Kenkichi Iwasawa (1917–1998)

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Kenkichi Iwasawa, whose ideas have deeply influenced the course of algebraic number theory in the second half of the twentieth century, died in Tokyo on October 26, 1998. He spent much of his mathematical career in the United States, but he remained quintessentially Japanese and incarnated many of the finest qualities of the traditional Japanese scholar. Iwasawa was born on September 11, 1917, in Shinshuku-mura near Kiryu in Gumma prefecture. After elementary school he was educated in Tokyo, where he first attended Musashi High School and then did his undergraduate studies at Tokyo University from 1937 to 1940. In 1940 he entered the graduate school of Tokyo University and became an Assistant in the Department of Mathematics, being awarded the degree of Doctor of Science in 1945. However, the war years were difficult for him. He became seriously ill with pleurisy in 1945 and was only well enough to return to his post at the university in April 1947. He was appointed Assistant Professor at Tokyo University from 1949 to 1955.

He travelled to the United States in 1950 to give an invited lecture at the International Congress of Mathematicians in Cambridge, Massachusetts, and then spent the two academic years 1950–52 at The Institute for Advanced Study in Princeton. While preparing to return to Japan in the spring of 1952, he received the offer of a post at the Massachusetts Institute of Technology and ended up staying there until 1967. In 1967 he moved to Princeton University as Henry Burchard Fine Professor of Mathematics and remained at Princeton until his retirement in 1986. In 1987 he and his wife returned to live in Tokyo.

He was awarded the Asahi Prize in 1959, the Prize of the Japan Academy in 1962, the American Mathematical Society Cole Prize in 1962, and the Fujiwara Prize in 1979. His principal mathematical legacy is a general method in arithmetical algebraic geometry, known today as *Iwasawa theory*, whose central goal is to seek analogues for algebraic varieties defined over number fields of the techniques which have been so successfully applied to varieties defined over finite fields by H. Hasse, A. Weil, B. Dwork, A. Grothendieck, P. Deligne, and others.

Cyclotomic Fields

Until about 1950, most of Iwasawa's papers were on questions of group theory, and we shall briefly discuss this aspect of his work later. However, he himself stated that he was interested in number theory from his student days, and all of his published papers from the early 1950s onwards are devoted to algebraic number theory. The dominant theme of his work in number theory is his revolutionary idea that deep and previously inaccessible information about the arithmetic of a finite extension F of $\mathbb{Q}$ can be obtained by studying coarser questions about the arithmetic of certain infinite Galois towers of number fields lying above F. This idea, whose power lies in subtly mixing p-adic analytic methods with Galois cohomology, has subsequently been applied to a much wider circle of problems in arithmetic algebraic geometry. But the origin and archetypical example of Iwasawa's

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Left to right: Kenkichi Iwasawa with John Coates, John Tate, and Ichiro Satake at the 1976 Symposium on Algebraic Number Theory in Kyoto.

theory is in the classical theory of cyclotomic fields when F is the field generated over $\mathbb{Q}$ by the p-th roots of unity, where p is a prime number, and the infinite tower above F is given by the fields generated by all p-power roots of unity. We shall now discuss it in some detail, emphasizing the new ideas introduced by Iwasawa.

We say that an odd prime number p is *irregular* if p divides the class number of the field $\mathbb{Q}(\mu_p)$, where μ_p denotes the group of p-th roots of unity. The first few irregular primes are given by $37, 59, 67, 101, \dots$. This notion was introduced by Kummer in his work on Fermat's Last Theorem. The fact that it is at all feasible to determine whether a prime p is irregular is because of a mysterious and unexpected connexion between irregularity and the values of the Riemann zeta function $\zeta(s)$. For $k = 2, 4, 6, \dots$, we have $\zeta(1-k) = -B_k/k$, where the B_k are the Bernoulli numbers defined by the expansion

