Professional Development of Mathematics Teachers

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General Background

"You can't teach what you don't know", but too many of our mathematics teachers may be doing exactly that: teaching what they don't know. This is one of the key findings of the landmark 1983 education document *A Nation at Risk*. What was true back in 1983 is even more true today. Since the only way to achieve better mathematics education is to have better mathematics teachers, this intolerable situation cries out for a radical reform. There is no mystery to the needed reform: university mathematics departments must do a better job of teaching their students (*preservice professional development*), schools of education must start emphasizing the importance of subject matter content knowledge, and state governments must embark on large-scale and systematic efforts to retrain the mathematics teachers already in the classrooms (*inservice professional development*). The difficulty lies in the execution.

The awareness by organizations of mathematicians of the need to coordinate university mathematics departments across the land in order to upgrade the preservice professional development of prospective teachers began to surface only in the past year. The Conference Board of the Mathematical Sciences (CBMS) has since appointed a Steering Committee for the Mathematics Education of Teachers Project to begin work in this direction. As to the problem with schools of education, its gravity cannot, in my opinion, be overstated (cf. [1]). Recently, mathematics educators began to call for a broad base of mathematical knowledge for all teachers, especially those in K–6 (kindergarten through sixth grade). The remarkable recent volume [2] of Liping Ma, for example, deals with this very issue. Unlike most technical writings in education, Ma's volume is easily accessible to mathematicians, and it also contains an ample listing of the relevant literature.

This article is concerned with the third component of the proposed remedy: inservice professional development. Because any improvement in education must start with improvement of the teachers already in the classroom, this topic is one of real urgency. In addition, some understanding of this topic is indispensable to a sound decision on how to approach preservice professional development. In one way or another, the latter directly influences the professional lives of most of the readers of the *Notices*.

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To the extent that the most important component of inservice professional development at the moment is to increase teachers' mathematical knowledge, inservice professional development is in one sense nothing more than the teaching of college-level mathematics. But teaching teachers in the field only through short sessions of a few weeks' duration has special concerns that are not shared by the usual teaching of college students. This article briefly discusses some of these special concerns.[1] The main body of the article is, however, devoted to the presentation of three specific examples of inservice professional development in order to illustrate some of the pitfalls that accompany attempts to cope with these concerns. Apart from some modifications, these examples are taken from a long report of my visits in 1997 to four summer institutes in California devoted exclusively to professional development for mathematics teachers [5].[2] Each example is an account of a mathematics presentation. It begins with a description of the content of the presentation and ends with my comments.

Some Observations about Inservice Professional Development

In the crudest terms, there are two kinds of inservice professional development: *enrichment* and *remediation*. The former is devoted to enlarging the mathematical knowledge of teachers who are already at ease with the mathematical demands in the classroom. The goal is to inspire them to even higher levels of achievement. The purpose of the latter is to ensure, as much as possible, that the teachers achieve an adequate understanding of standard classroom mathematics. Attention will therefore be focused on bread-and-butter topics in school mathematics, though they will be presented from a slightly more advanced point of view.

We concentrate on remedial professional development here.[3] Now, inservice professional development is about improving teachers' classroom performance, not just about improving their knowledge of mathematics. Thus even for remediation, there should be discussions of pedagogical issues in addition to mathematics. However, my view is that the major need for most teachers who attend professional development programs is for more robust mathematics background, so the first order of correction has to be about mathematics. Accordingly, my observations and comments will focus mainly on mathematics and will touch on pedagogy only sparingly. Higher-order corrections can deal with pedagogy.

[1] *Article [4] discusses a few others.*

[2] *More discussions along the lines of this article can be found in [6], [7], and [8].*

[3] *Note that [7] and [8] are about enrichment.*

Before going into the special features that distinguish inservice professional development from ordinary teaching, let me point out that there is a unifying theme that underlies all of them, and it is that inservice professional development in mathematics is essentially a race against time: how to do something in a mere three to four weeks in the summer, plus a handful of meetings in the subsequent year, that could overcome eighteen years or more of teachers' nonlearning or miseducation. Everything that follows is in one way or another colored by this severe time limitation. As to the special features themselves, it must be admitted that there is no universal agreement on what they are. The following is a minimal list from my own perspective, and it will serve as the point of reference for the remainder of this article.

