The Nonabelian Reciprocity Law for Local Fields

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This note reports on the Local Langlands Correspondence for GL_n over a p-adic field, which was proved by Michael Harris and Richard Taylor in 1998. A second proof was given by Guy Henniart shortly thereafter. The Local Langlands Correspondence is a nonabelian generalization of the reciprocity law of local class field theory. It gives a remarkable relationship between nonabelian Galois groups and infinite-dimensional representation theory. Posed as an open problem thirty years ago in [L1], its proof in full generality represents a milestone in algebraic number theory. The goal of this note, aimed at the nonspecialist, is to state the main result with some motivations and necessary definitions.

Reciprocity Laws

For any positive integer d, the law of quadratic reciprocity describes the primes p for which the congruence $x^2 \equiv d \bmod p$ has a solution. It implies, quite counterintuitively, that the existence of a solution to this congruence modulo p depends only on the residue class of p modulo $4d$. Despite the large number of different proofs, it remains one of the deepest and most mysterious results of elementary number theory.

The search for generalizations of quadratic reciprocity, beginning with work of Gauss and Eisenstein on cubic and higher reciprocity laws, motivated a great deal of number-theoretic research in the nineteenth century. However, it was not possible to formulate a unified and general reciprocity law until the notion of a reciprocity law itself was reformulated within the context of class field theory. This theory was initiated by Kronecker in the case of quadratic imaginary fields. It was developed into a general framework by Weber and Hilbert in the 1890s and was proven by Furtwangler, Takagi, and Artin in the first quarter of the twentieth century.

In simple terms, class field theory seeks to describe all of the finite abelian extensions of a number field F, that is, the finite Galois extensions K/F such that $\mathrm{Gal}(K/F)$ abelian.[1] The theory accomplishes its goal by establishing a deep relation between generalized ideal class groups attached to F and decomposition laws for prime ideals in abelian extensions K/F. This theory is elementary in the case $F = \mathbb{Q}$. According to the Kronecker-Weber theorem, every abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension $\mathbb{Q}(e^{2\pi i/N})$ for some integer N. On the other hand, the generalized ideal class groups of $\mathbb{Q}$ are just the multiplicative groups $(\mathbb{Z}/N\mathbb{Z})^*$ and their quotients. Class field theory yields the isomorphism between $(\mathbb{Z}/N\mathbb{Z})^*$ and the Galois group of $K = \mathbb{Q}(e^{2\pi i/N})$ in which a residue class $m \in (\mathbb{Z}/N\mathbb{Z})^*$ maps to the Galois automorphism sending $e^{2\pi i/N}$ to $e^{2\pi i m/N}$. Implicit in this isomorphism is the decomposition law: if p is a prime not dividing N and f is the order of p in $(\mathbb{Z}/N\mathbb{Z})^*$, then (p) factors in K as a product of

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[1] *Class field theory treats all* global *fields, i.e., finite extensions of* $\mathbb{Q}$ *or of the field* $\mathbb{F}_q(X)$ *of rational functions over a field of* q *elements. In this exposition, we will discuss only the number field case.*

$\varphi(N)/f$ distinct prime ideals in the ring of integers in K. Concretely, this means that $x^N - 1$ factors modulo p as a product of $\varphi(N)/f$ distinct irreducible polynomials and, in particular, the number of factors in the factorization modulo p depends only on p modulo N. The law of quadratic reciprocity follows from this fact in a natural way.

For an arbitrary number field F, the corresponding assertion is the Artin reciprocity law. It is most cleanly stated using "ideles" in place of ideals and a maximal abelian extension F^{ab} of F in place of finite abelian extensions. Here we work inside a fixed algebraic closure $\bar{F}$ of F and take F^{ab} to be the union of all finite abelian extensions K of F in $\bar{F}$. The "idele group" of F is defined in terms of the inequivalent completions of F, each of which is defined by a "place". Consider first the case $F = \mathbb{Q}$. The places v are the primes $p = 2, 3, 5, \ldots$ and one infinite place $v = \infty$. The completions relative to these places are the fields $\mathbb{Q}_p$ of p-adic numbers for v finite and $\mathbb{Q}_\infty = \mathbb{R}$ for $v = \infty$. For p prime let $\mathbb{Z}_p$ be the ring of p-adic integers within $\mathbb{Q}_p$. The *adele ring* $A_\mathbb{Q}$ is the subring of tuples $(x_v) = (x_\infty, x_2, x_3, x_5, \ldots)$ in the direct product $\prod_v \mathbb{Q}_v$ (product over all v) such that x_v is in $\mathbb{Z}_v$ for all but at most finitely many finite places v. The *idele group* is the group $\mathbb{A}_\mathbb{Q}^*$ of invertible elements in $\mathbb{A}_\mathbb{Q}$. Locally compact topologies [CF] may be defined for $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}^*$ so that $\mathbb{A}_\mathbb{Q}$ is a topological ring and $\mathbb{A}_\mathbb{Q}^*$ is a topological group. The definitions are such that the diagonal embedding $x \mapsto (x, x, x, x, \ldots)$ identifies $\mathbb{Q}$ and $\mathbb{Q}^*$ with discrete subgroups of $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}^*$, respectively.