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n t^n / n!.$$

Kummer proved the remarkable result that p is irregular if and only if p divides the numerator of at least one of the rational numbers

$$\zeta(-1), \cdots, \zeta(4-p).$$

Iwasawa was the first to realize, in a series of papers published in the 1960s, that the key to a deeper understanding of this result was to study a natural arithmetic representation of the Galois group of the cyclotomic field generated by all the p-power roots of unity over $\mathbb{Q}$. To explain his idea, we begin by defining the *Iwasawa algebra* of an arbitrary profinite abelian group G by

$$\bigwedge(G) = \varprojlim_{U} \mathbb{Z}_p[G/U],$$

where U runs over all open subgroups of G, $\mathbb{Z}_p$ denotes the ring of p-adic integers, and $\mathbb{Z}_p[G/U]$ denotes the ordinary group ring of the finite abelian group G/U with coefficients in $\mathbb{Z}_p$. The Iwasawa algebra is particularly useful because it has both an analytic and an algebraic interpretation. The analytic interpretation of $\bigwedge(G)$ is as the algebra of measures on G with values in $\mathbb{Z}_p$, allowing us to define the integral $\int_G f\,d\mu$ for all μ in $\bigwedge(G)$ and all continuous functions f from G to $\mathbb{Q}_p$, the field of p-adic numbers. The algebraic interpretation is that we can naturally extend the continuous action of G on any compact $\mathbb{Z}_p$-module X to an action of the whole Iwasawa algebra $\bigwedge(G)$.

To apply these notions to cyclotomic fields, we write μ_{p^∞} for the group of all p-power roots of unity; we define F_∞ to be the maximal real subfield of $\mathbb{Q}(\mu_{p^\infty})$, that is,

$$F_\infty = \mathbb{Q}(\mu_{p^\infty}) \cap \mathbb{R};$$

and we take G to be the Galois group of F_∞ over $\mathbb{Q}$. The principal analytic ingredient for Iwasawa's theory of cyclotomic fields is the p-adic analogue of the Riemann zeta function, whose existence was already known from the work of Kummer and T. Kubota, and H. Leopoldt, but for which Iwasawa gave a new construction in his 1969 paper [1] by an ingenious use of the classical Stickelberger elements.

We say that an element φ of the ring of fractions of $\bigwedge(G)$ is a *pseudo-measure* if $(\sigma - 1)\varphi$ belongs to $\bigwedge(G)$ for all σ in G (intuitively, one should think of such a φ as possibly having a simple pole at the trivial character of G). Let $G_{\mathbb{Q}}$ be the Galois group of a fixed algebraic closure of $\mathbb{Q}$, and write $\psi : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ for the homomorphism giving the action of $G_{\mathbb{Q}}$ on μ_{p^∞}, that is, $\sigma(\zeta) = \zeta^{\psi(\sigma)}$ for all σ in $G_{\mathbb{Q}}$ and ζ in μ_{p^∞}. By Galois theory G is a quotient of $G_{\mathbb{Q}}$, and the characters ψ^k, where k is any even integer, factor through G. Modulo a slight change of language and normalisation, Iwasawa proved that there exists a unique pseudo-measure ζ_p on G such that

$$\int_G \psi^k d\zeta_p = (1 - p^{k-1})\zeta(1-k) \quad (k = 2, 4, 6, \dots); \tag{1}$$

this makes sense, since one can integrate any nontrivial p-adic homomorphism of G against any pseudo-measure.

The principal algebraic and arithmetic ingredient is a compact G-module X_∞, which Iwasawa had introduced in his papers in the 1950s. Let M_∞ be the maximal abelian extension of F_∞ with the properties that it is unramified outside p and that its Galois group over F_∞ is a pro-p-group. Put $X_\infty = G(M_\infty/F_\infty)$ for the Galois group of M_∞ over F_∞. Clearly X_∞ is a compact $\mathbb{Z}_p$-module. In addition, X_∞ has a natural action of G given by

$$\sigma \cdot x = \tilde{\sigma} x \tilde{\sigma}^{-1}$$

for σ in G and x in X_∞, where $\tilde{\sigma}$ denotes any lifting of σ to the Galois group of M_∞ over $\mathbb{Q}$ (the maximality of M_∞ guarantees that it is Galois over $\mathbb{Q}$).