(A) *No extended lecturing.* It would not do if the mathematics instruction in professional development is delivered only in the unidirectional style from professor to students. Teachers need to be shown as often as feasible, by deeds rather than just by words, how mathematics is usually done: the zig-zag process to arrive at a solution by trial-and-error, the use of concrete examples to guide explorations, the need of counterexamples in addition to theorems in order to achieve understanding and, above all, the fact that mathematical assertions are never decreed by fiat but are justified by logical reasoning. Teachers have to witness this process with their own eyes before they learn to do mathematics the same way and, more importantly, before they can *teach their own students to do likewise*. Otherwise they cannot be effective teachers. There is no recipe for achieving miraculous results here, but in practice, successful professional development efforts use a judicious mixture of lecturing and the discovery method.

(B) *Keep the mathematics simple and relevant to K–12.* Professional development aims at increasing teachers' *understanding* of mathematics, so it must teach substantive mathematics and not just a collection of projects which can be easily modified for immediate use in a classroom. On the other hand, if the mathematics is too far removed from the teachers' classroom experience, they may not be motivated to learn, and very little professional development will take place. Thus both the choice of topics and the style of mathematical exposition should revolve around the teaching of K–12 mathematics. As an example, in order to teach the mathematical foundation of fractions to teachers, one might be tempted to start with the construction of the quotient field of an integral domain, because this approach leads to a deeper understanding of $\mathbb{Q}$. A little reflection reveals that this would not be an optimal way to use the limited time available and that a more suitable approach may be to do fractions directly and use the time to get teachers to understand why, for example, $(a/b)/(c/d) = (ad)/(bc)$. Another example is furnished by discrete mathematics: although its simplicity and easy accessibility are most seductive, it is not yet a staple in K–12, and its presence in professional development must therefore be kept in check.

(C) *There should be grade-level separation.* For convenience as well as financial reasons, it is not uncommon to lump teachers of all grades together (K–12) for instruction in professional development programs (see the second and third samples below). While such an arrangement on occasions can provide a valuable experience for teachers, overall the loss far outweighs the gain. For the purpose of teaching meaningful mathematics, mathematical presentations in professional development programs should be tailored to the needs of teachers of specific grade levels: say, elementary, middle, or high schools.

(D) *There should be year-round follow-up programs to monitor the teachers' progress.* Substantive knowledge, be it mathematical or otherwise, is not learned overnight. Teachers need mathematical reinforcement over an extended period of time (one day each month for a year, say). Moreover, observations by an experienced person in the teachers' own classrooms would help them find out if they are successfully putting the new mathematical knowledge to work.

(E) *Teachers should be paid for participating in professional development.* Teachers are generally not well paid. Because professional development typically cuts into their summer vacations and weekends, it often takes time away from a needed second income or interferes with family life. Unless we pay them to participate,[4] we will have no leverage to ask for their conscientious effort to learn. Needless to say, the success of any professional development effort is judged entirely by how much the teachers manage to learn.

The preceding two items, (D) and (E), are not strictly mathematical concerns. Because they will not surface again in the rest of the article, this may be the right place to append a few clarifying comments. What is at issue here is how to ensure that a professional development program succeeds in turning out better teachers. By tradition, there is no assessment of teachers' progress in such a program[5]; and even if there is, how can the failing of any kind of an exam be used as a deterrent to nonperformance in a professional development program? Therefore, all one can do is to offer generic encouragement, as in (E), and gentle guidance, as in (D), for lack of anything better. In addition, the issue of payment for teachers has a direct impact on the presentation of mathematics in professional development. When teachers come to a such a program as volunteers, it is difficult to ask them to work hard and do homework problems; appealing to their pride can go only so far. Under the circumstances, an instructor in professional development would likely overcompensate for teachers' lack of practice outside the program by concentrating on doing problems during each session. Without gainsaying the benefit that some teachers would reap from such an experience, one must recognize that an overemphasis on doing problems is not an optimal way to use the limited time available. In point of fact, I sensed such an overemphasis in all four sites I visited [5], and it raised the question in my mind of whether the fact that all those teachers were grossly underpaid[6] played a role. Readers may wish to keep this question in mind when they read the three examples below.

