphere $\hat{C}$, and the *deficiency* $\delta(a) = \delta(a,f)$, $a \in \hat{C}$, is

$$\delta(a) = 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r)}.$$

It is elementary that $T(r) \uparrow \infty$ and $0 \le \delta(a) \le 1$. Of course, $\delta(a) = 1$ if the equation $f(z) = a$ has no solutions. Nevanlinna's famous Second Fundamental Theorem implies that

$$\sum_a \delta(a) \le 2, \tag{1}$$

a deep generalization of the Picard theorem. Nevanlinna theory in the plane asks for refinements of (1), given other properties of f. Nevanlinna's $T(r)$ plays the role of the maximum modulus $M(r)$ in the special case that f is entire; we then simply have $\delta(\infty) = 1$. The standard references are [18] and [19].

Nevanlinna's key insight was his "lemma of the logarithmic derivative", which states that for most large r,

$$\int_0^{2\pi} \log^+ \left| \frac{f'}{f}(re^{i\theta}) \right| d\theta = o(\log(rT(r))). \tag{2}$$

Thus, the left side of (2) is negligible when compared to $T(r)$. Since in (2) we may replace f by $f - a$ for any complex number a, (2) indicates that if f is very close to a complex value a on a portion I of $\{|z| = r\}$, then f' must also be small on I.

Wolfgang's first works revisited the expression (2) in a direct manner, by estimating

$$M(I,F) \equiv \int_I \left| \frac{f'}{f}(z) \right| |dz|, \tag{3}$$

where I is any subarc of $\{|z| = r\}$. This integral is far more treacherous than (2); in fact, it diverges whenever I contains a zero or pole of f. His estimates appear in [14] and [15]; later Petrenko found the sharpest bounds.

Wolfgang obtained two striking applications from his estimates. To explain them, we need the notion of the *order* ρ of a meromorphic function f:

$$\rho = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}. \tag{4}$$

For example, $\exp(z^k)$ has order k. In [14] Wolfgang showed that when $\rho < \infty$, (1) alone does not describe the full situation: in addition, we have that

$$\sum \delta^{1/2}(a) < \infty, \tag{5}$$

which confirmed a conjecture of Teichmüller. Wolfgang considered this and the paper [8] (with Erdős, discussed below) his two best. Some years later, Weitsman, following insights of Hayman, showed that $1/3$ is the optimal exponent in (5). That (5) reflects special properties of functions of finite order became clear when, with Hayman, Wolfgang showed (cf. [18], Chapter 5, and [19], Chapter 4) that Nevanlinna's defect relation (1) is sharp among all entire functions.

Paper [15] proved a conjecture that G. Pólya posed in his famous paper [23]. Let f be entire and of order $\rho < \infty$ with power series development

$$f(z) = \sum a_k z^{n_k}. \tag{6}$$

If $n_k/k \to \infty$ as $k \to \infty$, then

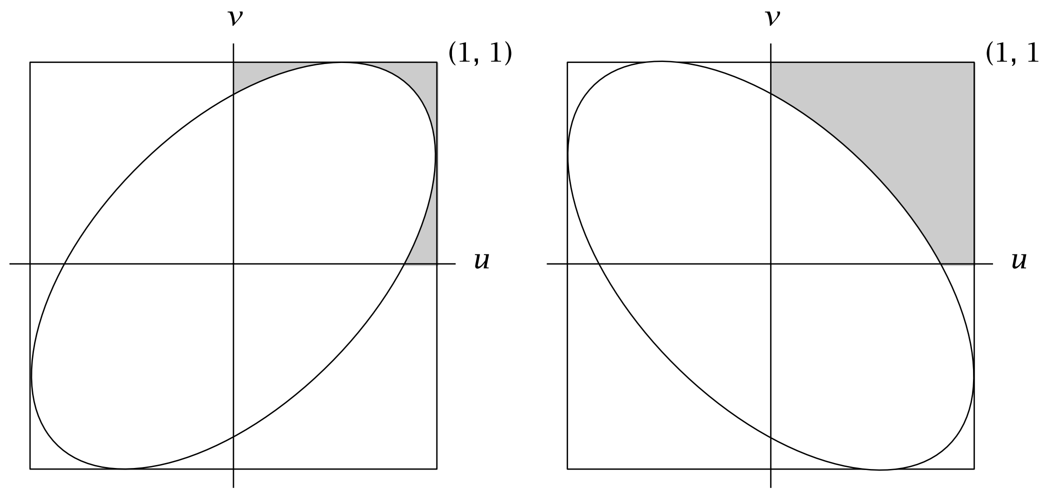
$$\limsup_{r\to\infty} \frac{L(r,f)}{M(r,f)} = 1,$$

where L and M are, respectively, the minimum and maximum modulus of f on $\{|z| = r\}$. Thus, in a very precise sense, these gap series behave as monomials for arbitrarily large values of r.

