Book Review

What Is Mathematics, Really?

Reviewed by Ed Dubinsky

What Is Mathematics, Really?
Reuben Hersh
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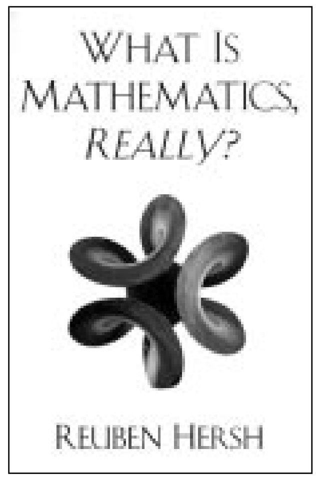
The title of this book refers to the classic *What Is Mathematics?* by Richard Courant and Herbert Robbins, a work to which Hersh reacts as do I, with "wonder and delight." Courant and Robbins approach the question by showing us, in exquisite exposition, a great deal of the content of mathematics. Hersh deals with it by exploring the nature of mathematics—where it comes from, what it is, really. In appealing to the philosophy of mathematics for this exploration, Hersh makes two important points. First, as one way of answering the question of Courant and Robbins, philosophy must be more than an attempt to establish a foundation for mathematics. Second, perhaps more than for most subjects, the purveyors of mathematics must be major players in the development of its philosophy.

In this lively and pleasant-to-read philosophical work, a serious and accomplished mathematician explores somewhat deeply, and rejects, what he considers to be the three main streams of mathematical philosophy: Platonism, formalism, and intuitionism or constructivism.[1] As an alternative he offers what he calls "humanism", the notion that "mathematics must be understood as a human activity, a social phenomenon, part of human culture,



historically evolved, and intelligible only in a social context." Hersh describes some of the standard issues of philosophy of mathematics, such as existence of finite and infinite mathematical entities, intuition, proof, and truth, and tries to show that his philosophy deals with these issues better than do the philosophies he rejects. Although I found the book very interesting and informative in many ways, I am not sure Hersh succeeds in making his case for his humanist philosophy.

Part One

The opening gambit of the book is presented as both "a worked exercise in Pólya's heuristic" and "an inquiry into mathematical existence." The problem is to count the various parts of a 4-dimensional cube and reflect on what kind of sense the calculations could make. In true pólyaesque spirit, Hersh switches immediately to the 3-cube and counts its vertices, edges, and faces. He does the same for the 2-cube and the 1-cube. The three sets of formulas show a clear pattern that is easily generalized to four dimensions. This leads to a list of questions about the existence of a 4-cube. If it exists, where is it? If it does not exist, how could we obtain such detailed information about it? What about a 3-cube? Does it exist in ordinary space, given that we can't produce a perfect 3-cube as a physical object? A little bit later, Hersh uses possible answers to these questions to help explain various philosophies of mathematics, including his own humanism.

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[1] *Unfortunately, the word "constructivism" is used in both philosophy of mathematics and the psychology of mathematics with two very different meanings. To avoid confusion, and because I happen to accept much of constructivism in its psychological sense, I prefer to use the term "intuitionism" in the philosophical context.*

After this introductory example, Hersh turns to the main point of the book, which is to explain why he rejects the three mainstream philosophies—Platonism, formalism, and intuitionism—as inadequate for a philosophy of mathematics and why he believes his humanism is superior. Along the way he considers a number of generally accepted properties of mathematics and tries to "debunk" them.

Mainstream Philosophies of Mathematics

Platonism, as Hersh explains it, is the idea that "mathematical entities exist outside space and time, outside thought and matter, in an abstract realm independent of any consciousness, individual or social." In my opinion it is impossible for anyone who has actually done mathematics—from the student trying to find the answer to the odd-numbered problems to the researcher trying to prove a theorem or find a counterexample—to avoid the feeling of trying to find something that is "out there", that has an existence independent of what anyone might be thinking or doing. When engaging in such activity, a Platonist view is almost a requirement. As Hersh points out, this does not mean that Platonism is adequate as a philosophy of mathematics. Indeed, Hersh rejects it as such for several reasons: it does not relate to material reality or make contact with flesh-and-blood mathematicians; it violates the empiricism of modern science; and it insists on acceptance of a "strange parallel existence of two realities—physical and mathematical," but does not explain how the two interact.

Formalism, according to Hersh, says that mathematics is an otherwise meaningless game played by explicit but arbitrary rules. Hersh's objections to formalism are more serious than are his concerns about Platonism. He argues that the rules are not arbitrary, but rather are "historically determined by the workings of society that evolve under pressure of the inner workings and interactions of social groups, and the physiological and biological environment of earth." Moreover, he asserts that this is not how mathematics is actually done, that "the notion of strictly following rules without any need for judgment is a fiction" and that it is "misleading to apply it to real life."