Finally, there is a nonacademic component to professional development that actually outstrips all others in importance: the amount of financial commitment by state governments to this task. Without rock-steady and generous financial backing, every phase of professional development becomes an adventure in desperation. For example, imagine trying to teach high school teachers about proofs in geometry in only ten days. Imagine doing it with a group of teachers in grades 6–12 because there is insufficient funding to separate them into two groups of grades 6–8 and 9–12. Imagine also never seeing the teachers again after the ten-day instructional session because there is no funding for any follow-up. What result can one expect in that case?

Our nation has to learn that its investment in education will come to naught if it does not also invest in its teachers.

The California Scenario

The following are three examples of mathematical presentations in professional development taken from the report [5] on my visits to four professional development sites within California in the summer of 1997. They have not been chosen for their exemplary execution of the basic principles (A)–(C) above. On the contrary, they were chosen because they give a fair representation of the state of professional development from one segment[7] of California, and through them we get to see how each

[4] *A minimal salary scale is $100 for each day of participation.*

[5] *It is well to note that, even with an elaborate assessment system for students in a normal classroom, we are still far from being able to determine whether learning does take place.*

[6] *The most generous of the four sites paid each teacher about $25 a day; one site actually required its teachers to pay for their attendance.*

[7] *But it is the major one: the four sites of [5] were established under the auspices of the California Mathematics Project, which is the official state agency in charge of inservice professional development for mathematics teachers.*

of (A)–(C) is (or is not) implemented in practice. It is to be noted that in the period 1990–98, mathematical professional development efforts in California were not known for their emphasis on mathematical content knowledge (cf. [3] for background). Part I of the report [5] discusses this issue at some length. I have made extended comments after each presentation. My overriding concern in these comments is whether the teachers are likely to become better mathematically informed as a result of attending the presentation. It would be futile to pretend that my comments are anything but subjective. Part of the reason is that there is as yet no such thing as a scientifically valid assessment method where any kind of teaching is concerned. What I have tried to do is to use (A)–(C) as basic criteria to judge whether a presentation would benefit the teachers from a mathematical standpoint. By making the assumptions behind my comments explicit, I hope to provoke further discussions on professional development.

First Sample

Topic: Discrete mathematics
Time Allowed: 90 minutes
Grade Level: High school teachers

The first twenty minutes or so were devoted to the computation of areas of triangles on a geoboard.[8] The emphasis was on either decomposing a given triangle into a disjoint union of right triangles with only vertical or horizontal legs (whose areas can therefore be immediately read off) or finding ways to represent it as the complement of the aforementioned kind of right triangles in a rectangle. Because there were only about ten teachers in this session, the presenter could pay special attention to each teacher in turn, and the conversation among the teachers was freely flowing. It was clear from the remarks overheard as well as from the questions raised that more than a few were not sure about the area formula of a triangle, and most of them seemed to find it challenging to compute the area of a triangle with vertices (say) at the lattice points $(3, 0)$, $(0, 2)$, and $(4, 1)$.

