The Mathematical Work of the 1998 Fields Medalists

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The Work of Richard E. Borcherds

James Lepowsky

Much of Richard Borcherds's brilliant work is related to the remarkable subject of Monstrous Moonshine. This started quietly in the 1970s when A. Ogg noticed a curious coincidence spanning two apparently unrelated areas of mathematics—the theory of modular functions and the theory of finite simple groups.

There are fifteen prime numbers p for which the normalizer of the congruence subgroup $\Gamma_0(p)$ in $SL(2,\mathbb{R})$ has the "genus-zero property"; that is, the compactification of the upper half-plane modulo this normalizer is a Riemann surface of genus zero, so that the field of modular functions invariant under this discrete group is generated by only one function (a *Hauptmodul*). The surprise was that these coincide with the fifteen primes that divide the order of the "Monster" sporadic finite simple group $\mathbb{M}$, a group of order about 10^{54} "discovered" by B. Fischer and R. Griess but not yet proved to exist at the time. Ogg offered a bottle of Jack Daniels whiskey for an explanation.

In 1978–79, J. McKay, J. Thompson, J. H. Conway, and S. Norton explosively enriched this numerology [CN] and in particular conjectured the existence of a natural infinite-dimensional $\mathbb{Z}$-graded representation (let us call it $V^\natural = \bigoplus_{n\geq -1} V_n^\natural$) of the conjectured group $\mathbb{M}$ that would have the following property: For each of the 194 conjugacy classes in $\mathbb{M}$, choose a representative $g \in \mathbb{M}$, and consider the "graded trace" $J_g(q) = \sum_{n\geq -1}(\mathrm{tr}\, g|_{V_n^\natural})q^n$, where $q = e^{2\pi i\tau}$, τ in the upper half-plane. Then the *McKay-Thompson series* J_g should be a (specified) Hauptmodul for a suitable discrete subgroup of $SL(2,\mathbb{R})$ with the genus-zero property, and, in particular, J_1 (corresponding to $1 \in \mathbb{M}$) should be the modular function $J(q) = q^{-1} + 196884q + \cdots$. This existence was soon essentially (and nonconstructively) proved by Thompson, A. O. L. Atkin, P. Fong, and S. Smith, and the problem was to uncover the deeper story.

Griess [Gr] then proved the existence of the Monster by constructing it as an automorphism group of a remarkable new algebra of dimension 196884. Later I. Frenkel, J. Lepowsky, and A. Meurman gave a construction, incorporating a vertex operator realization of the Griess algebra, of a "moonshine module" $V^\natural$ for $\mathbb{M}$ whose McKay-Thompson series for $1 \in \mathbb{M}$ was indeed $J(q)$. Only some, and far from all, of the McKay-Thompson series for this structure $V^\natural$ could be computed directly. This construction was reinterpreted by physicists, during the resurgence of string theory in the mid-

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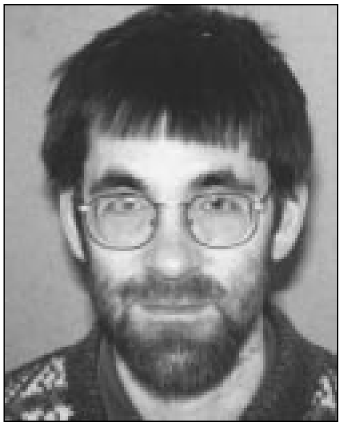

Richard E. Borcherds

1980s, as a "toy model" physical theory of a 26-dimensional bosonic string compactified on a 24-dimensional toral "orbifold" associated with the Leech lattice. Thus the Monster turned out to be the symmetry group of an idealized physical theory. The term "vertex operator" comes from the early days of string theory, when operators of this type were used to describe interactions at a "vertex". Affine Lie algebras were constructed via what turned out to be certain variants of physical vertex operators.

Then came a penetrating insight of Borcherds: He introduced his axiomatic notion of "vertex algebra" [B1] and perceived among many other things that the moonshine module could be endowed with an $\mathbb{M}$-invariant vertex algebra structure. The concept of vertex algebra is a mathematically precise algebraic counterpart of the concept of "chiral algebra" in two-dimensional conformal quantum field theory as formalized by A. Belavin, A. Polyakov, and A. Zamolodchikov (a physical theory foundational in string theory and in two-dimensional statistical mechanics). This fundamental notion reflects deep features of the traditional notions of commutative associative algebra and *at the same time* of Lie algebra. A vertex operator algebra structure (a variant of vertex algebra structure) on $V^\natural$ was given in [FLM] and allowed the possibility of characterizing $V^\natural$, by a still-unproved conjecture, as the unique vertex operator algebra satisfying a short list of natural conditions. The Fischer-Griess Monster, then, is the automorphism group of a (conjecturally) unique new kind of mathematical object. The nonclassical flavor of the theory of vertex algebras in mathematics can be thought of as analogous to the nonclassical flavor of string theory in physics.

Borcherds was meanwhile developing his theory of generalized Kac-Moody algebras [B2], now also called "Borcherds algebras". Kac-Moody algebras form a very important class of Lie algebras generalizing the class of finite-dimensional semisimple Lie algebras. Outside the fundamental class of affine Lie algebras, the infinite-dimensional Kac-Moody algebras have been notoriously difficult to construct concretely. Borcherds had the insight to study systematically the phenomenon of "imaginary simple roots", and the resulting algebras encompass a wide variety of striking examples whose root multiplicities Borcherds determined completely and which he in fact constructed directly from suitable vertex algebras. He established what are now called the "Weyl-Kac-Borcherds character" and "denominator formulas" for these algebras. Most importantly, he made the fundamental discovery that for suitable families of examples, the root multiplicities are exactly the coefficients of certain automorphic forms.

