From Schwarz to Pick to Ahlfors and Beyond

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In his pivotal 1916 paper [P], Georg Pick begins somewhat provocatively with the phrase, "The so-called Schwarz Lemma says…", followed by a reference to a 1912 paper of Carathéodory. Pursuing that lead, one finds a reference to the original source in the expanded notes from a lecture course at the Eidgenössische Polytechnische Schule in Zürich given by Hermann Amandus Schwarz during 1869–70 ([S], 108–132). The lecture notes start by stating the Riemann Mapping Theorem from Riemann's doctoral dissertation of 1851 and noting that Riemann's argument did not provide a fully rigorous proof. The goal of the lecture course is to provide the first complete proof for a general class of domains. The Riemann mapping theorem states that any simply connected plane domain other than the entire plane can be mapped one-to-one and conformally onto the interior of the unit circle. (The plane domains considered at the time appear to have been domains bounded by a simple closed curve.) The lecture notes prove the theorem for domains bounded by a closed convex curve.

Schwarz's proof is based on his earlier work on domains bounded by polygons and what is now known as the Schwarz-Christoffel formula. Schwarz's paper ([S], 65–83) appeared in 1869. In it he notes that back in 1863–64, when he attended a course by Weierstrass on the theory of analytic functions, he did not know of a single special case of a plane figure given in advance for which one could establish a conformal mapping onto the unit disk. He decides to start with the simplest case of a square. (It is in that context that he proves his famous "reflection principle" for analytic functions.) He goes on to give a general formula, noting that Christoffel developed it independently. He credits Weierstrass for filling in the details of showing that the arbitrary constants involved in the integral expression can be chosen to give the desired mapping for a polygon of any number of sides, whereas Schwarz himself had succeeded for just four sides.

Once in possession of a mapping for polygons, Schwarz proceeds to approximate an arbitrary convex domain by domains bounded by polygons and to show that the corresponding mappings converge to a limit mapping with the desired properties. This proof has long been forgotten, since the result was superseded by more general results, leading eventually to a proof of the full theorem. However, the first step in his argument for convex domains is precisely the statement and proof of an early version of what eventually became known as the "Schwarz Lemma".

***Lemma 1** (The Schwarz Lemma). Let $f(z)$ be analytic on a disk $D = \{|z| < R_1\}$, and suppose that $|f(z)| < R_2$ on D and $f(0) = 0$. Then*

$$|f(z)| \leq \frac{R_2}{R_1}|z| \quad \text{for } |z| < R_1. \tag{1}$$

It is also generally noted (although not originally by Schwarz) that strict inequality holds in (1) for every $z \neq 0$ unless f is of the special form

$$f(z) = \frac{R_2}{R_1}e^{i\alpha}z \quad \text{for some real } \alpha. \tag{2}$$

As immediate corollaries, one has:

***Corollary 1** (Liouville's Theorem). A bounded analytic function in the entire plane is constant.*

Proof. R_2 is fixed, and R_1 may be chosen arbitrarily large.

***Corollary 2.** If $R_1 = R_2$, then*

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$$(3) \qquad |f'(0)| \le 1.$$

A slightly less obvious, but still elementary corollary is

Corollary 3. *If $R_1 = R_2$ and if f maps the boundary to the boundary, then at any point b with $|b| = R_1$ where $f'(b)$ exists, one has*

$$(4) \qquad |f'(b)| \ge 1.$$

The proof follows immediately from the fact that distances to the origin are shrunk under f, and therefore distances from the boundary are stretched. More precisely, for t real, $0 < t < 1$, we have $|f(tb)| \le t|b|$, so that

$$|f(tb) - f(b)| \ge R_1 - tR_1 = |tb - b|$$

from which (4) follows.

We will return later to the possible significance of this elementary observation. Although we will not make use of it here, we note that a refinement of the above argument gives a stronger and sharp boundary equality; namely, with $R_1 = R_2 = 1$, if a single boundary point b maps to the boundary and if $f'(b)$ exists, then

$$|f'(b)| \ge 1 + \frac{1 - |f'(0)|}{1 + |f'(0)|}.$$

A proof may be found in [O3].