Intuitionism accepts the set of natural numbers as the fundamental datum of mathematics from which all meaningful mathematics must be obtained through a process of finite construction that does not make use of the law of the excluded middle. In objecting to this particular philosophy, Hersh adopts the anthropological point of view that the intuition of the natural numbers is simply not universal. His view is supported by the research of Piaget, who established that children construct in their minds a conception of the natural numbers based on their experiences and certain modes of thinking. For Piaget, as opposed to Kronecker, the natural numbers are not given by God (at least not before the age of seven for most children in Western culture), but are constructed in an individual's mind by coordinating the concepts of set inclusion and ordering.

A Humanist Philosophy of Mathematics

As an alternative to the mainstream philosophies of mathematics which he rejects, Hersh offers a humanist or sociohistorical point of view. He says: "There's no need to look for a hidden meaning or definition of mathematics beyond its social-historic-cultural meaning." In other words, one answers the big questions by looking at what is and has been done in the society of mathematicians and by people dealing with mathematical situations in everyday life. Thus, to the standard kinds of existence discussed by philosophers, the mental and the physical, Hersh adds a third, the social. To illustrate his view and compare it with other philosophies, Hersh considers a pair of examples: the meaning of the concept of "two" and a return to the issue of existence of the 4-cube.

According to Hersh a key to understanding the concept of "two" is to see that the word is used both as an adjective and as a noun. It is an adjective that represents a process (counting). If one looks at what people do, Hersh argues, then the set of counting numbers is actually finite, because no one can count to, say, $10^{10^{10}}$, and so that is not a counting number. On the other hand, "two" is also a noun. In Hersh's philosophy the existence of the object to which this noun refers comes from a social process of disconnecting "from 'real objects', to exist as shared concepts in the minds/brains of people who know elementary arithmetic."

This does not seem to me to be very different from the empiricist view that, as an object, "two" represents what is common in all situations in which there are two (the adjective here) objects. This view is opposed to the constructivist, or Piagetian, view that the object "two" is constructed using a mechanism called reflective abstraction applied to those situations. In both cases there is individual mental activity as well as social interaction, and one can focus on either. But in my view the more important distinction is between thinking of abstraction as extraction of common features versus thinking of it as construction of meaning. One reason I find it hard to accept Hersh's version of humanism is its focus on social issues as opposed to the struggle of an individual to make sense out of her or his experiences.

The example of the 4-cube can be used to illustrate some differences between various philosophies. Hersh points out that for the Platonist the 4-cube exists as a "transcendental, immaterial, inhuman abstraction," and our ideas about it are representations of this ideal; for the intuitionist as well as for the formalist, there is no real 4-cube, but only a representation "without a represented";

and for the humanist the 4-cube exists "at the social-cultural-historic level, in the shared consciousness of people…as a kind of shared thought or idea."

As a constructivist (in the sense of Piaget, not Brouwer/Bishop) I cannot resist suggesting another possibility. Using our senses, we experience certain phenomena that lead us to imagine (that is, make certain mental constructions of) what we call squares and cubes. In trying to understand these phenomena and mental images, we can apply a mathematical formalism to define a (unit) n-cube as a subset of $\mathcal{R}^n$, consisting of all n-tuples, each of whose components are either 0 or 1. Each individual n-tuple is a vertex; pairs of n-tuples differing in exactly one component are edges; quadruples of n-tuples with all but two components fixed are faces; and so on.

One might object that this is a formal, not constructive, point of view. I would argue, however, that what is formal here is Hersh's formulation in which a 4-cube is obtained purely by using an arithmetic pattern to obtain formulas analogous to those in lower dimensions. In the analysis I propose, the individual is going back and forth between mental constructions of points, lines, edges, etc., and the sets of n-tuples of 0's and 1's. The combination of mental constructions and formal expressions is what the individual uses to give meaning to the abstract concept of an n-cube. I would agree with Hersh that it is a shared conceptualization in the sense that these mental activities generally take place in a social context and emerging ideas are certainly negotiated, but in my view it is the mental constructions of individuals, not their social interactions, that is the basic mechanism through which the n-cube comes to exist.

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Infallibility and Other Conventional Wisdom about Mathematics

Whether one agrees with it or not, however, the real test for the humanist philosophy, or any philosophy for that matter, is its ability to serve as a tool for investigating important questions. In Chapters 3 and 4, Hersh tries to use it as a basis for investigating, and rejecting, a number of commonly held ideas about mathematics.