Borcherds's remarkable achievement concerning moonshine followed, in his strikingly original proof [B3] that all of the McKay-Thompson series for $V^\natural$ do in fact agree with the 194 series written down by Conway and Norton and in particular satisfy the genus-zero property; that is, the Conway-Norton conjecture holds for $V^\natural$. His strategy was to tensor $V^\natural$ with a rank-two vertex algebra to form a rank-26 vertex algebra on which $\mathbb{M}$ acts canonically, and he drew on a rich variety of ideas, among them ideas from vertex algebra theory, the theory of Borcherds algebras (particularly his singularly interesting "Monster Lie algebra"), string theory (especially, critical 26-dimensional string theory and the "no-ghost theorem" of R. C. Brower, P. Goddard, and C. Thorn), and modular function theory. He established a twisted denominator formula for the Monster Lie algebra by exploiting the homology of a suitable subalgebra, and he concluded that the series for $V^\natural$ satisfy the "replication formulas" of Conway-Norton and thus, as a result of a verification of initial data, agree with the Conway-Norton series.

The fact that the root multiplicities of the Monster Lie algebra are the coefficients of $J(q)$ and the relation of this fact to the denominator formula are just the tip of an iceberg: When Borcherds pursued this idea for a wide range of Borcherds algebras, he discovered a powerful and unexpected correspondence between certain classical modular functions and meromorphic modular forms associated with arithmetic subgroups of $SO(n, 2)$. The resulting infinite product expansions led to striking new results on moduli spaces of certain varieties. J. Harvey, G. Moore, Borcherds, and others have developed potentially far-reaching connections with mirror symmetry and string duality.

Borcherds's insights have influenced a wide range of works. For example, the value of having a conceptual notion of vertex (operator) algebra has been immense. It becomes possible to formulate new questions and to address new problems. Here are some notable examples: the initiation of a program to construct (geometric) conformal field theory using vertex operator algebras (I. Frenkel), solution of the problem of constructing "tree-level" conformal field theory in the sense of G. Segal and M. Kontsevich (Y.-Z. Huang), a vertex-operator-algebra-theoretic proof of modular transformation properties of "characters" of modules (Y. Zhu), and a natural approach to the construction of vertex (operator) algebras and their modules (developed systematically by H. Li and others).

The deepest mysteries of moonshine are still not fully resolved. Some notable works in this direction are Borcherds's and A. Ryba's investigation of moonshine over finite fields and work of C. Dong,

H. Li, and G. Mason on Norton's generalized moonshine conjectures.

Only part of Borcherds's important work has been touched on here. Discussions and treatments of many facets of his accomplishments and his ideas can be found in, for example, [FLM], [Ge], [JLW], [K], and Goddard's and Borcherds's talks at the International Congress [Go], [B4]. These works include listings of many of Borcherds's papers.

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The Work of William Timothy Gowers

Joram Lindenstrauss

William Timothy Gowers has worked in two areas: Banach space theory and combinatorics. The main tools he used in his work in Banach space theory are also combinatorial in nature. I shall present here four of his main research achievements.

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William Timothy Gowers

1. Counterexamples to the main open problems on the structure of infinite-dimensional Banach spaces.

It has become clear in recent decades that there is a rich structure theory for Banach spaces of a high finite dimension, as exemplified by the theorem of A. Dvoretzky on the existence of almost Euclidean sections. On the other hand, progress on the structure theory of infinite-dimensional Banach spaces was rather slow till recently, and many natural problems (most of them going back to Banach and his school in the 1930s) remained open.

Let me recall some background material. A sequence of vectors $\{x_i\}_{i=1}^{\infty}$ is said to be a (Schauder) *basis* of a Banach space X if every $x \in X$ has a unique representation of the form $x = \sum_{i=1}^{\infty} a_i x_i$ with a_i scalars. The basis is called *unconditional* if for every choice of signs $\theta = \{\theta_i\}_{i=1}^{\infty}$ the series $\sum_{i=1}^{\infty} \theta_i a_i x_i$ converges whenever $\sum_{i=1}^{\infty} a_i x_i$ does (or equivalently, the operator

$$T\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^{\infty} \theta_i a_i x_i$$

is bounded). In the common separable Banach spaces it is quite easy to find bases, and it is also easy to prove that every infinite-dimensional Banach space has an infinite-dimensional subspace with a basis. A famous result of P. Enflo is that not every separable Banach space has a basis (and in fact does not even have the so-called approximation property which is implied by the existence of a basis). As for unconditional bases, it is not hard to prove that several common spaces (like $L_1(0,1)$ or $C(0,1)$) fail to have unconditional bases. It was an open problem whether every infinite-dimensional Banach space has an infinite-dimensional subspace with an unconditional basis. For a long time it was hoped that every infinite-dimensional Banach space might even have the stronger property of containing a subspace isomorphic to c_0 or to ℓ_p for some $1 \le p < \infty$ (in some sense this could have been viewed as an infinite-dimensional version of Dvoretzky's theorem). This hope was put to rest by B. S. Tsirelson, who constructed a reflexive space not containing a subspace isomorphic to ℓ_p for any $1 < p < \infty$. His construction had the remarkable feature that the space (or the norm) was not defined explicitly by some formula (as were all examples of Banach spaces till then), but rather by an implicit equation. (All the examples mentioned below are also constructed by a procedure of this type.) All this was known by the mid-1970s.