These considerations then led smoothly into the second topic: Pick's theorem. Let P be a polygon with vertices on the *lattice points* (i.e., those (x, y) where both x and y are integers) of the coordinate plane. Let B be the number of lattice points on the boundary of P, and let I be the number of lattice points in the interior of P. Then Pick's theorem asserts that the area of P can be computed by the formula: $I + \frac{1}{2}B - 1$. However, the teachers were not shown this formula but were asked, for the case that P is a triangle, whether they could guess a relationship among I, B, and the area of P. After a short period of trial and error, a few could guess the formula correctly, though without being able to articulate the underlying reason. The presenter then pointed out a systematic way based on inductive reasoning to approach this question that would eventually lead to the correct formula in general. There were murmurs of appreciation. A short write-up of a guided proof of the theorem was then handed out. Finally, the teachers were asked to guess a formula for the number of segments with *distinct* lengths in an $n \times n$ square, where the vertices of the square and the endpoints of the segments are all lattice points. It is natural to guess that this number is $\frac{1}{2}(n^2 + n)$. For $n \leq 4$, this is correct. However, when $n = 5$, it is strictly less than $\frac{1}{2}(5^2 + 5)$, because duplication of the lengths of such segments occurs due to the appearance of Pythagorean triples in this range. For instance, the lengths of the segment joining $(0, 0)$ to $(0, 5)$ and of that joining $(0, 0)$ to $(3, 4)$ are both 5. Of course, it then follows that the conjectured formula fails for all $n \geq 5$. Nobody got this part. This is a good lesson in not jumping to conclusions on the basis of limited experimentation, and it also shows why proofs are important. The general formula is in fact unknown, but there is apparently an asymptotic estimate of its order of magnitude, connected to the number of representations of n as the sum of two squares, as $n \to \infty$.

COMMENTS: *The fact that the geoboard was needed to help these high school teachers with area computations was a bit surprising, because the formula for computing the area of a triangle, as well as its simple proof, should be second nature to them. After all, the area formula of a triangle T can be explained very simply in the following way. Fix one side of T as base; then by reflecting T across another side, one obtains a parallelogram P whose area is twice that of T. Note that P has the same base and height as T. The area of P, on the other hand, is the same as that of the rectangle R with the same base and same height as T: by looking at a picture of P and R, one sees easily that R is obtained from P by subtracting a triangle from one side and adding another one congruent to it on the other side. So the area of T is half that of R, and the latter is equal to the product of the base and the height of T. However, if we accept the fact that this simple argument was either not known or not understood by these teachers, then the inevitable conclusion is that they would have benefited more from learning basic materials than something like Pick's theorem or the counting of segment lengths in a square. The latter two items are hardly foundational K–12 material, and they do not lead to any new understanding of the fundamental issues of area. There is also the danger that teachers with a limited exposure to mathematics might have been*

[8] *A board with pegs which are placed evenly in both the vertical and horizontal directions. Rubber bands are then hooked onto pegs to make shapes.*

misled into thinking that either result is central in mathematics.

Nevertheless, these criticisms do not contradict the impression that this presentation is in many ways a good demonstration of how to reach out to the teachers and enhance their understanding of mathematics in the process. In a span of 90 minutes, they were exposed to mathematics that is at once simple and nontrivial, as well as shown both the virtues and limitations of experimentation. The judicious use of a handout to bring closure to the proof of Pick's theorem also shows an effective way of circumventing the time limitation.

Second Sample

Topic: Connections
Times Allowed: Two and a half hours
Grade Level: K–12 teachers

The presenter announced the theme of the presentation: connections. He asked the teachers to share their thoughts of what this could mean. People volunteered their reactions: connections to real-world applications; connections between mathematical ideas, between topics, between grade levels, between activities and ideas, etc.

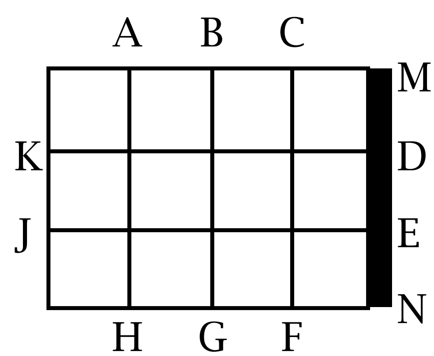
The Königsberg bridges problem was posed and the teachers, divided into groups, were asked to try their hand at a solution. The presenter went around the room nudging people on; the point at which one should focus attention on what happens at each region did not come easily. At some point, the presenter decided—perhaps on the basis of his observations?—that it was time to bring closure to this investigation. He wrote clearly on the overhead projector: The problem could not be solved because "(i) each region had an odd number of bridges; and (ii) you need an even number of bridges connected to each region, because if you start from a region, you must go out-in, out-in, …, or if you don't start there, you must go in-out, in-out, …etc." There was, however, no mention of the fact that if there is an even number of bridges connected to each region, then the bridges problem would have a solution.