For an arbitrary number field, there are finitely many infinite places v, for which F_v is $\mathbb{R}$ or $\mathbb{C}$, and infinitely many finite places, for which F_v is a finite extension of $\mathbb{Q}_p$ for some prime p. The adele ring A_F and idele group A_F^* are defined in terms of the localizations F_v as in the case $F = \mathbb{Q}$. Again, the respective diagonal embeddings of F and F^* have discrete images.

The Artin reciprocity law is a certain continuous, surjective homomorphism

$$r : \mathbb{A}_F^*/F^* \to \mathrm{Gal}(F^{ab}/F)$$

defined explicitly in terms of "Frobenius automorphisms". It is from this explicit definition of r that it is possible (with some work) to deduce fundamental number-theoretic information about F. In the case $F = \mathbb{Q}$, $\mathbb{A}_\mathbb{Q}^*/\mathbb{Q}^*$ can be identified with $\mathbb{R}_+^* \times \prod_p \mathbb{Z}_p^*$, and r amounts to the identification of $\prod_p \mathbb{Z}_p^*$ with $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ coming from the construction of $\mathbb{Q}^{ab}$ by adjoining all roots of unity. This formulation puts together all of the class field theory isomorphisms for the extensions $\mathbb{Q}(e^{2\pi i/N})$ in one package.

The theory just described is called *global* class field theory because it deals with a number field. *Local* class field theory is concerned with abelian extensions of a *local field* F, i.e., a locally compact nondiscrete field. In characteristic zero, the only such fields are $\mathbb{R}$, $\mathbb{C}$, and finite extensions of the field $\mathbb{Q}_p$. The latter examples are called *p-adic fields.* For a local field F, the reciprocity law takes a similar form, with the multiplicative group F^* taking the place of the idele group. More precisely, the local reciprocity law is a continuous homomorphism

$$r : F^* \to \mathrm{Gal}(F^{ab}/F).$$

As in the global number-field case, the map r provides tangible nontrivial information about the arithmetic object $\mathrm{Gal}(F^{ab}/F)$ in terms of the elementary object F^*. If F is p-adic, r is injective and its image is dense, but it is no longer surjective. In particular, r extends by continuity to an identification of $\mathrm{Gal}(F^{ab}/F)$ with the profinite completion of F^*, and thus we obtain a simple description of $\mathrm{Gal}(F^{ab}/F)$. For example, if $F = \mathbb{Q}_p$, where p is an odd prime, then $\mathbb{Q}_p^*$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/p)^* \times \mathbb{Z}_p$ and $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ is isomorphic to $\hat{\mathbb{Z}} \times (\mathbb{Z}/p\mathbb{Z})^* \times \mathbb{Z}_p$, where $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. Of course, local class field theory consists of much more than an abstract group isomorphism. See [S] for a list of the properties of r relating the structure of F^* to the arithmetic of F^{ab}.

Nonabelian Reciprocity

In the second of his *Two Lectures on Number Theory, Past and Present* ([W], vol. III, p. 301), André Weil summed up as follows the state of affairs in the 1920s and 1930s after the main results of class field theory had been established:

> Artin's reciprocity law, which in a sense contains all previously known laws of reciprocity as special cases, deals with a strictly commutative problem. It establishes a relation between the most general extension of a number-field with a commutative Galois group on the one hand, and on the other hand the multiplicative group over that field. Where do we go from there? Well—of course we take up the noncommutative case.

The group $\mathrm{Gal}(F^{ab}/F)$ is equal to $\mathrm{Gal}(\bar{F}/F)^{ab}$, i.e., the quotient of $\mathrm{Gal}(\bar{F}/F)$ by the closure of its commutator subgroup. A noncommutative generalization of Artin's law would, at the least, provide a description of the full Galois group $\mathrm{Gal}(\bar{F}/F)$ in some way that expresses the decomposition law for primes in finite extensions of F. It was a major obstacle, however, to find the right language for fomulating such a law. One gets a sense of the difficulty of the problem from a remark of Weil[2] to

[2]*Note [1971a] in [W], vol. III, p. 457.*

the effect that E. Noether, E. Artin, and H. Hasse had hoped in vain that their theory of simple algebras would lead to a nonabelian theory but that by 1947 Artin confided that he was no longer sure that such a theory existed.[3] The breakthrough came out of Langlands's discovery in the 1960s of the conjectural "Principle of Functoriality", which included a formulation of a nonabelian reciprocity law (both global and local) as a special case.

It turns out that in the nonabelian case, a key shift of viewpoint is necessary: one describes the representations of $\mathrm{Gal}(\bar{F}/F)$ rather than $\mathrm{Gal}(\bar{F}/F)$ itself. Thus, the global reciprocity law is formulated as a general conjectural correspondence between Galois representations and automorphic forms. In fact, in the 1960s Serre and Shimura derived nonabelian reciprocity laws of this type using the cohomology of modular curves. However, the Langlands reciprocity law, like Artin's abelian law, posits a correspondence independent of any particular geometric construction.