Hersh suggests that mathematics has a "front", which consists of polished results that we show to the world (including our students) and a "back", which consists of what we do to obtain those results. According to Hersh, mainstream philosophy relates only to the front, whereas humanism insists that we focus on the back. When he does look to the rear, Hersh finds that mathematics is not infallible, because mathematicians make mistakes. Indeed, he points out that some proofs are so long and complex that it is not clear that *anyone* can say for sure that they are correct.

Humanism also argues that mathematics is not unique, because in many mathematical situations mathematicians do not understand each other and it does happen that different mathematicans can develop different approaches to study the same phenomena. Euclid's proofs are incomplete, people cannot fathom his axioms, and there are alternatives to Euclidean geometry, so "mathematics does not contain truths about the universe that are clear and indubitable." Hersh claims that formalism does not describe where a mathematical result comes from, because, he asserts, the mathematician always "knows" the result *before* he or she writes down a formal proof.

Intuition

An important issue that every philosophy of mathematics has to consider is mathematical intuition, and here Hersh provides some interesting insights. For the Platonist, intuition is the mechanism for accessing the postulated ideal world, for connecting human awareness with mathematical reality. How is intuition acquired or developed? Why does it vary from individual to individual? Does it directly perceive an ideal reality, as our eyes perceive visible reality? Hersh criticizes the mathematical Platonist for not even trying to answer such questions.

For the intuitionist, the source of the natural numbers (from which all mathematics is supposed to originate) is intuition. But, Hersh argues, this view violates historical, pedagogical, and anthropological experience. As I indicated above, this "innate intuition" appears to be unavailable to Western children until a certain age and does not appear at all in certain cultures. Again the criticism is that the intuitionists do not deal with such questions.

For the formalist, intuition is the source of the correct theorems for which formal proofs are devised. Hersh suggests, somewhat sarcastically, that according to formalism "Cauchy knew Cauchy's integral theorem, even though…he didn't know the meaning of any term in the theorem. He didn't know what is a complex number, what is an integral, what is a curve; yet he found the complex number represented by the integral over this curve!" This is unsatisfactory for Hersh because of its variance with the experience, for example, of so many mathematicians who make conjectures they can't prove.

For the humanist, mathematics is the study of mental objects with reproducible properties, and intuition is the faculty by which we consider or examine these internal, mental objects. Intuition is the effect in the mind/brain of, at very early stages, manipulating concrete objects; at a later stage, of making marks on paper; and still later, manipulating mental images, doing problems, and discovering things for ourselves. Thus, according to Hersh, in the humanist viewpoint intuition consists in the mental representations of mathematical objects acquired through repeated experiences. These representations and the ideas we have about them are checked by interaction with teachers, fellow students, and mathematical colleagues. Thus, intuition is a set of shared concepts, what Hersh calls "mutually congruent mental representations."

The Existence of Infinite Objects

For me, philosophy of mathematics becomes interesting and important when it tries to explain the most essential aspects of mathematical experience. Intuition is certainly one of these. Infinity is another. Unfortunately, I find Hersh's treatment of this important topic somewhat superficial, occasionally naive, and overall unsatisfactory. Of course, as Hersh points out, infinity is different from physical reality and comes out of our heads. But the brain, he tells us, is a finite object and cannot contain anything infinite. Here we have a contradiction that surely must be dealt with, but Hersh gives us only: "It's not the infinite that our minds/brains generate, but *notions* of the infinite." I am not sure what the difference is between "the infinite" and "notions of the infinite," but I am sure that I do not feel very enlightened by this explanation. He returns to infinity several times throughout the book, but, unfortunately, he eschews analysis in favor of "sound bites", such as "Euclid had finite line segments, never an infinite line," "'Infinite' isn't a number," and "The biggest computer ever built doesn't have space for even one infinite non-terminating nonrepeating decimal number."

Let me consider the last comment for an alternative way of trying to go more deeply into the matter. Hersh would argue that one could not store the following number, for example, in a computer:

$$x = 0.1101000100000001000000000000\,000100\ldots,$$

and the only way of dealing with it is by approximation through truncation. But here is something one *can* store in a computer:

```
x := func(n); $ n is a nonnega-
tive integer

if exists i in [0..n] | n=2**i
then return "1"; else return "0";
end;

end;
```

This is a program that produces, given n, the n^{th} digit in the decimal expansion of the number in question. Moreover, if after writing this program one gives the instruction

```
"0."+%+[x(i) : i in [0..100]];
```

one gets the computer to print x correct up to 100 places. Thus this program gives the computer the *capacity* for calculating x correctly to any number of decimal places.

I would argue that in this way one can store quite a few irrational numbers in a computer. Of course, one can store only some in this way (perhaps all those which intuitionists accept as numbers), but the mind can do better; I wonder what implications such a point of view would have for Hersh's notions of infinity. I am disappointed that Hersh appears to be satisfied with an almost superficial analysis of such questions.