Thus far, 40 minutes had passed: 30 on the discovery process and 10 on the summary and write-up.

Next, a completely different topic was brought up: abstract graphs. *Eulerian circuit* in a graph was introduced, and teachers were asked to test the existence or nonexistence of such a circuit in some simple graphs that were handed out. Another period of discovery by the teachers followed, and this time most of the teachers seemed to have an intuitive grasp of the solution fairly quickly. Again, the presenter summarized: "The graphs for which there is an even number of paths connected to each vertex have an Eulerian circuit. If one vertex has an odd number of paths, there is no Eulerian circuit." Again, no mention was made of the fact that if every vertex has an even number of paths connected to it, then there would be an Eulerian circuit. This part took 25 minutes.

Now the *connection* was pointed out between the bridges problem and the abstract graphs: "region" and "bridge" of the former correspond to "vertex" and "edge" of the latter respectively. This discussion took 10 minutes.

The last problem to be taken up was this: What is the circuit of minimum length that goes through every street of the following street map at least once (all street blocks are assumed to be of equal length)?



Call this the *mailperson problem*, for obvious reasons. Again, the teachers were given ample time to work on the problem while the presenter walked around the room giving hints and encouragement. Clearly some streets would have to be traversed more than once because there are ten vertices with odd degrees, but the insight needed for the solution is not so trivial this time around. I could detect no evidence from observing my neighbors that any of them had a full understanding of what it takes to solve the problem. The generous hints given by the presenter did seem to lead many teachers to the correct conclusion that they had to add paths in order to set up an Eulerian circuit of minimum length. This observation was born out when, at the end, four teachers were invited to the front of the room and each offered the solution obtained by his or her group. It came out of the explanations, though without emphasis, that the way to do this was to add paths to each vertex of odd degree (A, B, C, D, E, F, G, H, J, and K) to make their degrees even and try to keep the total number of added paths to a minimum. By coincidence, although the four solutions differed in minor details, they all ended up adding only 7 additional paths, e.g., AB, CM, MD, EN, NF, HG, JK. It was somehow agreed without any comments or discussion that this was the solution to the mailperson problem. But beyond describing the added paths, none of the four teachers took the trouble to explain *why* the paths would lead to a circuit, much less the fact that it would be a circuit of minimum length. The presenter did not address these gaps in the mathematical reasoning in his summary either, but did point out that this way of adding paths to solve the problem represented algebraic thinking. The preceding took 45 minutes.

In conclusion, the presenter went back to the theme of the presentation: connections. The last problem is a different problem from the bridges problem, yet via algebraic thinking, we saw the connection between the two.

As soon as the session was over, a teacher leader[9] in my group asked me why, by adding those paths, the street graph would acquire an Eulerian circuit. I decided that I would first gain her confidence by explicitly drawing an Eulerian circuit on the augmented street graph and then explain to her the reason for my success. But as I started to draw the Eulerian circuit and tried to convince her that the drawing was so easy as to be computer programmable, I found that communication was difficult because there was too much background to cover.

[The actual time devoted to mathematics was thus two hours, according to my record. The remaining thirty minutes were accounted for by the usual reasons: the presentation started a bit late; time was used in the initial warm-up, in passing out instructional materials, in regrouping the teachers between topics, and in the final summation, etc.]