The standard proof of the Schwarz Lemma—not the proof that Schwarz himself originally gave—consists of observing that the condition $f(0) = 0$ implies that the function $g(z) = f(z)/z$ is a regular analytic function in the disk $|z| < R_1$; apply the maximum principle to $g(z)$ in each disk $|z| \le r$, and take the limit as $r \to R_1$. That proof, according to Carathéodory (in his 1952 book *Conformal Representation*, p. 114, Note 13) is due to Erhard Schmidt, but was first published by Carathéodory in 1905. Carathéodory also notes that a similar proof had been given by Poincaré back in 1884.

Corollary 1 above, Liouville's Theorem, is a fundamental result of complex function theory, with many important consequences. The relation of the Schwarz Lemma to Liouville's Theorem is a prototypical example of what is often known as "Bloch's Principle", whose author, André Bloch, is probably best known on three counts: one of them is "Bloch's Theorem", which we discuss below; a second is "Bloch's Principle"; and the third is the fact that both of these, as well as a considerable amount of other interesting mathematics, were obtained while Bloch was in a psychiatric hospital. He had been confined there after murdering his brother and his aunt and uncle at the end of World War I; he had served for three months at the front and then had to be discharged as a result of a fall from the top of an observation post (see the 1988 articles in French and English by Henri Cartan and Jacqueline Ferrand).

Bloch's Principle is more a heuristic device than a result. It states, in essence, that whenever one has a global result such as Liouville's Theorem, there should be a stronger, finite version from which the general result will follow. The move from Liouville's Theorem to the Schwarz Lemma is a perfect example of such a result. Bloch himself gave another example, by proving a finite result that not only implies Picard's Theorem, but was a simple "elementary proof" of Picard's Theorem, not relying on the use of the elliptic modular function or the Koebe uniformization theorem.

Theorem 1 *(Bloch's Theorem). There is a universal constant B, $B > 0$, with the property that for every value of $R < B$, every function $f(z)$ analytic in the unit disk D and normalized so that $|f'(0)| = 1$ maps some subdomain of D one-to-one conformally onto a disk of radius R.*

The largest value of B for which Bloch's Theorem holds is known as *Bloch's Constant*.

We shall return to Bloch's Theorem and Bloch's Constant later. Let us now resume our story with the paper of Pick [P] referred to at the outset. In that paper, Pick made a crucial observation relating the Schwarz Lemma to hyperbolic geometry which one might have thought would have been made earlier by Klein or Poincaré.

Lemma 2 *(Schwarz-Pick Lemma). Let $f(z)$ be a holomorphic map of the unit disk D into the unit disk. Then*

$$(5) \quad \hat\rho(f(z_1), f(z_2)) \le \hat\rho(z_1, z_2) \quad \text{for all } z_1, z_2 \in D,$$

where $\hat\rho$ refers to distances measured in the hyperbolic metric in D.

For future reference let us note the explicit form of these quantities. We shall use the unit disk model of the hyperbolic plane, in which the hyperbolic metric is given by

$$(6) \qquad d\hat{s}^2 = \left(\frac{2}{1 - |z|^2}\right)^2 |dz|^2$$

and its Gauss curvature $\hat{K}$ satisfies

$$(7) \qquad \hat{K} \equiv -1.$$

Integrating (6) yields

$$(8) \qquad \hat\rho(0, z) = \log \frac{1 + |z|}{1 - |z|} = 2 \tanh^{-1} |z|.$$

What Pick observed was that we can compose f with linear fractional transformations of D onto D, taking z_1 to 0 and $f(z_1)$ to 0. These linear fractional transformations preserve the metric (6) and are therefore isometries of the hyperbolic plane. Hence (5) reduces to

$$(9) \qquad \hat\rho(0, f(z_2)) \le \hat\rho(0, z_2).$$

But hyperbolic distance to the origin, given by (8), is a monotonic function of Euclidean distance, so that (9) is equivalent to (1) when $R_1 = R_2 = 1$. Furthermore, equality holds in the Schwarz Lemma if and only if distances to the origin are preserved; hence the same holds for hyperbolic distances. It follows that equality holds in the Schwarz-Pick Lemma if and only if f is an isometry of the hyperbolic plane, which is to say, a linear fractional transformation of the unit disk onto itself.