The recent development was started by a paper of T. Schlumprecht, who modified Tsirelson's example and produced a space that is "arbitrarily distortable" (I do not define this notion, since it will not be needed in the sequel). Using Schlumprecht's result, Gowers and B. Maurey [7] constructed a separable Banach space Y that does not have any infinite-dimensional subspace with an unconditional basis. In fact, this space Y turned out to have a stronger property: If Z is any subspace of Y, then any bounded linear projection P on Z is trivial (i.e., either $\dim PZ < \infty$ or $\dim Z/PZ < \infty$). If a space Z has an unconditional basis $\{z_i\}_{i=1}^{\infty}$, then there are many nontrivial bounded projections on Z (for any subset M of the integers

$$P_M\left(\sum_{i=1}^{\infty} a_i z_i\right) = \sum_{i \in M} a_i z_i$$

is a bounded linear projection). A space Y with such a property was called in [7] *hereditarily indecomposable* (H.I.). Before [7] it was a well-known problem whether there exists at all an infinite-dimensional Banach space X that cannot be represented as a direct sum $X = X_1 \oplus X_2$ with $\dim X_1 = \dim X_2 = \infty$ (i.e., on which there is no nontrivial bounded linear projection).

H.I. spaces have many additional unexpected properties. As shown in [7], if Y is an H.I. space and T is a bounded linear operator from Y into itself, then $T = \lambda I_Y + S$ for some scalar λ and a strictly singular S (an operator is called *strictly singular* if its restriction to any infinite-dimensional subspace is not an isomorphism). Thus every such T is either a Fredholm operator with index 0 (if $\lambda \neq 0$) or is strictly singular and thus not Fredholm (if $\lambda = 0$). Consequently, an H.I. space is not isomorphic to any of its proper subspaces. This solves in particular the classical "hyperplane problem", which asks whether any infinite-dimensional Banach space is isomorphic to its hyperplanes (it is trivial that all hyperplanes are mutually isomorphic, and it is easy to see that the common infinite-dimensional spaces are isomorphic to their hyperplanes). The hyperplane problem was actually first solved by Gowers [1] before it became clear that the example in [7] does the same. The example in [1] does have an unconditional basis and other interesting properties.

It is a relatively easy result due to R. C. James that a space with an unconditional basis is either reflexive or it contains a subspace isomorphic to c_0 or ℓ_1. A considerable amount of work was done on the question of whether every infinite-dimensional Banach space has a subspace that is either reflexive or c_0 or ℓ_1. Several promising positive results were found on this question. However, Gowers [2] showed that in general the answer is negative and thus gave a stronger version of the result in [7] on unconditional bases.

Another well-known open problem in Banach space theory was the following: Assume that X and Y are Banach spaces, and assume that each of them is isomorphic to a complemented subspace (i.e., a subspace on which there is a bounded linear projection) of the other. Must X be isomorphic to Y? With mild additional assumptions it was known that the answer is positive (and this has many applications). Gowers [3] showed, however, that in general the answer is negative. He produced a Banach space X so that X is isomorphic to $X \oplus X \oplus X$ but not to $X \oplus X$.

In [8] Gowers and Maurey develop a general method for constructing Banach spaces for which a certain given class of linear maps (say, a shift in a sequence space) are bounded but so that any bounded linear operator on the space essentially belongs to the algebra generated by those given operators. This method has far-reaching implications and will certainly be of much use in the future. It is shown in [8] how to derive the results of [1] and [3] from this general method.

2. The dichotomy theorem.

In [4] Gowers proved a general dichotomy theorem for Banach spaces which, in particular, says the following: Every infinite-dimensional Banach space has a subspace that is either an H.I. space (i.e., a space with very few operators) or a space with an unconditional basis (i.e., a space with a very rich structure). The proof is combinatorial, using infinite Ramsey theory. The theory of finite-dimensional Banach spaces mainly uses arguments based on measure (volume) and probability. These tools are not naturally available in the infinite-dimensional setting. Ramsey type arguments turn out to be an important tool in the infinite-dimensional setting, where they can sometimes replace arguments using measure.

The dichotomy result makes it clear that H.I. spaces are of importance in the structure theory of general Banach spaces (and not just pathological counterexamples). The dichotomy result led immediately to the solution (this time in the positive direction) of the classical "homogeneous space" problem. A Banach space X is called *homogeneous* if it is isomorphic to all its infinite-dimensional subspaces. A short time before the dichotomy result was proved, R. Komorowski and N. Tomczak-Jaegermann proved the following result: A Banach space X of "finite cotype" that does not contain a subspace isomorphic to ℓ_2 must have a subspace without an unconditional basis. By combining this result with the dichotomy theorem, one deduces that the only homogeneous Banach space is ℓ_2. Indeed, if X is homogeneous it must have "finite cotype" by known arguments related to the approximation property. Hence, if it is not isomorphic to ℓ_2 (and thus does not contain ℓ_2), it must contain (and thus be) an H.I. space. But H.I. spaces are not isomorphic to any proper subspace of themselves and thus are certainly not homogeneous.

3. Szemerédi's theorem.
The theorem states the following: If $\delta > 0$ and an integer k are given, then there is an $N(k, \delta)$ so that every subset of $\{1, 2, \cdots, n\}$ containing more than δn elements must contain an arithmetic progression of length k whenever $n \geq N(k, \delta)$. The theorem was proved for $k = 3$ by K. Roth using tools from analytic number theory. For $k > 3$ the theorem was proved first by E. Szemerédi using an extremely intricate and ingenious combinatorial argument. Some years later another proof was found by H. Fürstenberg using the structure theory of ergodic measure preserving transformations.