COMMENTS: *This presentation has obvious strengths. The choice of three different problems to illustrate nontrivial mathematical connections was good. An even more significant strength was the way the presenter made use of the discovery method to get everyone actively involved and yet never failed to clearly summarize after each episode what he wanted each teacher to learn from it. In my opinion, the discovery method imposes on the instructor the obligation to bring mathematical closure after each investigation.*

On the mathematical side, the built-in liabilities of addressing a mixed audience of K–12 teachers cannot be ignored. At the most basic level, the preceding account makes clear the enormous amount of time given to waiting for all teachers to do their own explorations, e.g., 40 minutes for the Königsberg bridges problem. Moreover, it may be assumed that the preoccupation with making the presentation accessible to everyone distracted the presenter from his full engagement with the mathematics at hand, which resulted in serious gaps in the mathematical exposition. Thus, while he explained carefully why the condition of each vertex having even degree was necessary for the existence of an Eulerian circuit, its sufficiency was never mentioned, much less proved. (Could this be because he knew that the difference between necessity and sufficiency is not likely to be well understood by K–6 teachers?) Some may argue that in a presentation to such a mixed audience, getting across the mathematical idea that there is *a relationship between the degree of a vertex and the existence of an Eulerian circuit is the important thing and the details do not matter. However, the incident with the teacher leader after the end of the session clearly shows that details do matter. She might have understood the solution to the mailperson problem much better had the* sufficiency *been properly emphasized earlier.*

In the case of the mailperson problem, the fact that the minimality of the length of the asserted circuits was never proved is another instance of the failure to provide mathematical closure. It was likely that many of the teachers had a vague and intuitive understanding of why adding strategically placed paths would lead to an Eulerian circuit of minimum length. But is it not an important part of professional development to help teachers articulate such intuitive feelings in precise and clear mathematical terms? In this case, the articulation never took place. Even more significant was the failure to carefully bring out the reasoning underlying the solution of the problem, i.e., *the idea that each time a street block is retraced we can keep a record of the retracing by adding a new path to that block. This idea changes the seemingly difficult search for a circuit of minimal length to a simple algebraic problem of how to add the smallest number of paths to make the degree of each vertex even. This is exactly the kind of* higher-order thinking skill *that the teachers would do well to acquire.*

The preceding comments should not be interpreted to mean that every mathematical presentation must have no gaps and every detail must be accounted for. There are times when a presenter is impelled, for pedagogical considerations, to tell the truth but not the whole truth. Nevertheless, when this happens, all the gaps should be clearly identified so that those teachers who are sufficiently prepared know exactly where they stand—mathematically.

Finally, one must ask once again whether teaching a group of K–12 teachers some standard topics in discrete mathematics is an optimal way to do professional development. As usual, it depends on how much is taught and to whom. In the present context, my judgment on the basis of personal contact and observations is that the teachers in this particular group were more in need of remediation than enrichment (on this point, see [5]). If this judgment is correct, then making them aware of some topics in graph theory—while not without merit—would not be nearly as beneficial to them as putting them at ease about symbolic computations or showing them why basic algebra (e.g., fractions and polynomials) is good mathematics and not just a col-

[9] *A past participant of the summer institute chosen to help with the running of the institute.*

lection of formulas to be memorized. Indeed, most of the teachers there had little symbolic manipulative skill, and many had an inadequate understanding of algebra. It is far from clear how a person with such fundamental deficiencies could make himself or herself into a more effective teacher just by learning a few pleasant facts in graph theory.

Third Sample

Topic: Technology
Time Allowed: Two hours
Grade Level: K–12 teachers

This presentation was the fifth in a series of eight on technology. The goal of this series was to introduce the teachers to the effective use of computers in mathematics classrooms. Throughout the presentation, each teacher was seated in front of a computer and was too preoccupied with trying out new computer commands to engage in the discovery method. Consequently, the presentation was essentially one from the presenter to the teachers.

The topic of discussion was prime numbers. The lead question was, Is 899 a prime? It was observed that to test the primality of n, one needs to use only primes $\leq \sqrt{n}$; no explanation of this observation was offered at this time, however, and the presenter was to return to it in a special case later. The first goal was to obtain a list of all the small prime numbers, and the sieve of Eratosthenes was mentioned. Some people indicated that they had heard of the sieve, but no discussion of what it does was given. (Earlier, there was a question about the definition of a prime.) The presenter suggested that the spreadsheet Claris on the computers in front of the teachers would be useful for this purpose. The teachers were then taught a command which would make the computer list all the integers divisible by 3 up to 100, and those divisible by 5, etc. These lists would then be used to strike out the composite integers from the list of all integers, leaving behind the primes. There followed a discussion of how to check directly whether a small integer is a prime. For example, to check whether 7, 11, or 30 is a prime, one needs only check with primes up to 3, 5, and $\sqrt{30}$, respectively. Finally, the primality of 899 was checked: using only primes up to $\sqrt{899} < 30$, it was found that $899 = 29 \cdot 31$.