By our general remarks above, X_∞ then has a natural structure as a module over the Iwasawa algebra $\bigwedge(G)$. It can easily be shown that X_∞ is finitely generated over $\bigwedge(G)$. Moreover, as G is a p-adic Lie group of dimension 1 with no element of order p, a simple structure theory is known for finitely generated $\bigwedge(G)$-modules. By combining arithmetic arguments with this structure theory, Iwasawa proved that there exist an integer $\tau_p \geqslant 1$ and elements $f_1, \ldots, f_{\tau_p}$ of $\bigwedge(G)$ that are not divisors of zero, such that we have an exact sequence of $\bigwedge(G)$-modules

$$(2) \qquad 0 \to X_\infty \to \bigoplus_{i=1}^{\tau_p} \bigwedge(G) \Big/ f_i \bigwedge(G) \to D \to 0,$$

where D is a $\bigwedge(G)$-module of finite cardinality. Via a remarkable series of papers in the 1960s, Iwasawa was led to the startling idea that there should be a simple relation between the analytic object given by (1) and the algebraic object defined by (2). The precise statement, which became known as *the main conjecture on cyclotomic fields*, is the assertion that

$$(3) \qquad \zeta_p \bigwedge(G)_0 = f_1 \cdots f_{\tau_p} \bigwedge(G),$$

where $\bigwedge(G)_0 = \ker(\bigwedge(G) \to \mathbb{Z}_p)$ is the augmentation ideal of $\bigwedge(G)$. Although it is not obvious, Kummer's criterion mentioned above is a consequence of (3) (the starting point for proving this is the classical argument showing that p is irregular if and only if the maximal real subfield of $\mathbb{Q}(\mu_p)$ possesses a cyclic extension of degree p that is unramified outside p and that is distinct from the maximal real subfield of the field generated by the p^2 roots of unity). Much deeper results due to J. Herbrand and K. Ribet about the eigenspaces for the action of the Galois group of $\mathbb{Q}(\mu_p)$ over $\mathbb{Q}$ on the p-primary subgroup of the ideal class group of $\mathbb{Q}(\mu_p)$ are also corollaries of (3).

The honour of giving the first proof of (3) for all primes p fell in 1984 to B. Mazur and A. Wiles using modular curves, but there is still great interest in studying the evolution of Iwasawa's ideas leading to (3), especially in his wonderful paper published in 1964 in [2]. In this paper Iwasawa used cyclotomic units to construct a compact $\bigwedge(G)$-module Y_∞ together with a natural $\bigwedge(G)$-homomorphism

$$(4) \qquad \varphi : Y_\infty \to X_\infty.$$

He then proved, by a very ingenious use of an explicit reciprocity law going back to E. Artin and H. Hasse, that we have an isomorphism

$$(5) \qquad Y_\infty \simeq \bigwedge(G) \Big/ \zeta_p \bigwedge(G)_0;$$

here we are implicitly using the construction of ζ_p given in the later paper [1].

For the sake of completeness, the precise definition of Y_∞ is as follows. Let F_n be the maximal real subfield of the field generated over $\mathbb{Q}$ by the p^{n+1}-th roots of unity ($n = 0, 1, \cdots$). The group of cyclotomic units C_n of F_n is defined to be the intersection with the unit group of the ring of integers of F_n of the multiplicative group generated by all conjugates of $1 - \zeta_n$, where ζ_n is a primitive p^{n+1}-th root of unity. There is a unique prime ν_n of F_n above p, and we write U_n for the group of local units in the completion of F_n at ν_n that are congruent to 1 mod ν_n. Finally, we define C_n to be the completion in the ν_n-adic topology of $C_n \cap U_n$. Then Y_∞ is the projective limit of the U_n/C_n taken with respect to the norm maps.

The beauty of (5) is that, via (1), it makes the values of the Riemann zeta function at the odd negative integers appear naturally in the arithmetic of cyclotomic fields. If we assume that the class number of the maximal real subfield of $\mathbb{Q}(\mu_p)$ is prime to p, Iwasawa showed that φ is an isomorphism, and hence (5) implies (3). In 1990 K. Rubin showed that one could use V. Kolyvagin's ideas on Euler systems derived from cyclotomic units to prove just enough about the structure of the kernel and cokernel of the map φ appearing in (4) to be able to derive the main conjecture (3) from Iwasawa's theorem (5) without any restriction on the prime p.