The proof of Roth gave a reasonable estimate for $N(3, \delta)$. The proof of Szemerédi gave an enormous bound for $N(k, \delta)$. One reason for this is that Szemerédi used in his proof the (much weaker) Van der Waerden theorem. This theorem (in its qualitative form) states that in any partition of the integers into two sets one of those sets must contain arbitrarily long arithmetic progressions. The original proof of this result gave (in its quantitative form) huge upper bounds—what the logicians call an Ackerman function. A more recent proof of Van der Waerden's theorem, due to S. Shelah, gave a much improved upper bound which was, however, still enormous. Fürstenberg's proof of Szemerédi's theorem gave no estimate on $N(k, \delta)$.

By using the basic approach of Roth, Gowers gave in [6] a proof of Szemerédi's theorem for every k. His main tool is a deep theorem of G. A. Frieman on the structure of sets of integers A so that the cardinality $|A + A|$ of the sum set $A + A$ is at most $C|A|$ for some constant C. The proof of Gowers gives an estimate for $N(k, \delta)$ of the form

$$2^{2^{\log|\delta|^{2^{2^{k+C}}}}}$$

for some constant C. This gives in particular the first "reasonable" estimate for the constant in Van der Waerden's theorem.

The proof of Fürstenberg led to the creation of an entire theory and to many generalizations of Szemerédi's theorem. It is expected that Gowers's proof will have a similar impact but in a somewhat different direction.

4. Szemerédi's uniformity lemma.
One of the tools used by Szemerédi in his proof of the result mentioned above is a result on partitioning general graphs into "uniform subsets", and it is called Szemerédi's uniformity lemma. This lemma found many other important applications in graph theory. The statement of the lemma involves three (small) parameters $\varepsilon, \delta, \eta$ and an estimate $K(\varepsilon, \delta, \eta)$ for the number of sets in the partition. The upper bound found by Szemerédi in the case $\varepsilon = \delta = \eta$ is a tower of 2's of height proportional to ε^{-6}.

In most cases in combinatorics such large estimates are just a consequence of the method of proof but do not describe the actual situation. This is the case, e.g., in Van der Waerden's theorem mentioned above. Very surprisingly, Gowers obtained in [5] a lower bound for $K(\varepsilon, \varepsilon, \varepsilon)$ which is also of the form of a tower of 2's (but of a smaller height, proportional to $\log|\varepsilon|$). The existence of such large lower bounds is of great significance in combinatorics and in the theory of complexity of computations.

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The Work of Maxim Kontsevich

Yuri I. Manin

The mathematical achievements of Maxim Kontsevich have received worldwide recognition. He has influenced a considerable body of research in mathematical physics, topology, and algebraic geometry. What follows is a brief report on some of his work.

Kontsevich's most famous paper is probably "Intersection theory on the moduli spaces of curves and the matrix Airy function" (*Comm. Math. Phys.*, **147** (1992), 1–23). It contains a complete proof of Witten's conjecture on the generating function of a family of characteristic numbers defined on the moduli spaces of curves with marked points. Such

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Maxim Kontsevich

a generating function appeared in the context of topological quantum field theory, and Witten's identities reflected a highly speculative conjecture that different approaches to the quantization lead to identical results.

To state a part of Kontsevich's results, we need some notation. Let $\overline{M}_{g,n}$ be the moduli space of stable n-pointed curves of genus g. The intersection theory of these spaces is understood in the sense of orbifolds, or stacks. The algebro-geometric study of the Chow ring of $\overline{M}_{g,0}$ was initiated by D. Mumford. Put

$$\Psi_{n,i} := \xi_i^*(c_1(\omega_{C/M})) \in H^2(\overline{M}_{g,n}, Q),$$

where $\xi_i : \overline{M}_{g,n} \to C_n$ are the structure sections of the universal curve.

Following the notation of Witten, the integrals of top degree monomials in $\Psi_{n,i}$ are denoted

$$\langle \tau_{a_1} \dots \tau_{a_n} \rangle = \int_{\overline{M}_{g,n}} \Psi_{n,1}^{a_1} \dots \Psi_{n,n}^{a_n}.$$

Kontsevich's Main Lemma gives an infinite family of identities that allows one to calculate these numbers algorithmically and to sum an appropriate generating function (I omit this part, but the result is extremely beautiful). The identities have the following structure. Fix (g, n), put $d = 3g - 3 + n$, and choose n independent variables $l_1, \dots, l_n$. Then

$$\sum_{d=d_1+\cdots+d_n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle \prod_{i=1}^{n} \frac{(2d_i - 1)!!}{l_i^{2d_i+1}}$$
$$= \sum_{\Gamma \in G_{g,n}} \frac{2^{-|V_\tau|}}{\mathrm{Aut}\,\tau} \prod_{e \in E_\tau} \frac{2}{l'(e) + l''(e)}.$$

Here $G_{g,n}$ is the set of the isomorphism classes of triples $\Gamma = (\tau, c, f)$ where:

(i) τ is a connected graph with all vertices $v \in V_\tau$ of valency 3 and with no tails;

(ii) c is a family of cyclic orders on all $F_\tau(v)$ where $F_\tau(v)$ is the set of flags adjoining v;

(iii) f is a bijection between $\{1, \dots, n\}$ and the set of all cycles of τ. A cycle is a cyclically ordered sequence of edges (without repetitions) $(e_1, e_2, \dots, e_k)$ such that for every i, e_i and e_{i+1} have a common vertex v_i and the flag (e_i, v_i) follows the flag (e_{i+1}, v_i) as specified by c;

(iv) for any edge $e \in E_\tau$, $\{l'(e), l''(e)\} = \{l_a, l_b\}$, where $\{a, b\} \subset \{1, \dots, n\}$ are the f-labels of the two cycles to which e belongs.