The first joint work of Edrei and Fuchs ([5] and [6]) completely characterized entire functions f of finite order for which $\sum \delta(a) = 2$; i.e., equality holds in (1). Not only is ρ a positive integer (this was shown earlier by Pfluger), but the global asymptotic behavior and Taylor expansion of f are completely described. In particular, only a finite number of nonzero terms can appear in the deficiency sum (1). These conclusions were obtained as limiting cases when the difference $2 - \sum \delta(a)$ is sufficiently small. One significant problem arising from this work remains open to this day: If ρ is "close to" an integer k and $\sum \delta(a)$ is "nearly" equal to 2, can there be only finitely many nonzero terms in (1)?

Their paper [7] introduced two major ideas that have transformed much future research. First, they made a serious study of "Pólya peaks" and showed that these peaks gave a new, elegant, and unified way to interpret the hypothesis that the order ρ in (4) be finite. This led to the common principle that a function f should be studied by comparing

Figure 1. The possible values of (u,v) must always be inside the first quadrant. According to the "ellipse theorem", they are also limited to the portion outside the ellipse $u^2+v^2-2uv\cos\pi\rho=\sin^2\pi\rho$. The shaded regions here show the set in question for $\rho=.33$ and $\rho=.7$.



its characteristic $T(r)$ to simple local comparison functions defined intrinsically in terms of $T(r)$. Before [7] authors were forced to create many ad hoc comparisons between $T(r)$ and r^ρ, but after [7] most of these notions were forgotten.

The derivation by Edrei and Fuchs in [7] of a key inequality of Goldberg provided an essential foundation on which A. Baernstein later could build his $*$-function. The $*$-function continues to have a major impact on symmetrization and geometric function theory.

While the impact of [7] on later work was enormous, its main conclusion should not be ignored. The "ellipse theorem" gives a complete relation between any two terms that appear in the deficiency sum (1) and the order ρ of a function f when $0\le\rho\le 1$. Thus let a and b be fixed in $\hat{C}$, and set $u=1-\delta(a)$, $v=1-\delta(b)$. Any understanding of the pair (u,v) sheds light on any two terms appearing in (1). It is clear that (u,v) is always confined to the square $0\le u,v\le 1$. Edrei and Fuchs prove that, in addition, the point (u,v) must lie on or outside the ellipse

$$u^2+v^2-2uv\cos\pi\rho=\sin^2\pi\rho$$

and this condition is best possible in all cases. See Figure 1.

This result is nearly forty years old. Except in some trivial cases, when $\rho>1$ there is no complete description of the possibilities of $\{(u,v)\}$ as f ranges over all meromorphic functions of order ρ. For entire functions, the "trivial case" occurs when ρ is an integer k, in which case (u,v) can lie anywhere in the square, with $f_k(z)=\exp(z^k)$ being extremal.

Closure Problems

Although Wolfgang wrote fifteen papers during his stay in Britain, he is probably best remembered today for his paper [11] on the closure of the functions $\{e^{-t}t^{a_\nu}\}$ in $L_2(0,\infty)$. The subject begins with Weierstrass's theorem that polynomials are dense in $C[a,b]$. H. Müntz proved in 1914 that if S is any subset of the positive integers, then the linear span of $\{t^m;\, m\in S\}$ is dense in $L_2(0,1)$ if and only if $\sum_S m^{-1}=\infty$. This result suggested that approximation of rather general functions might be possible by using specific subclasses that have attractive structures and led to a wide development. A more refined analysis is needed in [11]; on the finite interval $(0,1)$ it corresponds to a study of the closure of $\{(\log(1/x))^{a_\nu}\}$. In addition, the possibility that the a_ν's are positive but not necessarily integers raises many technical complications. Let

$$\psi(r)=2\sum_{a_\nu<r}a_\nu^{-1} \tag{7}$$

for $r>a_1$. Wolfgang's theorem shows that the system $\{e^{-t}t^{a_\nu}\}$ is complete in $L_2(0,\infty)$ if and only if

$$\int_{a_1}^{\infty}[\exp\psi(r)]r^{-2}\,dr=\infty.$$

Roughly speaking, this says that if $a_\nu\sim\alpha\nu$ as $\nu\to\infty$, then the system is complete if $\alpha\le 1/2$ and incomplete if $\alpha>1/2$.