An equivalent formulation of the Schwarz-Pick Lemma states that every holomorphic map of the unit disk into itself is either linear fractional—hence a non-Euclidean isometry—or else it shrinks the hyperbolic length of every curve.

Among the many generalizations of the Schwarz-Pick Lemma, probably the single most influential one was that of Ahlfors [A1] in 1938 (or [A2], pp. 350–355).

Lemma 3 *(Schwarz-Pick-Ahlfors Lemma). Let f be a holomorphic map of the unit disk D into a Riemann surface S endowed with a Riemannian metric ds^2 with Gauss curvature $K \le -1$. Then the hyperbolic length of any curve in D is at least equal to the length of its image. Equivalently,*

(10)
$$\rho(f(z_1), f(z_2)) \le \hat{\rho}(z_1, z_2) \quad \text{for all } z_1, z_2 \text{ in } D$$

or

$$\|df_z\| \le 1 \quad \text{everywhere,} \tag{11}$$

where the norm is taken with respect to the hyperbolic metric on D and the given metric on the image, and ρ denotes distances on S with respect to that metric.

For proof, Ahlfors offers a clever but elementary argument based on the fact that the Laplacian of a real function must be nonnegative at a local minimum. Pulling back a conformal metric on the image surface S under the conformal map f, one obtains a conformal metric on the unit disk, to be compared with the original hyperbolic metric.

When his collected papers [A2] were published in 1982, Ahlfors had the opportunity to evaluate his earlier work with the wisdom of hindsight. He confesses that his generalization of the Schwarz Lemma "has more substance than I was aware of," but still says that "without applications my lemma would have been too lightweight for publication" ([A2], p. 341). Of the applications he gave, the most striking is an elementary new proof of Bloch's Theorem (Theorem 1 above) with an explicit lower bound for Bloch's constant B, namely $B \ge \frac{\sqrt{3}}{4}$. Despite many attempts over the years, only slight improvements on this lower bound have been obtained; a more detailed discussion of all these matters can be found in the article [O1].

Over the course of the twentieth century, a whole line of investigations has extended the approach of Pick and Ahlfors to the Schwarz Lemma, where a key factor is that the unit disk is complete in the hyperbolic metric. (See, for example, Theorems 3 and 4 below.) However, in the past decade another approach has allowed a return to the original Schwarz Lemma, in which one has a map of a finite disk into a finite disk. Let us start with some examples of such results.

A *geodesic disk of radius R* on a surface is the diffeomorphic image of a Euclidean disk of radius R under the exponential map. Equivalently, one has geodesic polar coordinates:

$$ds^2 = d\rho^2 + G(\rho, \theta)^2 \, d\theta^2, \tag{12}$$

where ρ represents distance to the center of the disk, and

$$G(0, \theta) = 0, \quad \frac{\partial G}{\partial \rho}(0, \theta) = 1, \quad G(\rho, \theta) > 0 \tag{13}$$

for $0 < \rho < R$.

