An Imaginary Tale: The Story of $\sqrt{-1}$

Reviewed by Brian E. Blank

An Imaginary Tale: The Story of $\sqrt{-1}$
Paul Nahin
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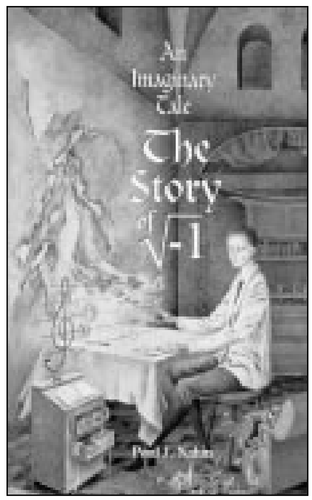
Paul Nahin's *An Imaginary Tale* begins with a cartoon strip in which an imaginary tiger professes to instinctively understand imaginary numbers. If only real humans were so blessed. Each year a new crop of high school students becomes acquainted with the Quadratic Formula and, along with it, negative discriminants. The number *i* is magically invoked to resolve the difficulty. Is it a real number? The correct answer is Clintonesque: "It depends on how you define 'real'." So we compromise: we say that it is an *imaginary* number, but we make sure that it is on the exam—that will make it seem real enough. Happily for most students, imaginary numbers are often no more than a fleeting nuisance. Those who do continue beyond high school algebra pass into an imaginary-free zone called calculus. Because we are often successful at extending this respite through linear algebra and ordinary differential equations, many mathematics majors never see an imaginary number during their entire college careers.

It should be conceded that mankind saw no need for algebraically closed fields for several millennia. Although the Quadratic Formula has become the instrument for exposing students to complex numbers, it did not at any time since its discovery some 4,000 years ago inspire the introduction of imaginary numbers into mathematics. When there arose a problem that resulted in a negative discriminant, it was considered insoluble. Such an interpretation was quite sensible in the Greek and Arab schools of algebra, in which algebraic equations were generally expressions of geometric relationships. The need for imaginary numbers did not manifest itself until the sixteenth century discovery of Cardan's Formula for the roots of the cubic equation.



The seeds for imaginary numbers were planted in the twelfth century when Arab algebra was introduced into Italy through the Latin translation of al-Khwārizmī's great treatise, *Al-jabr wā'l-muqābala*. In 1225 Leonardo of Pisa (Fibonacci) published an approximate solution of a specific cubic and showed that the exact solution could not have a certain form. Incorrect solutions of the cubic were published in 1328 and 1344 by Paolo Gerardi and by Maestro Dardi of Pisa. By the end of the fourteenth century, however, a crucial step was taken. In two anonymous Florentine manuscripts there appears the linear change of variable that transforms a general cubic equation into the so-called *depressed cubic*

$$x^3 + px = q. \tag{1}$$

Progress in the fifteenth century was subtle but essential. Notation improved, abstraction increased. Early in the sixteenth century Scipione del Ferro found the general solution of equation (1):

Brian E. Blank is professor of mathematics at Washington University in St. Louis. His e-mail address is `brian@math.wustl.edu`.

$$x = \sqrt[3]{q/2 + \sqrt{\Delta}} + \sqrt[3]{q/2 - \sqrt{\Delta}}\,,$$
$$\Delta = q^2/4 + p^3/27.$$

Cardan's Formula, as this expression came to be known, presented an immediate conundrum. Consider the equation

$$x^3 - 15x = 4 \tag{2}$$

in which all three roots are real. In this case Cardan's Formula becomes

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}. \tag{3}$$

Rejecting complex numbers here would have meant rejecting three *real* solutions, solutions which were found only after three centuries of struggle.

It is from this juncture that Paul Nahin begins his tale in earnest. Brimming with enthusiasm, he recounts the history of complex numbers through to Cauchy's 1814 memoir on contour integration. In doing so, Nahin interweaves formulaic mathematics with historical narrative, somewhat in the manner of [5]. The story unfolds in three distinct parts: the introduction of and early reactions to complex numbers, the attempts to realize complex numbers geometrically, and the use of complex numbers in the service of elegant mathematics. Along the way the reader will encounter several of the most beautiful exact formulas of classical mathematics: product formulas of Viète, Wallis, and Euler; the reflection formula for the gamma function; the functional equation of the zeta function; and a grab bag of pretty integral and series formulas.