If τ is embedded into a closed Riemann surface X that is oriented compatibly with c, the cycles of τ become the boundaries of the oriented connected components of $X \setminus |\tau|$ (2-cells). Then f labels these cells, and $\{a, b\}$ become the labels of the cells adjoining e.

To prove this, Kontsevich analyzes the remarkable cell complex representation of the moduli space and reinterprets complex analytic integrals via combinatorial data. A paradoxical property of his identity is the cancellation of poles in the right-hand side, which is not obvious even in the simplest cases.

The ideas contained in this paper were developed in many works of physicists; the matrix Airy function Kontsevich introduced in order to sum the generating function is an important ingredient of what are now called Kontsevich's models.

On the other hand, the experience he acquired in dealing with the geometry of moduli spaces allowed him to introduce several decisive ideas in the very active area of quantum cohomology and the Mirror Conjecture.

The Mirror Conjecture is by now a series of stunning, interrelated insights in the geometry of complex manifolds with vanishing canonical class, which are motivated by a conjectural duality in quantum string theory. Only some of these insights are formulated as precise mathematical conjectures. The first of them was what we call the Mirror Identity: an equality of two formal series, one of which is a generating function for the number of rational curves of various degrees on a three-dimensional quintic, and another is produced from periods of the mirror dual variety.

Kontsevich's paper "Enumeration of rational curves via torus actions" (*The Moduli Space of Curves*, R. Dijkgraaf, C. Faber, and G. Van der Geer, eds., *Progress in Mathematics*, vol. 129, Birkhäuser, Boston, 1995, pp. 120–139) consists of two parts. The first part contains the definition and study of what are now called Kontsevich stable maps. These are systems $(C; x_1, \dots, x_n; f)$ where C is a projective curve with only cusps as singularities, x_i are pairwise distinct smooth points on it, and $f : C \to V$ is a map without infinitesimal automorphisms to a smooth projective manifold V. Such maps with a fixed image $\beta = f_*(C) \in H_2(V)$ form a Deligne-Mumford stack $\overline{M}_{g,n}(V, \beta)$. Kontsevich's insight was that such a stack carried a Chow class that he called a virtual fundamental class. Using the image of this class as a correspondence between V^n and $\overline{M}_{g,n}$, he defined the very strong motivic version of Gromov-Witten invariants. Some of these invariants are essentially numbers, among which the numbers of rational curves of various degrees on a quintic threefold are contained. The conjecture that an appropriate generating function for them expresses a variation of Hodge structure for the dual quintic was the first case of the Mirror Conjecture.

The second part of Kontsevich's paper gives explicit formulas for Gromov-Witten numbers for complete intersections in (products of) projective spaces, in particular for quintics. The structure of the formulas is similar to the one above (summation over trees). However, their origin is very different.

One can get recursive formulas for Gromov-Witten numbers of, say, a projective space by using degenerations of stable maps of rational curves to it. However, on for example a quintic, rational curves tend to be rigid, so that there is nothing to degenerate. Kontsevich's idea is to degenerate the quintic itself, replacing it by a union of five hyperplanes. The problem of how to calculate the weights of individual stable maps to such a simplex is solved by the creative application of Bott's residue formula to the stack of stable maps, corresponding to the standard torus action on the ambient projective space. It so happens that only very degenerate curves mapping onto the 1-skeleton of the simplex contribute. Their combinatorial types are marked trees.

Kontsevich thus exhibited a precise formula for the left-hand side of the conjectural Mirror Identity (counting curves). It is worth stressing that he has supplied the first ever *algebro-geometric definition* of this function: all previous work on the Mirror Conjecture dealt with a vague "physical" notion of it. Kontsevich stopped short of proving this case of the Mirror Conjecture, which he reduced to an explicit identity. A. Givental completed the proof, introducing new ideas: in particular, the technique of equivariant cohomology.

In the paper "Vassiliev's knot invariants" (*Adv. in Soviet Math.* **16** (1993), 137–150), Kontsevich invented a generalization of Gauss's integral formula for linking numbers, thereby supplying simultaneously all of Vassiliev's invariants of knots. A parametrized knot is a map $K : S^1 \to R^3$. The space R^3 will be represented as $C_z \times R_t$. Put

$$Z(K) = 1 + \sum_{m=1}^{\infty} (2\pi i)^{-m} \times \int_{t_1 < \ldots < t_m} \sum_P \pm D_P \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'}.$$

Here P runs over the set of "good pairings" (z_i is paired with z_i'); we omit the definition of "good" and of the sign; D_P is the so-called chord diagram naturally associated with K and P, so that the whole series is a formal linear combination of chord diagrams. Now denote by ∞ the trivial knot embedded as this figure (without self-intersection), and put

$$\tilde{Z}(K) = \frac{Z(K)}{Z(\infty)^{c/2}},$$

where c is the number of critical points of K (invertibility of the denominator can be easily checked). This is the Kontsevich invariant of K. It does not change under arbitrary deformations of K, and it contains all invariants of finite order.

In a recent preprint, "Formal quantization of Poisson manifolds", Kontsevich solved a longstanding problem showing that any Poisson manifold admits a formal quantization. In the flat case he produced an explicit quantization introducing a beautiful new class of integrals. This work has great potential.

The Work of Curtis T. McMullen

John Milnor

Curt McMullen has made important contributions to the study of Kleinian groups, hyperbolic 3-manifolds, and holomorphic dynamics. Indeed, following the lead of Dennis Sullivan, he clearly regards these three areas as different facets of one unified branch of mathematics. Following are descriptions of a few selected topics. I hope that these will illustrate the variety and depth of his work. However, by all means the reader should look at the original papers, since he is a master expositor. See especially his two books and his Berlin lecture [Mc10].