Returning to the module X_∞ defined above, Iwasawa also conjectured that X_∞ is a finitely generated $\mathbb{Z}_p$-module, and this was proven in 1979 by B. Ferrero and L. Washington. It then follows from (2) that X_∞ is a free $\mathbb{Z}_p$-module of finite rank, say t_p. It is perhaps of interest to note that the largest value of t_p for $p < 4 \times 10^6$ is $t_p = 7$, and this is achieved for the single prime $p = 3,238,481$ (see the paper by Buhler, Crandall, Ernvall, and Metsankla in volume 61 (1993) of *Mathematics of Computation*). It should also be noted that (3) says nothing about the value of the integer τ_p appearing in (2). However, if the class number of the maximal real subfield of $\mathbb{Q}(\mu_p)$ is prime to p, then Iwasawa's theorem (5) and the fact that φ is then an isomorphism show that we can take $\tau_p = 1$. No theoretical approach is known for showing that the class number of the maximal real subfield of $\mathbb{Q}(\mu_p)$ is prime to p, but it has been verified numerically for $p < 4 \times 10^6$ (see the above paper).

These brief remarks highlight only several facets of Iwasawa's varied work on cyclotomic fields. For example, another theme of his research, growing out of his proof of (5), was his discovery of a beautiful explicit formula for the Hilbert norm residue symbol in the field generated over $\mathbb{Q}_p$ by the p^n-th roots of unity for all $n \geqslant 1$. His paper [3] has inspired a large body of work on explicit reciprocity laws in general over the last twenty years.

Left to right: Shigeru Iitaka, Ichiro Satake, André Weil, and Kenkichi Iwasawa. November 1, 1994.

$\mathbb{Z}_p$-Extensions

As we have already stressed, the deepest aspect of Iwasawa's work on cyclotomic fields was the marriage of algebraic and analytic objects expressed by the main conjecture (3). However, already from his papers published in the late 1950s, Iwasawa saw that many of the algebraic aspects of his theory were not special to cyclotomic fields and could be established in the more general setting of $\mathbb{Z}_p$-extensions of number fields. Let F be a finite extension of $\mathbb{Q}$. A $\mathbb{Z}_p$-*extension* of F is an infinite tower of fields

$$(6) \qquad F = F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots,$$

where, for each $n \geqslant 0$, F_n is a cyclic extension of F of degree p^n. The reason for this terminology is that if we define F_∞ to be the union of all the $F_n (n \geqslant 0)$, then F_∞ is a Galois extension of F whose Galois group over F is topologically isomorphic to the additive group of the p-adic integers $\mathbb{Z}_p$. It is easy to see that every F has a unique $\mathbb{Z}_p$-extension contained in $F(\mu_{p^\infty})$, which is called the *cyclotomic* $\mathbb{Z}_p$-extension of F. However, if F is not totally real, class field theory shows that F admits infinitely many non-cyclotomic $\mathbb{Z}_p$-extensions. Iwasawa's first general result about arbitrary $\mathbb{Z}_p$-extensions was the following asymptotic formula. For each $n \geqslant 0$, let p^{e_n} denote the order of the p-primary subgroup of the ideal class group of F_n. Then there exist integers λ, μ, and ν, depending on the $\mathbb{Z}_p$-extension F_∞/F, such that, for all sufficiently large n, we have

$$(7) \qquad e_n = \lambda n + \mu p^n + \nu.$$

What lies behind the proof of (7) is the fact that, if Γ denotes the Galois group of F_∞ over F, then a simple structure theory is known for finitely generated modules over the Iwasawa algebra $\bigwedge(\Gamma)$. Iwasawa originally gave an ad hoc proof of this structure theory, but J.-P. Serre pointed out that it followed from known results in commutative algebra, since $\bigwedge(\Gamma)$ is isomorphic to the ring $\mathbb{Z}_p[[T]]$ of formal power series in an indeterminate T with coefficients in $\mathbb{Z}_p$. Iwasawa wrote a whole series of papers (see, in particular, [4]) in which he exploits this structure theory to study various compact Γ-modules attached to an arbitary $\mathbb{Z}_p$-extension F_∞/F, which are important for arithmetic questions. The assertion (2) is typical of the rather general results he obtains in this direction.