This blend of engaging history and sparkling mathematics has been written for students who have finished freshman calculus, to be "read as a supplement to the more standard presentations of mathematics." By this the author means, one may presume, presentations written by mathematicians: texts that feature accurate statements and carefully reasoned deductions. Nahin, it must be noted, writes from the perspective of the electrical engineer that he is. Be prepared for a notation that expresses the relationship $i = \exp(i\pi/2)$ as $i = 1\angle 90^\circ$. Expect formal manipulations with power series without regard to convergence.

The lack of rigor, one must admit, is defensible: an overly rigorous approach would not be appropriate for Nahin's intended readers. Furthermore, most of the action covered by the book, including that part of Cauchy's work that is discussed, took place before Cauchy introduced the ϵ and δ to analysis. By granting the author some discretion in the matter of rigor, however, we should not forsake our expectations of "honest" mathematics. Readers should not be misled about the validity of an inadequate argument. They should not be confused because the author is reluctant to confront difficult concepts. They should not be kept in the dark about the ideas that lie behind the formulas. Unfortunately, Nahin often does not meet these pedagogical responsibilities.

Consider del Ferro's solution of equation (1). The first step is to substitute $x = u + v$ into equation (1) to obtain

$$u^3 + v^3 + (3uv + p)(u + v) = q.$$

Nahin states that this single equation "can be rewritten as two individually less complicated statements: $3uv + p = 0$ which then says that $u^3 + v^3 = q$." The reader who forgives the confused wording of the thought must still overcome the confusion (between necessity and sufficiency) of the thought itself. Referring to the expression of the solution as a sum of two terms, Nahin asks, "How did del Ferro know to do this?" Responding to his own question, he asserts that the answer lies in the distinction Mark Kac made between an ordinary genius and a magician: with a magician, we do not know how he came to do something even after we have watched him do it. A pretty distinction, but I am sorry: we all know magicians, and del Ferro was no magician. A glance at the Quadratic Formula provides the rationale for del Ferro's approach. Moreover, the incorrect solutions that Gerardi and Dardi published in the fourteenth century had the forms $u + \sqrt{w}$ and $u + \sqrt[3]{w}$ respectively. There is no mystery here. The author ill serves the student when he mystifies the deductive process in mathematics.

In labeling del Ferro a magician, Nahin sets the tone for his book. By the time we reach its end, we are learning "wizard mathematics." One after another the results presented are said to be "astonishing" or "astounding." One mundane result is deemed to be a "bombshell." Fine. Only a killjoy would purge this sort of hyperbole from an elementary exposition. But in mathematics we are not content to allow facts to *remain* astonishing; we try to get to the bottom of them. After solving the equation $(z + 1)^n = z^n$, Nahin observes that all the roots lie on the vertical line $x = -1/2$. He describes this as "rather surprising" *and moves on.* It is not enough for an author to stand back in awe of this apparent coincidence; the author must look to the equation to explain why the proven result is neither a coincidence nor surprising, once it has been understood. To quote E. T. Bell, "So long as there is a shred of mystery attached to any concept, that concept is not mathematical." Words for a mathematical expositor to live by.

A recurring pedagogical concern in *An Imaginary Tale* is that mathematical facts are not put into perspective. The book's contents are an eccentric mix of the essential and the inconsequential sitting cheek by jowl, all treated with equal relish. Students require a stronger guiding hand. They also need a structure in which to frame the facts that they learn. For example, the lengthy discussion of Cardano's *Ars Magna*, in which appeared the first published solution of the cubic, finds no room to remark that the treatise also contained the first solution of the biquadratic. Galois is not mentioned. Abel appears once and then only to say that he called Cauchy a bigot. In short, once the theory of equations has produced i, it has become dispensable.