Curtis T. McMullen

Solving the Quintic

His first work was on Smale's theory of purely iterative algorithms. By definition, these are numerical algorithms which can be carried out by iterating a single rational function, without allowing any "if $\cdots$, then $\cdots$" branching. In [Mc1] he showed that the roots of a polynomial of degree n can be computed by a generally convergent, purely iterative algorithm if and only if $n \le 3$. With Peter Doyle [DMc] he showed that these roots can be computed by a tower of finitely many such algorithms if and only if $n \le 5$.

A Fat Julia Set

The Julia set J of a rational map f from the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself can be described roughly as the compact set consisting of all points $z \in \hat{\mathbb{C}}$ such that the iterates of f, restricted to any neighborhood of z, behave chaotically. It is not known whether such a Julia set can have positive area without being the entire Riemann sphere. However, McMullen [Mc2] produced very simple examples for the more general question, in which we replace the rational map by a transcendental function, such as the map $z \mapsto \sin(z)$ of

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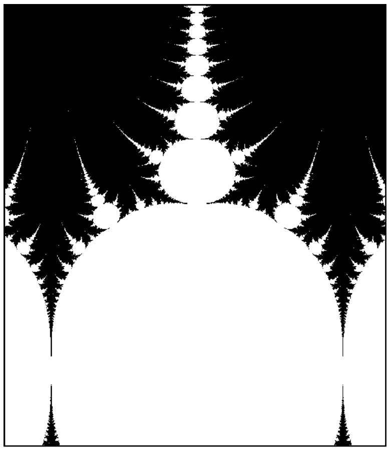

Figure 1. The Julia set for $z \mapsto \sin(z)$, shown in black, has positive area but no interior points.

Figure 1. (A different and more complicated example was given in [EL].)

The Kra Conjecture

To any Riemann surface X one can associate the Banach space $Q(X)$ consisting of all holomorphic quadratic differentials $\Phi = \phi(z)\,dz^2$ for which the norm $\|\Phi\| = \int |\phi|\, dz\, d\overline{z}$ is finite. Any covering map $f : X \to Y$ induces a push-forward operation $f_* : Q(X) \to Q(Y)$, where the image differential at a point y is obtained by summing over the points of $f^{-1}(y)$. This operation can never increase norms, $\|f_*(\Phi)\| \le \|\Phi\|$. In the special case of the universal covering $f : \widetilde{Y} \to Y$ of a hyperbolic surface of finite area, Irwin Kra conjectured in 1972 that there is always some definite amount of cancellation between the different preimages of a point of Y, so that $\|f_*\|$ is strictly less than 1. This was proved by McMullen [Mc3, Mc5]. In fact, McMullen considered a completely arbitrary covering map $f : X \to Y$, showing that $\|f_*\| < 1$ if and only if this covering is nonamenable.

Cusps in the Boundary of Teichmüller Space

Next, a problem in Kleinian groups. In 1970 Lipman Bers compactified the Teichmüller space of complex structures on a hyperbolic Riemann surface of finite area by adding an ideal boundary consisting of algebraic limits of associated Kleinian groups. He conjectured that the "cusps", corresponding to ideal limits in which some simple closed curve has been pinched to a point, are everywhere dense in this boundary. (Compare Figure 2.) This was proved by McMullen [Mc4] in 1991, using a careful estimate for the change in the associated group representation $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ as some simple closed curve in the surface S shrinks to a point.

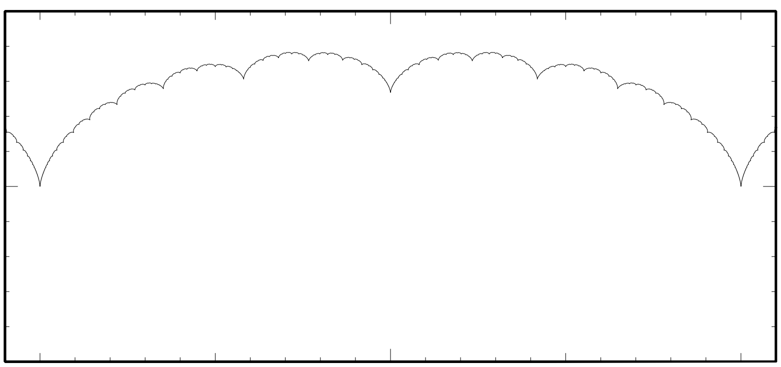

Figure 2 (courtesy of David Wright). Dense cusps in the boundary of Teichmüller space for a punctured torus, using the Maskit embedding. Teichmüller space is the region underneath this boundary curve.

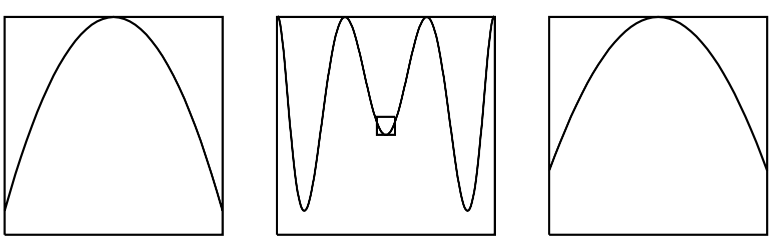

Figure 3. A quadratic map f, its iterate $f^{\circ 3}$, and its renormalization $\mathcal{R}_3(f)$. The right-hand box is obtained from the small box in the middle by magnifying and rotating $180°$.