We shall use the following notation throughout this article: D_R denotes the disk $\{|z| < R\}$, and $d\hat{s}^2$ is a Riemannian metric on D_R. Let f map D_R into a geodesic disk centered at $f(0)$ on a surface S with metric ds^2. Then

$$\rho(p) = \text{ distance on } S \text{ from } f(0) \text{ to } p \tag{14}$$

$$\hat{\rho}(z) = \text{ distance on } D_R \text{ from } 0 \text{ to } z. \tag{15}$$

Example 1. *Let f be a holomorphic map of $|z| < R_1$ into a geodesic disk of radius R_2 centered at $f(0)$ on a surface S with Gauss curvature $K \le 0$. Then*

$$\rho(f(z)) \le \frac{R_2}{R_1} |z| \quad \text{for } |z| < R_1. \tag{16}$$

Note that this is a direct extension of the original Schwarz Lemma, and it has exactly the same consequences:

Corollary 1. *Any holomorphic map of the entire plane into a geodesic disk on a surface with $K \le 0$ must be constant.*

Corollary 2. *If $R_2 \le R_1$, then $\|df_0\| \le 1$.*

Corollary 3. *If $R_2 = R_1$ and if at some point z with $|z| = R_1$, $\rho(f(z)) = R_1$ and df_z exists, then*

$$\|df_z\| \ge 1. \tag{17}$$

Remarks. (1) This example is a slightly more general form of the first part of Lemma 6 of Ros [R]; his proof goes through without change. (2) Corollary 1 is false for $K > 0$; the inverse of stereographic projection is a nonconstant conformal map of the entire plane onto a geodesic disk consisting of the sphere minus a point.

Example 2. *Let f be a holomorphic map of $\{|z| < r < 1\}$ into a geodesic disk of radius ρ_2 centered at $f(0)$ on a surface S whose Gauss curvature satisfies $K \le -1$. Let ρ_1 be the hyperbolic radius of $|z| = r$; i.e.,*

$$\rho_1 = \log \frac{1+r}{1-r}$$

by (8). If $\rho_2 \le \rho_1$ and $d\hat{s}^2$ is the hyperbolic metric on $|z| < 1$, then

$$\rho(f(z)) \le \hat{\rho}(z) \quad \text{for } |z| < r. \tag{18}$$

Corollary 1. *Under the same hypotheses,*

$$\|df_0\| \le 1. \tag{19}$$

Corollary 2. *If, furthermore, $\rho_2 = \rho_1$ and f maps the boundary into the boundary, then at any point z on $|z| = r$ where df_z exists,*

$$\|df_z\| \ge 1. \tag{20}$$

Note that in both these examples we can assert only distance shrinking from the center, unlike in the Schwarz-Pick Lemma and its descendants. In fact, as (17) and (20) indicate, the reverse is likely to be true near the boundary. However, one can show that the original Ahlfors extension of the Schwarz-Pick Lemma is in fact a consequence of the finite version in Example 2 (see [O2]).

Before stating the general shrinking lemma, let us note some of the generalizations of the Ahlfors Lemma that have been made.

Theorem 2. *Yau ([Y], 1973). Let the surface $\hat{S}$ be complete, with Gauss curvature $\hat{K} \ge -1$, and let f be a holomorphic map of $\hat{S}$ into S, with $K \le -1$. Then $\|df_p\| \le 1$ for all p in $\hat{S}$; i.e., the length of every curve in $\hat{S}$ is greater than or equal to the length of its image.*

Theorem 3. *Troyanov ([T], 1991), Ratto-Rigoli-Véron ([RRV], 1994). Let $\hat{S}$ be complete, and let f map $\hat{S}$ holomorphically into S. Suppose that*

$$K(f(p)) \le \hat{K}(p), \tag{21}$$

that

$$K(f(p)) \le 0, \tag{22}$$

and that certain further restrictions hold on K and $\hat{K}$. Then $\|df_p\| \le 1$ for all p in $\hat{S}$.

We refer to the original papers for the exact hypotheses in each case. What is of interest here is condition (21), which represents the natural culmination of the line of investigation initiated by Ahlfors. The underlying philosophy is that the more negative the curvature in the image domain, the more a holomorphic map will shrink distances and curve lengths. Note that we are really comparing two metrics on the same domain: the original metric $d\hat{s}^2$ and the pullback of the metric ds^2 under f. In fact all of the Ahlfors-type lemmas may be stated as comparison theorems between two conformally related metrics, and, again, the philosophy is that the more negative the curvature the shorter the curve lengths in the metric.