At times the divide between the engineer and the mathematician is especially deep. To "demonstrate dramatically" the power of De Moivre's Theorem, the author uses the theorem to write out trigonometric expressions for the roots of $z^5 - 1 = 0$. He then presents Lagrange's algebraic derivation of these same roots *in terms of radicals*. Oblivious to their significance, Nahin gets out his calculator and shows that the floating point approximations of the two sets of solutions agree. He then notes that in contrast to De Moivre's Theorem "Lagrange's clever algebraic substitution that works so well for the degree 5 equation does not work in the general case." He suggests $n = 97$. Is this irony? No mention is made of Gauss's work on the solution of the cyclotomic equation by means of radicals.

The treatment of exponents throughout the book is perplexing. This is not a topic in which intuition alone may serve as a substitute for accurate definitions. Without knowing the precise meaning of complex exponents, how can a student resolve Clausen's "infuriating puzzle," which purports to show that $\exp\left(-4\pi^2\right) = 1$? Yet that is just what Nahin challenges the student to do. He warns that this riddle "should keep you sleepless for a few nights," but withholds its solution. (The nineteenth century mathematician Eugène Catalan stripped the algebraic misdirection from Clausen's puzzle to better unmask the nub. In Catalan's condensation, the equation $e^{-\pi} = e^{\pi}$ is obtained by raising each side of the genuine equality $e^{2\pi i} = e^{-2\pi i}$ to the power $i/2$.)

This is tricky material. My reservations concerning the use of powers, however, arise with the most basic material at the very beginning of the book. Although the student is informed that any cubic equation has three roots (and is advised to read an appendix containing a statement of the Fundamental Theorem of Algebra), the student is not explicitly told that a complex number has three cube roots. Calculating that the student has not made this deduction, Nahin *deliberately* conceals it. Let us reconsider equation (2). Cardan's Formula (3) in this case reduces to 4, $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$ *when the three cube roots are used*. Nahin pretends to be unaware of this when, in order to show that Cardan's Formula yields the obvious root 4, he writes "it is sufficient to see that $\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1}$ and $\sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1}$." There is no mention that the left sides of these supposed equalities are not uniquely defined. Shortly thereafter, when, speaking of a class of cubics of which equation (2) is an example, Nahin remarks that there is "just one positive root; that is, *the* root given by the Cardan formula" (reviewer's emphasis). Continuing the pretense, Nahin counsels the student to complete the solution of the cubic by dividing by the linear factor provided by Cardan's Formula and then applying the Quadratic Formula. Two chapters later he includes one sentence to redress these deceptions. By that time the derivation of Cardan's Formula has long been forgotten. Any reasonably sharp student may well wonder why formula (3) does not provide *nine* solutions to equation (2).

Heretofore I have cited aspects of Nahin's presentation that might confuse the student. Of equal importance are the numerous instances in which the student might be deceived by a vacuous explanation. For example, the infinite series $\sum (-1)^{n+1}/n^z$ is said to converge for $\mathrm{Re}(z) > 0$ "precisely because of the alternating signs," as if the terms were real-valued. A few pages later Nahin begins to prove the reflection formula for the gamma function. Requiring the evaluation of a certain integral, the proof is suspended until the discussion of complex integration in the final chapter. Unfortunately, when the time comes to complete the deferred evaluation, Nahin does not choose the appropriate contour. Only a trivial case of the reflection formula is actually proved.

Because *An Imaginary Tale* is largely told as history, I cannot let pass several historical inaccuracies. Nahin ascribes to Cardano the method by which the quadratic term is eliminated from a cubic equation: "This was a major achievement in itself, and it is all Cardan's." As mentioned above, the necessary transformation was discovered at least one and a half centuries before Cardano's rediscovery.

After discussing the Fibonacci numbers, Nahin comments "Such recurrences often occur…but in Leonardo's time they were brand new. Indeed, Leonardo's recurrence was the first time such a thing had been encountered." Not so! Consider the following scheme that the ancient Greeks devised for approximating $\sqrt{2}$: let $s_1 = d_1 = 1$ and, for $n > 1$, let $s_n = s_{n-1} + d_{n-1}, d_n = 2s_{n-1} + d_{n-1}$. The approximation to $\sqrt{2}$ is d_n/s_n, but that is beside the point here. After separation we find that both sequences satisfy the Fibonacci-like recurrence relation $g_n = 2g_{n-1} + g_{n-2}$ for $n > 2$.