Up to this point, the mathematics behind these activities was never discussed.

Instruction was given concerning the commands that would force the computer to list all the primes up to 100; this took a while. Heuristic arguments were given as to why the four primes 2, 3, 5, and 7 are sufficient to test the primality of all integers up to 100, but no proof was given. A command was then introduced to display all the primes up to 300 on the computer screen. Apparently, some schools do have Claris in the school computers, so many teachers became aware of the possibility of making use of the computer in mathematics lessons.

The presenter brought to the teachers' attention the fact that 29 and 31 are consecutive odd integers that are primes. The term *twin primes* was introduced. The teachers were asked to explore for themselves the possibility that the number of twin primes is infinite. As background, the well-known theorem on the infinity of primes was recalled, and a proof was sketched. Then the question was raised about the existence of "triplet primes". The answer is *no* because "one of every triple of consecutive odd integers is divisible by 3". No explanation was given. Finally the use of primes in cryptography was briefly mentioned, as was the definition of Mersenne primes.

COMMENTS: *By acquainting the teachers with Claris, the presentation added a new weapon to their pedagogical arsenal: at least in one instance, the computer is at their service. This is the positive aspect of the presentation. Is there perhaps room for improvement?*

There is already a ferocious debate on record as to how much technology should be used in mathematics education, and how soon. I believe students should be taught the proper use of technology, and technology is an integral part of mathematics education, at least starting with the fifth or sixth grade. In this particular instance, however, I would venture the opinion that Claris had not been used properly. There is a tactile *component in the learning of mathematics that cannot be replaced by technology or any other shortcut. By using Claris to churn out the list of primes up to* 100*—all multiples of a prime being deleted by the typing of a computer command—it is quite possible that the understanding of the sieve would stay at the stage of button-pushing. It would have been more educational to the teachers if they had used the computer to display all the odd numbers from* 1 *to* 200 *but proceeded to eliminate all the composite numbers among them* by hand. *This tactile experience with prime numbers would have had a better chance of making them understand what the sieve is about.*

The benefit of technology should not be just to save labor. It can enhance mathematics education if it is put to creative uses such as instant and easy *experimentations with new ideas or providing test cases for conjectures. But a prerequisite for using technology this way is an adequate understanding of the relevant mathematics, and this is why one must always include the pertinent mathematics in any technological presentation. For this reason, I found the absence of any mathematical discussion something of a surprise. In particular, three assertions were made imprecisely and without justification in the course of the presentation: (i) If n is composite, then there must be a factor k of n so that $k \leq \sqrt{n}$, (ii) the number of primes is infinite,* *and (iii) at least one of three consecutive odd integers is divisible by* 3*. Did the mixed audience of K–12 teachers make such mathematical explanations impractical? After the presentation was over, I engaged the eighth-grade teacher sitting next to me in conversation and offered to show her the proofs of (i)–(iii). To my delight, she accepted. I went through all three proofs slowly, pausing at each step to make sure she was with me. Then, since the infinity of primes had been mentioned, I thought of widening her horizon further by telling her about Dirichlet's theorem on the infinity of primes in arithmetic progressions. She was fascinated. The whole discussion took less than ten minutes. If such explanations were given to the whole class instead of in a one-on-one setting, it might have taken twenty minutes. In the context of a two-hour presentation (as this one was), such a mathematical discussion would be eminently feasible.*

However, above and beyond this concern with lumping all K–12 teachers together for professional development, I have one suggestion: items (i) and (ii) listed above are so basic that this would be the right opportunity to make every teacher (including K–6 teachers) understand the explanation. The encounter with the eighth-grade teacher after the session was over suggests that teachers' interest in mathematics should not be underestimated.

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