This type of result seems oddly reminiscent, but in apparent reverse, of the standard comparison theorems from Riemannian geometry, which say roughly that the more negative the curvature the more certain curves are stretched. Specifically, one has:

Lemma 4 *(Riemannian comparison lemma). Let ds^2 and $d\hat{s}^2$ be metrics given in geodesic polar coordinates by*

$$ds^2 = d\rho^2 + G(\rho,\theta)^2\, d\theta^2$$
$$d\hat{s}^2 = d\rho^2 + \hat{G}(\rho,\theta)^2\, d\theta^2.$$

If

$$K(\rho,\theta) \le \hat{K}(\rho,\theta) \quad \text{for } 0 < \rho < \rho_0, \tag{23}$$

then

$$\frac{1}{G}\frac{\partial G}{\partial \rho} \ge \frac{1}{\hat{G}}\frac{\partial \hat{G}}{\partial \rho} \tag{24}$$

and

$$G(\rho,\theta) \ge \hat{G}(\rho,\theta) \quad \text{for } 0 < \rho < \rho_0. \tag{25}$$

Note that

$$G(\rho_1,\theta) = \frac{ds}{d\theta} \text{ along the geodesic circle } \rho = \rho_1, \tag{26}$$

so that (25) implies that

$$L(\rho_1) \ge \hat{L}(\rho_1) \quad \text{for } 0 < \rho_1 < \rho_0, \tag{27}$$

where $L(\rho)$ and $\hat{L}(\rho)$ refer to the lengths in their respective metrics of geodesic circles of radius ρ.

An obvious question is what relation, if any, exists between the Ahlfors-type lemmas as in Lemma 3 and the Riemannian comparison lemma (Lemma 4). The answer is twofold: First, there is a heuristic argument, based on (17) and (20), that provides a link between the two. Second, we can use the Riemannian comparison lemma to prove a general finite shrinking lemma that contains our Example 2 above as a special case and therefore provides a new route to proving the original Ahlfors Lemma.

Let us start with a brief look at the heuristic argument relating the two forms of comparison. We have a geodesic disk $\hat{D}$ of radius ρ_1 on a surface with Riemannian metric

$$d\hat{s}^2 = d\hat{\rho}^2 + \hat{G}(\hat{\rho},\theta)^2\, d\theta^2,$$

where for any point P in $\hat{D}$, $\hat{\rho}(P)$ is the distance between P and the center O of the disk. We map $\hat{D}$ conformally by f into a surface S with metric ds^2 and assume that the image lies in a geodesic disk D of the same radius centered at the point $f(0)$. Under suitable curvature restrictions we wish to show that

$$(28) \qquad \rho(f(P)) \le \hat{\rho}(P) \quad \text{for all } P \text{ in } D,$$

where $\rho(Q)$ is the distance on S from $f(0)$ to Q. We introduce geodesic polar coordinates

$$ds^2 = d\rho^2 + G(\rho, \theta)^2\, d\theta^2,$$

with $0 \le \rho < \rho_1$ and $0 \le \theta < 2\pi$, on the image, and we assume that the curvature relation is

$$(29) \qquad K(\rho, \theta) \le \hat{K}(\hat{\rho}, \theta) \quad \text{when } \rho = \hat{\rho};$$