Thurston Geometrization

The still unproved Thurston Geometrization Conjecture asserts that every compact 3-manifold can be cut up along spheres and tori into pieces, each of which admits a simple geometric structure. Here eight possible geometries must be allowed. For six of these eight geometries, the problem is now well understood, but difficulties remain in the hyperbolic case, while the spherical case, including the classical Poincaré conjecture, is still intractable. For references, compare [Mi2].

Thurston outlined proofs that a 3-manifold admits a hyperbolic structure in two important special cases. First suppose that M is a *Haken manifold*, that is, suppose that M is S^2-irreducible and can be built up inductively from 3-balls by gluing together submanifolds of the boundary, taking care that no essential simple closed curve in this submanifold bounds a disk in the manifold. Thurston showed that M can be given a hyperbolic structure if and only if every $\mathbb{Z} \oplus \mathbb{Z}$ in its fundamental group comes from a boundary torus. McMullen [Mc6] used his work on the Kra conjecture to give a new and explicitly worked out proof of this theorem. (The details are quite complicated.) The second case handled by Thurston concerned 3-manifolds which fiber over the circle. Again, McMullen gave a new proof, which will be discussed below.

Renormalization

Let f be a smooth even map from the closed interval $I = [-1, 1]$ into itself with a nondegenerate critical point at the origin and with no other critical points. We will say that f is *renormalizable* if there is an integer $n \ge 2$ so that the n-fold iterate $g = f^{\circ n}$ maps the subinterval $\{x\,;\,|x| \le |g(0)|\}$ into itself with only one nondegenerate critical point. If we rescale by setting $\hat{f}(x) = g(\alpha x)/\alpha$ where $\alpha = g(0)$, then $\hat{f}$ will be a new map from the interval I into itself satisfying the original hypothesis. This $\hat{f}$ is called the *renormalization* $\mathcal{R}_n(f)$. In 1978 Mitchell Feigenbaum, and independently Pierre Coullet and Charles Tresser, considered the special case $n = 2$ and studied maps f which are *infinitely renormalizable*, so that we can form a sequence of iterated renormalizations f, $\mathcal{R}_2 f$, $\mathcal{R}_2^{\circ 2} f$, ..., each mapping I to itself with one critical point. They observed empirically that this sequence of maps always seems to converge to a fixed smooth limit map. Their ideas, motivated by renormalization ideas from statistical mechanics and by attempts to understand the onset of turbulence in fluid mechanics, now occupy a central role in one-dimensional dynamics, since the infinitely renormalizable maps are the most difficult ones to understand.

This construction was extended to the complex case by Douady and Hubbard, using the idea of a *quadratic-like map*, that is, a proper holomorphic map $f : U \to V$ of degree two, where U and V are simply connected open sets in $\mathbb{C}$ and $\overline{U}$ is a compact subset of V. This has led to important work by mathematicians such as Dennis Sullivan, Curt McMullen, and Mikhail Lyubich. Most subsequent progress in understanding the real case has been based on complex methods. (One exception is Martens [Ma].) McMullen's book [Mc7] provided the first careful presentation of the foundations of renormalization. As one example, he was the first to notice the possibility of an aberrant form of "crossed renormalization" in the complex case which does not fit into the usual pattern. He used his work on renormalization to obtain partial results on the generic hyperbolicity conjecture for real quadratic maps, that is, the conjecture that every such map can be approximated by one with an attracting periodic orbit. For example, he showed that every component of the interior of the Mandelbrot set which meets the real axis is hyperbolic. The full conjecture was later proved by Lyubich [L1] and by Graczyk and Świątek [GS].

McMullen's second book [Mc8] developed renormalization theory further and tied it up with Mostow rigidity and also with Thurston geometrization. (See also [Mc10].) He introduced the concept of a "deep point" in a fractal subset $X \subset \mathbb{C}$. By definition, p is *deep* if there are positive constants ϵ and c so that the distance from an arbitrary point q to X is at most $c|p-q|^{1+\epsilon}$. Taking $p = 0$ for convenience, we can understand this concept by zooming in on the origin so as to magnify the set X by some large constant λ. Replacing X by the magnified copy λX, we can replace the constant c by c/λ^{ϵ}, which tends to zero as $\lambda \to \infty$. In other words, these magnified images will fill out the complex plane more and more densely, with gaps which become smaller and smaller as λ becomes large. (Compare [Mi1].)

Now consider a hyperbolic 3-manifold which may have infinite volume. The *convex core* of such a manifold M can be described as the smallest geodesically convex subset which is a strong deformation retract of M. Assuming both upper and lower bounds for the injectivity radius at points of the convex core, McMullen's inflexibility theorem asserts that two such manifolds M and M' which are "pseudo-isometric" must actually be related by a diffeomorphism which becomes exponentially close to an isometry as we penetrate deeper into the convex core. Closely related is the statement that actions of $\pi_1(M)$ and $\pi_1(M')$ on the 2-sphere at infinity for hyperbolic 3-space are quasiconformally conjugate and that this quasiconformal conjugacy is actually conformal at every deep point of the limit set for this action.

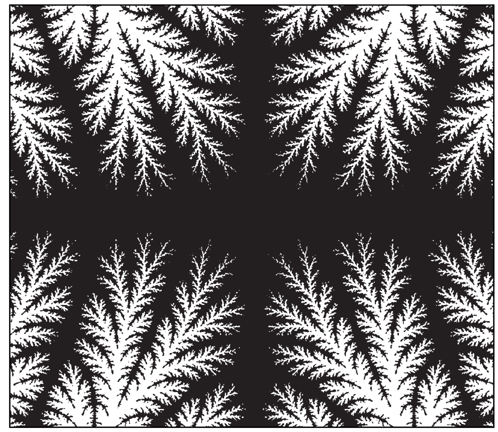

Figure 4. The critical point (at the center of symmetry) is a deep point of the Julia set for the Feigenbaum infinitely renormalizable map $z \mapsto 1 - az^2$, where $a = 1.401155189\cdots$. This Julia set has no interior points.