Referring to the product formula for the zeta function, Nahin proclaims "…it gave Euler an entirely new proof of the infinity of the primes, the first since Euclid's from two thousand years before." In fact, Goldbach anticipated Euler by several years. In a letter that he wrote to Euler in 1730, Goldbach established the infinitude of primes by observing that the Fermat numbers are pairwise relatively prime.

In telling the story of the zeta function, Nahin jumps from Euler to Riemann, neglecting Riemann's teacher, Dirichlet, who played a crucial intermediate role. Nahin also neglects to inform the reader that Euler discovered (but did not prove) an equivalent form of the functional equation of the zeta function [7]. By stating that Riemann *attempted* to find a formula for the prime counting function, $\pi(x)$, and leaving it at that, Nahin may give the erroneous impression that Riemann was not successful in his search. (What Riemann was not successful at was using his formula for $\pi(x)$ to prove the Prime Number Theorem.)

Finally, we have long since passed the point when the author of a historically informed work should mention the Leibniz-Gregory series without acknowledging that the name reflects a Western view of history. When the author writes that the Maclaurin series of $\sin(x)$ and $\cos(x)$ were known "at least since Newton's time," he may not be wrong, but his lower bound is several hundred years too great. The Gregory series for $\arctan(x)$, the Leibniz series for $\pi/4$, and the two Maclaurin series just cited were all known to and recorded by the Indian mathematician Mādhavan (ca. 1340–1425).

Permit me a final criticism that concerns both pedagogy and history. Nahin takes complex numbers as a given. Many mathematicians consider it absolutely fundamental that the complex numbers *must* be constructed. To understand the necessity, consider the charming words (unfortunately not found in Nahin's book) of Descartes, in coining the term *imaginary*: "For any equation one can imagine as many roots [as its degree would suggest], but in many cases no quantity exists which corresponds to what one imagines." As late as 1770 Euler stated that imaginary numbers are *impossible*. If these are the opinions of an ordinary genius and a magician, then how can we deny the necessity of constructing such quantities? Even Cauchy, a hero of Nahin's book, felt obliged to construct $\mathbb{C}$ and did so in 1847 as $\mathbb{R}[x]/\left(x^2+1\right)$. Lest there be any doubt as to his purpose, Cauchy remarked, "We completely repudiate the symbol $\sqrt{-1}$, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it." Of this *An Imaginary Tale* makes no mention. But Nahin *does* condescendingly dismiss William Rowan Hamilton's prior algebraic construction of $\mathbb{C}$. In fact, the disdainful tone with which the author lessens Hamilton's entire scientific career is irksome. Perhaps it is relevant to note that Nahin has already written a biography of Oliver Heaviside, electrical engineer and scourge of quaternionists. Nahin is not the first to disparage Hamilton, but one is astonished that such attitudes persist.

Because a book published by Princeton University Press tends to bring with it a presumption of excellence, I have found it appropriate to discuss the shortcomings of *An Imaginary Tale* in detail. That does not mean that the book lacks merit. Nahin set out with a worthy idea, brought tremendous energy to its development, and presented the results with evangelical zeal. He took great care in the execution and recording of his calculations. He has conveniently collected much interesting material, some of which will be new to many readers—I am thinking in particular of two enlightening sections on electronic circuits. Unfortunately, in the transition from promising manuscript to published book advance praise was secured while more important tasks were left undone. *An Imaginary Tale* is not a bad book, but it might have been so much better. Those who teach complex variables may wish to dip into it for ideas to enrich their lectures, but because of its frequent lack of clarity and correctness they may hesitate to recommend it to their students. Fortunately there are several good alternatives: [3], [5], and [10] for history; [4], [8], and [11] for applications of complex variables; [5] and [6] for the work of Euler; and [1] for a serious treatment of the gamma and zeta functions that is within the reach of well-prepared undergraduates. For the history of complex analysis, specialists can turn to [2] and [9] as well as the primary sources.

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