Iwasawa himself has said that the discovery of the asymptotic formula (7) suggested to him that there might be analogies between the p-primary subgroup of the ideal class group of a $\mathbb{Z}_p$-extension F_∞ of F and the p-primary subgroup of the group of points on the Jacobian variety of a curve over a finite field. Such an analogy would suggest that the invariant μ appearing in (7) is zero, and Iwasawa conjectured that this should always be the case if F_∞ is the cyclotomic $\mathbb{Z}_p$-extension of F. This conjecture was proven by B. Ferrero and L. Washington (and a rather different proof was given by W. Sinnott) when F is an abelian extension of $\mathbb{Q}$, but it remains open in general. In the early 1970s Iwasawa discovered the first examples of non-cyclotomic $\mathbb{Z}_p$-extensions where $\mu > 0$ (see [6]). Today there remain many interesting open questions about Iwasawa's invariants λ and μ, including the following two conjectures about the cyclotomic $\mathbb{Z}_p$-extension F_∞ of F. If F is totally real, R. Greenberg has conjectured that we always have $\lambda = \mu = 0$. Secondly, if we fix F and vary p over all primes, it is conjectured, by analogy with the Jacobian variety of a curve over a finite field, that λ is bounded. This last conjecture is known for no other number field than $F = \mathbb{Q}$, where we have $\lambda = 0$ for all p.

Group Theory

Iwasawa's contributions to group theory, and particularly to the theory of locally compact groups, have also had a lasting impact. His paper [5] gave an essential step towards the solution of Hilbert's fifth problem, which asked whether any locally Euclidean topological group is necessarily a Lie group. One of the many new results about topological groups and Lie groups that are proven in this paper is what is now known as the *Iwasawa decomposition* of a real semisimple Lie group. He also shows that any connected Lie group is topologically the product of a compact Lie group and a Euclidean space. In another direction, he proves that if a locally compact group G has a closed normal subgroup N such that both N and G/N are Lie groups, then G itself is a Lie group. Iwasawa recorded many years later his pleasure as a young man at receiving a letter from C. Chevalley praising this paper. He also was one of the pioneers of employing methods from the theory of locally compact groups to number theory. In particular, independently of J. Tate, he discovered the adelic approach to E. Hecke's L-functions with Grossencharacters, foreshadowing a vast amount of subsequent work done in this direction from the point of view of automorphic forms.

Influence and Legacy

It is not easy to discern who influenced Iwasawa most in the early stages of his mathematical career. He himself cited S. Iyanaga and Z. Suetuna as having played an important role in assisting his initial research in group theory. When he came to the United States, E. Artin and A. Weil were the dominant influence in the shift of his interests to algebraic number theory. He had relatively few research students of his own, and the best known of these are B. Ferrero, R. Greenberg, and L. Washington.

Although nearly all of Iwasawa's work in number theory was concerned with $\mathbb{Z}_p$-extensions of number fields, he was very conscious of the parallels with curves over finite fields and was aware that his ideas might be fruitfully applied to a much wider circle of problems. The first step in this direction was carried out by B. Mazur, who initiated, in the late 1960s, the study of the arithmetic of abelian varieties over $\mathbb{Z}_p$-extensions of number fields. Many others followed, and today it is no exaggeration to say that Iwasawa's ideas have played a pivotal role in many of the finest achievements of modern arithmetical algebraic geometry on such questions as the conjecture of B. Birch and H. Swinnerton-Dyer on elliptic curves; the conjectures of B. Birch, J. Tate, and S. Lichtenbaum on the orders of the K-groups of the rings of integers of number fields; and the work of A. Wiles on the modularity of elliptic curves and Fermat's Last Theorem. In all of these problems, values of L-functions play a central role, and a key step in progress on them has been to establish at least some partial analogue of Iwasawa's main conjecture (3) for the relevant L-function.

It would have been anathema to Iwasawa's great personal modesty to write at too great a length about the way in which his ideas have influenced a whole generation of number theorists in Europe, Japan, and the United States. It is sufficient perhaps to recall Narihira's beautiful tanka about the flowering influence of the Fujiwara family in Heian Japan

Saku hana no	Longer than ever before
Shita ni kakururu	Is the wisteria's shadow -
Hito o oomi	So many are those
Arishi ni masaru	Who find shelter
Fuji no kage kamo	Beneath its blossoms!

and say that it can very aptly be applied to Iwasawa's deep and lasting influence on the whole field of arithmetical algebraic geometry today.

Photograph courtesy of Hideo Wada.

Iwasawa on the occasion of his 77th birthday in Tokyo, June 4, 1994, with (left to right) Mrs. Koizumi, Mrs. Satake, and Mrs. Iwasawa.

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