As an application, McMullen gave a new proof of the second Thurston geometrization theorem. To any surface diffeomorphism $\psi : S \to S$ we can associate the *mapping torus* T_ψ, that is, the quotient of $S \times \mathbb{R}$ under the $\mathbb{Z}$ action which is generated by $(x, t) \mapsto (\psi(x), t + 1)$. If S has genus two or more and ψ is pseudo-Anosov, then Thurston showed that T_ψ is a hyperbolic 3-manifold. McMullen proved this by using his inflexibility result to construct a hyperbolic structure on $S \times \mathbb{R}$ which is invariant under the given $\mathbb{Z}$ action.

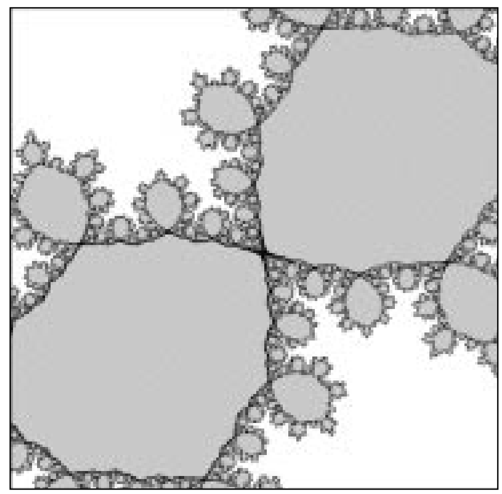

Figure 5. Filled Julia set associated with the golden mean Siegel disk, with rotation number $\rho = 1/(1 + 1/(1 + 1/(1 + \cdots)))$. The Siegel disk is the large region to the lower left.

Next he applied these ideas to renormalization. One basic result is a rigidity theorem for bi-infinite "towers" of renormalizations. We can think of such a tower as a bi-infinite sequence $(\cdots, q_{-1}, q_0, q_1, q_2, \cdots)$ of quadratic-like maps $q_j : U_j \to V_j$, where each q_{j+1} is a renormalization $\mathcal{R}_{n_j}(q_j)$. He showed that if the renormalization periods n_j are bounded and if the annuli $V_j \smallsetminus \overline{U}_j$ have modulus bounded away from zero, then the entire tower is uniquely determined up to a suitable isomorphism relation by its quasiconformal conjugacy class.

Consider an infinitely renormalizable real quadratic map f with periodic combinatorics. Using the complex theory, McMullen showed that the successive renormalizations converge exponentially fast to a map which is periodic under renormalization. Closely related is the statement that the critical point is a deep point for the Julia set of f. (Figure 4.)

Now consider a quadratic map $f(z) = z^2 + c$ which has a Siegel disk of rotation number ρ. That is, choose the constant c so that the derivative of f at one of its two fixed points is equal to $e^{2\pi i\rho}$, where $\rho \in \mathbb{R} \setminus \mathbb{Q}$ satisfies a suitable Diophantine condition. McMullen used similar ideas in [Mc9] to show that this Siegel disk is "self-similar" about the critical point 0 (the central point in Figure 5) if the continued fraction expansion of ρ is periodic. In fact, his argument can be used to show that the entire Julia set J of f is *asymptotically self-similar* in the following sense: There is a scale factor λ with $|\lambda| > 1$ so that the magnified images $\lambda^n J$ converge to a well-defined limit set $\hat{J} = \lambda\hat{J} \subset \hat{\mathbb{C}}$ as $n \to \infty$, using the Hausdorff topology for compact subsets of the Riemann sphere. (In this particular example, $\lambda = 1.8166\cdots$ is real.) The corresponding limit for the boundary of the Siegel disk is a quasicircle contained in $\hat{J}$, while the corresponding limit for the filled Julia set $K(f)$ (the union of bounded orbits for f) is the entire sphere $\hat{\mathbb{C}}$.

There has been very significant subsequent work in renormalization, based in part on McMullen's ideas. Compare the discussion in [Me]. Note in particular [L2], which implies that the boundary of the Mandelbrot set is asymptotically self-similar about the Feigenbaum point, and [L3], which proves existence of a full horseshoe structure for the real renormalization operator and uses it, together with work of Martens and Nowicki, to prove that every real quadratic map outside a set of measure zero has either a periodic attractor or an absolutely continuous asymptotic measure.

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About the Cover

Curt McMullen has shown that a quadratic Siegel disk is asymptotically self-similar about its critical point whenever the continued fraction expansion for its rotation number is periodic. (Compare the discussion on this page.) The cover figure (a color version of Figure 5) illustrates this result by showing part of the filled Julia set (the union of bounded orbits) for the quadratic polynomial map $f(z) = z^2 - (0.3905\cdots + i \cdot 0.5867\cdots)$, which has a Siegel disk with rotation number equal to the golden mean $\rho = (\sqrt{5} - 1)/2$. The critical point lies at the center of this picture, while the Siegel disk is the large region to the lower left with emphasized boundary. Under f, this disk maps homeomorphically onto itself with rotation number ρ, and the symmetric region to the upper right folds onto it. If we expand this figure repeatedly by a fixed scale factor of $1.8166\cdots$, keeping the center point fixed, then the expanded images will converge to a well defined limiting shape.

—*John Milnor*