The basic problem is that the integral may converge without the $\{a_\nu\}$ satisfying the Blaschke condition $\sum a_\nu^{-1}<\infty$. This necessitates consideration of a function of the form

$$H(z)=\prod_{\nu=1}^{\infty}\left(\frac{z-\lambda_\nu}{z+\lambda_\nu}\exp(-2z/a_\nu)\right) \tag{8}$$

to cancel out the zeros of a certain function $G(z)$, analytic in the right half-plane. The paper, written before the contribution of functional analysis to closure problems in complex analysis was fully appreciated, involves a highly sophisticated application of the Ahlfors Distortion Theorem and shows Wolfgang already at the height of his analytical powers.

Although (8) is regular only in the right half-plane, Wolfgang uses an inverse integral transform to obtain the desired function orthogonal to the family $\{e^{-t}t^{a_\nu}\}$.

The product (8) itself has many uses. It provided a key ingredient for [12], which was so admired by

Ph.D. Students of Wolfgang Fuchs:
Tseng-Yeh Chow (1953)
Alan Schumitzky (1965)
Linda R. Sons (1966)
David Drasin (1966)
Virginia W. Noonburg (1967)
M. A. Selby (1970)
I-Lok Chang (1971)
Subinoy Chakravarty (1975)

Agnew. Let f be of exponential type k (we write $f \in \mathcal{E}_k$): this means that $\log M(r) = O(kr)$ as $r \to \infty$, with $M(r)$ equal to the maximum modulus as above. Let us call a sequence $\{a_\nu\}$ of positive real numbers a *determining sequence corresponding to* $\mathcal{E}_k$ if the conditions $a_{\nu+1} - a_\nu > c > 0$ for all ν, $f \in \mathcal{E}_k$, and $f(a_\nu) = 0$ for $\nu = 1, 2, \dots$ imply that $f \equiv 0$. The most famous theorem of this type is due to F. Carlson: If $k < \pi$, then $\{\nu; \nu \geq 0\}$ is a set of uniqueness for $\mathcal{E}_k$, and the example $f(z) = \sin \pi z$ shows that this bound on k is exact. The contribution of [12] is to give a condition both necessary and sufficient for any k: If ψ is constructed as in (7) from the $\{a_\nu\}$, then

$$\limsup_{r\to\infty} \psi(r) r^{-2k/\pi} = \infty.$$

Several other papers are in this vein and are extensively discussed in the monographs of Mandelbrojt [22] and Boas [4]. These problems, with more general weights than e^{-t}, were also considered in the thesis of Malliavin, who related them to "Watson's problem". In 1967 Wolfgang showed that his original approach led to an elegant solution to one result in Malliavin's thesis.

In the 1950s Malliavin carried the ideas of (8) much further and deduced the converse to Pólya's maximal density theorem concerning gap series. In [23] Pólya had proved that if a power series of the form (6) has radius of convergence one and the (Pólya) maximal density of the nonzero coefficients in (6) is D, then every arc of $\{|z| = 1\}$ of length greater than $2\pi D$ contains a singularity of f. This richly amplifies the well-known fact that the circle of convergence of any power series has at least one singularity.

While the precise definition of maximal density is too complicated to be reproduced here, any subset of integers does have such a density (this density is defined in terms of a lim sup). Thus it is natural to ask if the Pólya density is the precise notion needed to guarantee Pólya's theorem. Malliavin was influenced by [13] to develop an extensive theory, which among other things showed that Pólya's notion of density was exact. In [20], Chapter 9, Koosis uses Malliavin's arguments to establish this converse directly from [13]. This discussion also provides an exhaustive explanation of the significance of products such as (8).

Additive Number Theory

Erdős shared Wolfgang's enthusiasm for their joint paper [8]: "[It] certainly will survive the authors by a few centuries" (quoted in [24]). An excellent exposition is in [21], Chapter II.