Implications for Education

The book contains a number of instances of carelessness. One of particular interest to me is the promise in the preface that there is a discussion of teaching in Chapter 1. I could not find it. The preface also claims that this book can "assist education reform by helping mathematics teachers and educators understand what mathematics is." However, Hersh supplies few details about how the ideas in the book apply to education. As nearly as I can tell, his main comments on this topic don't come until the end of the book, in Chapter 13, where Hersh devotes about a page to assertions about education that are not very convincing. We read, for example, that "if other factors are compatible, adoption by teachers of a humanist philosophy could benefit mathematics education." This may well be, but I could not find even an attempt to justify this claim, much less anything that warrants the comment that follows: "It's not unexpected that a philosophy [Hersh's humanist philosophy] epistemologically superior is educationally superior."

I cannot judge if the contents of the 237 pages preceding this assertion warrant the claim of epistemological superiority. I can say, however, that there is little or nothing written here that gives

Hersh the right to make his educational assertion. It appears to be an example, depressingly common, of a mathematician expressing an opinion about education and thinking that the fact that he or she has this view means that it has been established for all. We are trying in mathematics education to develop better standards of discourse.

Part Two

For any mathematician just beginning to be seriously interested in the philosophy of mathematics, Hersh performs a great service in the second part of this book. He gives us a capsule account, ranging from a few sentences to a few pages, of the philosophical thinking of nearly fifty individuals, from Aristotle to Wittgenstein. It is an invaluable introduction, and I recommend it to any mathematician for her or his first reading in the philosophy of mathematics. It is particularly helpful because it is written from the point of view of a mathematician. The work of Orestes and the Scholastics is omitted, but this may be because Hersh is focusing on the analysis strand of mathematics and so is less interested in the arithmetic/number theory/algebra line of development.

Hersh pays greater attention to the work of Piaget than do most who write about the philosophy of mathematics. He acknowledges the tremendous impact of Piaget's writings on cognitive psychology. He seems to feel that Piaget's notions of stages are based on maturation rather than cognitive development and so misunderstands its role in education. I heartily agree with Hersh's focus on Piaget's epistemology, but I am disappointed that he rejects the book Piaget wrote with Beth on this topic because of his disagreement with the philosophical position taken in the first half of the book. What Hersh misses is that this first half, on foundations of mathematics, was written by Beth. The second half was written by Piaget and is a wonderful account of his epistemology. In it he considers in some depth all of the questions Hersh is interested in and more. It would have been very helpful to read Hersh's view of this thinking.

Philosophy and Theology

In this 150-page outline of 3,000 years of philosophical thinking, does Hersh tell a coherent story? I think that he does, and it goes something like this. For almost all of this period, mathematics was intimately connected, as was all scientific thinking, with theology. Most of the objections to Platonism can be dealt with by thinking of its ideal world as residing in the mind of God. Thus, mathematics exists as the thought of God, and therefore any knowledge of it provides eternal truths about the universe. Imagine the glory of a mathematician whose discovery of a nontrivial theorem provides an insight into the thinking of an eternal, all-powerful, all-knowing deity.

This point of view held, with dissenters of course, until about the end of the eighteenth century, when two things started happening. First, the development of alternatives to Euclidean geometry cast doubt on the eternal truth status of one of the main pillars of mathematics. Second, some mathematicians and scientists did not believe in the existence of God, and others who did wanted to think of mathematics as independent of any divine intervention. This removed one of the main arguments in support of Platonism. So where does mathematics come from if not from the mind of God? The answer, many thought, was in the logical foundations of the subject. For others, it was in formalism; for still others, intuitionism. For Hersh these philosophical alternatives fail to deal satisfactorily with the main questions of philosophy of mathematics, and this is what brings him to the humanist point of view.

Conclusion

I think this book is very valuable for its sketch of a number of philosophical ideas relating to mathematics and its introduction to Hersh's humanist philosophy of mathematics. As a (psychological) constructivist, I am disappointed that Hersh says so little about a set of ideas that has been receiving a great deal of attention in recent years. But my most serious criticism is that, in spite of a great deal of "table talk" about his social-historic-humanist approach, Hersh does not use it very much—or, in my opinion, very effectively—to attack the great problems of the philosophy of mathematics. His analyses and criticisms of other philosophies are very useful, but they don't rely on the humanist point of view and could easily have been made without it. As just one important example, consider a major criticism that Hersh makes of Platonism. He points out that it posits an ideal world independent of human thought or activity but tells us nothing about the mechanisms by which humans can interact with that world. True enough, but Piaget already applied his constructivist viewpoint to make this same criticism thirty-five years ago. This reader is left wondering what new results in the philosophy of mathematics can be obtained from Hersh's humanist approach.