that is, for each fixed θ, the curvature of the image geodesic disk is at most equal to the curvature of the original at the same distance from the center. Then what we want to show, inequality (28), is that each geodesic disk $\hat{\rho} < c$, for $c < \rho_1$, maps into the geodesic disk $\rho < c$ in the image. Heuristically, the images of the interior disks are likely to be largest when f maps $\hat{D}$ *onto* the full disk D. So let us assume that f is such a map and f takes the boundary $\hat{\rho} = \rho_1$ to the boundary $\rho = \rho_1$. Let us further assume that f is defined and conformal in a slightly larger disk, $\hat{\rho} < \rho_0$. Then the Riemannian comparison lemma applies, and we have inequality (27), which tells us that *globally* the map f takes the geodesic circle $\hat{\rho} = \rho_1$ of length $\hat{L}(\rho_1)$ onto a geodesic circle of greater or equal length $L(\rho_1)$; *locally*, by virtue of (26), the inequality (25) tells us that under the map of $\hat{\rho} = \rho_1$ to $\rho = \rho_1$ that relates points with the same angular coordinate θ, we have

$$(30) \qquad \frac{ds}{d\hat{s}} \ge 1.$$

However, f will not in general preserve θ, so that inequality (27) tells us only that (30) holds *on average*, where s and $\hat{s}$ represent arclength along $\rho = \rho_1$ and $\hat{\rho} = \rho_1$ *under the map* f. The final heuristic assumption is that (30) holds along the whole curve $\rho = \rho_1$, under the map f. Then conformality of f implies that the same inequality also holds in the radial direction, so that along each "radius" $\theta = \theta_0$ of $\hat{D}$, we have

$$(31) \qquad \left.\frac{d\rho}{d\hat{\rho}}\right|_{\hat{\rho}=\rho_1} \ge 1.$$

Here $\rho(\hat{\rho})$ is the function whose value is $\rho(f(P))$ at the point P in $\hat{D}$ with coordinates $(\hat{\rho}, \theta_0)$. Suppose now that we have *strict* inequality in (31), so that points in $\hat{D}$ near the boundary $\hat{\rho} = \rho_1$ move *farther* from the boundary $\rho = \rho_1$ of D and therefore move *closer* to the center of D. Then (28) holds, in fact with strict inequality, for points P in some annular region near the boundary of $\hat{D}$. We are then back to our original situation on a disk of smaller radius in $\hat{D}$, and we may expect the same kind of contraction (28) to extend.

In brief, the heuristic connection is that an equality like (29) on Gauss curvature implies an *expansion* of the boundary $\hat{\rho} = \rho_1$ to $\rho = \rho_1$, which by conformality of f implies an expansion in the radial direction from the boundary, or a movement of points *toward* the center, and therefore a *contraction* in the sense of (28).

We have not been able to turn this heuristic argument into a complete proof under the full generality of (29), but we have been able to obtain the result for a very broad class of metrics, including those of Examples 1 and 2, namely, for all metrics $d\hat{s}^2$ which have circular symmetry.

***Theorem 4** (General Finite Shrinking Lemma). Let $\hat{D}$ be a geodesic disk of radius ρ_1 with respect to a metric $d\hat{s}^2$. Assume that $d\hat{s}^2$ is circularly symmetric, so that*

$$(32) \qquad d\hat{s}^2 = d\hat{\rho}^2 + \hat{G}(\hat{\rho})^2\, d\theta^2 \quad \text{for } 0 \le \hat{\rho} < \rho_1,$$

where $\hat{G}$ depends on $\hat{\rho}$ only and not on θ. Let f be a holomorphic map of $\hat{D}$ into a geodesic disk D of radius ρ_2 on a surface S, with center at the image under f of the center of $\hat{D}$. If $\rho_2 \le \rho_1$ and if

$$(33) \qquad K(\rho, \theta) \le \hat{K}(\hat{\rho}) \quad \text{for } \rho = \hat{\rho},$$

then

$$(34) \qquad \rho(f(P)) \le \hat{\rho}(P) \quad \text{for all } P \text{ in } \hat{D}.$$

For details of the proof we refer to [O2].

There are several remarks to be made concerning this result.

First, as stated it does not immediately include the full form of Example 1 above, because of the assumption $\rho_2 \le \rho_1$. However, the proof yields a more general statement (Theorem 2 of [O2]) without that assumption; Example 1 then appears as a special case.