Thus, let $A = \{a_k\}$ be a nondecreasing sequence of nonnegative integers, and for $n \in \mathbb{Z}$ let $r(n; A)$ be the number of solutions to the inequality $a_i + a_j \leq n$ with $a_i, a_j \in A$, using any consistent method of enumeration. Special techniques are available when $A = \mathcal{Q} = \{m^2,\ m \geq 0\}$; in this case $r(n; A)$ is simply the number of points of the integral lattice in $\{|z| \leq n^{1/2}\}$, and so $r(n; \mathcal{Q}) \sim \pi n$. In classical work dating back to Hardy in 1915, it was shown that this asymptotic relation cannot be attained too rapidly: when $A = \mathcal{Q}$ and $c = \pi$, then

$$\limsup_{n\to\infty} \frac{|r(n;A) - cn|}{\Phi(n)} > 0, \tag{9}$$

where $\Phi(n) = \{n \log n\}^{1/4}$.

These arguments were heavily based on the interpretation of $r(n; A)$ when $A = \mathcal{Q}$. The contribution of [8] is that such limitations are, in H. Halberstam's words from 1988, "a law of nature." In fact, if A is any such sequence, then there is a universal $\Phi(n) \uparrow \infty$ such that (9) must hold for any $c > 0$. Of course, if we allow $c = 0$, then a sufficiently sparse A allows that $r(n; a) n^{-1} \to 0$ as rapidly as desired. Erdős-Fuchs show that $\Phi(n) = n^{1/4} \log^{-1/2} n$ gives (9) for any A.

References

[1] W. Arveson et al., *Letter*, Notices Amer. Math. Soc. **37** (1990), 263.
[2] R. Askey et al., *Letter*, New York Rev. of Books **44** (March 27, 1997), 50.
[3] K. F. Barth and D. F. Shea, eds., *Complex variables*, Special Issue Dedicated to Albert Edrei and Wolfgang Heinrich Johannes Fuchs, vol. 12, 1989.
[4] R. P. Boas, *Entire functions*, Academic Press, New York, 1954.
[5] A. Edrei and W. H. J. Fuchs, *On the growth of meromorphic functions with several deficient values*, Trans. Amer. Math. Soc. **33** (1959), 292–328.
[6] ____, *Valeurs déficientes et valeurs asymptotiques des fonctions méromorphes*, Comment. Math. Helv. **33** (1959), 258–295.
[7] ____, *The deficiencies of meromorphic functions of order less than one*, Duke Math. J. **27** (1960), 233–250.
[8] P. Erdős and W. H. J. Fuchs, *On a problem of additive number theory*, J. London Math. Soc. **31** (1956), 67–73.
[9] W. Feit, *A mathematical visit to China*, Notices Amer. Math. Soc. **24** (1977), 110–113.
[10] W. H. J. Fuchs, *A uniqueness theorem for mean values of analytic functions*, Proc. London Math. Soc. **48** (1945), 35–47.
[11] ____, *On the closure of* $\{e^{-t} t^{a_\nu}\}$, Proc. Cambridge Philos. Soc. **42** (1946), 91–105.
[12] ____, *A generalization of Carlson's theorem*, J. London Math. Soc. **21** (1946), 1057–1059.
[13] ____, *On the growth of functions of mean type*, Proc. Edinburgh Math. Soc. Ser. **29** (1954), 53–70.

[14] ____, *A theorem on the Nevanlinna deficiencies of meromorphic functions of finite order*, Ann. of Math. **68** (1958), 203–209.

[15] ____, *Proof of a conjecture of Pólya concerning gap series*, Illinois J. Math. **l7** (1963), 661–667.

[16] ____, *Topics in the theory of functions of one complex variable*, Van Nostrand, Princeton, 1967.

[17] ____, *Théorie de l'approximation des fonctions d'une variable complexe*, Université de Montréal, 1968.

[18] A. A. Goldberg and I. V. Ostrovskii, *Distribution of values of meromorphic functions* (Russian), Nauka, Moscow, 1970.

[19] W. K. Hayman, *Meromorphic functions*, Oxford Univ. Press, Oxford, 1964.

[20] P. Koosis, *The logarithmic integral, II,* Cambridge Univ. Press, Cambridge, 1992.

[21] H. Halberstam and K. F. Roth, *Sequences, Vol. I*, Oxford Univ. Press, Oxford, 1966.

[22] S. Mandelbrojt, *Séries adhérentes, régularison des suites, applications*, Gauthier–Villars, Paris, 1952.

[23] G. Pólya, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*, Math. Z. **29** (1929), 549–640.

[24] C. Pomerance, *Paul Erdős: Number theorist extraordinaire*, Notices Amer. Math. Soc. **45** (1998), 19–23.