Second, the results described in Theorem 3 above represent the natural culmination of a century-long process starting with Carathéodory's 1905 publication of what we now call the "Schwarz Lemma", through the Pick interpretation, and the successive generalizations by Ahlfors and Yau (Theorem 2 above), with the overall philosophy that a holomorphic map of one surface into another whose curvature is more negative will shrink distances. All of the earlier results, prior to those of the papers [T] and [RRV] referred to in Theorem 3, required a uniform bound below on curvatures in the domain of the map that dominates a global bound above in the image. What Theorem 3 shows is that, under suitable hypotheses, pointwise bounds will suffice.

Theorem 4 completes in a certain sense this circle of ideas by going back to the original Schwarz-type lemma for a map of a finite disk into a finite disk, in contrast to the Schwarz-Pick-Ahlfors-Yau-Troyanov-Ratto-Rigoli-Véron versions, all of which require the domain of the map to be provided with a complete Riemannian metric. At the same time, it applies to maps in which one has a pointwise comparison of Gauss curvatures. However, whereas the hypotheses of the papers [T] and [RRV] described in Theorem 3 compare the curvatures at each point of the image with that at the preimage of the point under a given holomorphic map, Theorem 4 compares curvatures in the image with curvatures in the domain at comparable distances from fixed points, independent of the map. In cases such as the original Ahlfors Lemma and the Yau generalization, where there are global bounds on curvatures, there is no difference between the two types of comparison.

We conclude with two final notes.

First, the Schwarz Lemma may be pictured as the progenitor of a huge family tree, branching out in many directions. In this article, we have followed just one of those branches, but there are many others. To name just two, there are the many generalizations to higher dimensions and the "Discrete Schwarz-Pick Lemma" for circle packings proved by A. F. Beardon and K. Stephenson in 1991, which was applied in 1996 by Z.-X. He and O. Schramm to give a new proof of Thurston's innovative approach to the Riemann mapping theorem via circle packing.

Second, the particular branch we have pursued here has blossomed in a most remarkable way in a recent result [BE] announced by M. Bonk and A. Eremenko. We can describe their main result as follows.

Consider a triangulation of the Riemann sphere consisting of four equilateral triangles with vertices at the points of an inscribed regular tetrahedron. Let f be a conformal map of a Euclidean equilateral triangle onto one of those four spherical triangles. By successive reflections, f can be extended to a meromorphic function in the entire plane whose image will be an infinite-sheeted Riemann surface over the sphere with simple branch points at each of the vertices of the triangulation. Each circular disk on the sphere whose boundary circle passes through three vertices of the triangulation will have an infinite number of unbranched sheets of the surface lying over its interior. Said differently, every such circular disk on the sphere is the one-one conformal image under f of (infinitely many) simply connected regions in the domain.

What Bonk and Eremenko assert is that, for any smaller disk on the sphere, *every* meromorphic function in the plane has the property that its image contains an unbranched disk of at least that size. In other words, the surface described above is the extremal surface, giving the precise value of another Bloch-type constant, analogous to the one in Bloch's Theorem (Theorem 1 above). Furthermore, the authors show that their result implies the original Bloch Theorem, as well as its striking generalization by Ahlfors to the "five-island theorem", stating that for any five Jordan domains on the sphere whose closures are disjoint, at least one of them must be simply covered by the image of any nonconstant meromorphic function in the plane.

A key idea in the Bonk-Eremenko proof is to introduce a metric on a branched Riemann surface over the unit sphere that is the ordinary spherical metric away from branch points and has infinite negative curvature at the branch points. One considers the surface to be the union of spherical triangles satisfying certain conditions. Then the idea, in the authors' words, is that "if the triangles are small enough, then the negative curvature concentrated at the vertices dominates the positive curvature spread over the triangles. Thus on a large scale our surface looks like one whose curvature is bounded above by a negative constant."

And so the fundamental insight of Ahlfors concerning the Schwarz-Pick Lemma continues to bear fruit in the most beautiful and unexpected new ways.

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