The Principle of Functoriality has led more generally to a web of interrelated results and conjectures which together make up the *Langlands Program*. This program has both a global and local aspect. See [A], [G], [L2], [K], [R] and the references they contain for a general overview of this program and its consequences.[4] The article [D] mentions the relation between the Langlands Program and the modularity of elliptic curves.

The Weil Group

To describe the local nonabelian reciprocity law, we introduce the *Weil group* $W(\bar{F}/F)$.[5] Assume that F is a p-adic local field, i.e., a finite extension of $\mathbb{Q}_p$ for some prime p. Let the ring of integers be $\mathcal{O}_F$. This ring has a unique maximal ideal, necessarily principal, and we let ϖ be any generator, a so-called *prime element*. Denote by q the cardinality of the *residue field* $k_F = \mathcal{O}_F/(\varpi)$. One knows that a Galois automorphism $\tau \in \mathrm{Gal}(\bar{F}/F)$ induces an automorphism $\bar{\tau}$ of the residue field $\overline{k_F}$ of $\bar{F}$, and the map $\tau \to \overline{\tau}$ from $\mathrm{Gal}(\bar{F}/F)$ to $\mathrm{Gal}(\overline{k_F}/k_F)$ is surjective. The Galois group $\mathrm{Gal}(\overline{k_F}/k_F)$ is isomorphic to $\hat{\mathbf{Z}}$ and the *Weil group* $W(\bar{F}/F)$ is defined as the dense subgroup of Galois automorphisms τ such that $\bar{\tau}$ is of the form $x \to x^{q^m}$ for some integer m.

[3]*Coincidentally, 1947 was the year that Harish-Chandra received his Ph.D. in physics under P. Dirac from Cambridge University. See [A] elsewhere in this issue of the* Notices *for a discussion of Harish-Chandra's work and its relation with the Langlands Program.*

[4]*For historical documents and commentary on functoriality as well as Langlands's papers in downloadable form, visit the Web site* `http://sunsite.ubc.ca/DigitalMathArchive/Langlands/`.

[5]*See Tate's article in [CO], Part II, for an excellent discussion of Weil groups in the local and global cases.*

The kernel of the map $\tau \to m$ is a closed subgroup of $\mathrm{Gal}(\bar{F}/F)$ called the *inertia subgroup* I_F of F. An element τ mapping to $m = 1$ is called a *Frobenius automorphism*. In any case, we have a (noncanonical) isomorphism $W(\bar{F}/F) \simeq \mathbb{Z} \times I_F$, and we use this to make $W(\bar{F}/F)$ into a topological group, taking the product of the discrete topology on $\mathbb{Z}$ and relative topology from $\mathrm{Gal}(\bar{F}/F)$ on I_F. With this definition, the restriction r' of the reciprocity map induces an isomorphism of topological groups

$$r' : F^* \to W(\bar{F}/F)^{ab}. \tag{1}$$

The Dual Perspective

As mentioned above, the key to the nonabelian reciprocity law is to dualize the reciprocity isomorphism (1). Thus, for F a p-adic field, we consider the isomorphism of character groups[6]

$$\sigma_1 : \mathrm{Hom}(F^*, \mathbb{C}^*) \to \mathrm{Hom}(W(\bar{F}/F), \mathbb{C}^*) \tag{2}$$

induced by r'.

Since the nonabelian generalization of a character is an irreducible representation, it is reasonable to consider the set $\mathcal{G}_n$ of equivalence classes of irreducible n-dimensional representations of $W(\bar{F}/F)$ for all $n \geq 1$. Then

$$\mathcal{G}_1 = \mathrm{Hom}(W(\bar{F}/F), \mathbb{C}^*),$$

and so from this point of view, the generalization of (2) would be some map of the form

$$\sigma_n : \boxed{???} \to \mathcal{G}_n$$

with target $\mathcal{G}_n$. What is much less obvious is the correct source for σ_n.

It turns out that the source of σ_n is the set of equivalence classes of *supercuspidal representations* of $GL_n(F)$, which are a type of infinite-dimensional representation that we define below. The global version relates irreducible n-dimensional representations of the Galois group and cuspidal representations of the group $GL_n(\mathbb{A}_F)$. It is more difficult to state precisely [CL], [L1], [R].

In hindsight it is not surprising that the nonabelian theory was not developed until the 1960s. The theory of infinite-dimensional representations of reductive groups was first developed actively in the 1950s, and the p-adic case was not studied intensively until the 1960s. The existence of supercuspidal representations for GL_2 over a p-adic field was apparently first observed by F. Mautner.[7] They were first studied in greater generality in the work of Jacquet and Harish-Chandra.

[6]*Here and below we consider only continuous characters, but omit the continuity in the notation.*

[7]*According to an oral communication from J. Shalika.*

Admissible Representations

Let $G = GL_n(F)$, and let (π, V) be a representation of G on a complex vector space V. Since F is a p-adic field, G has a family of open compact subgroups $\{K_m\}_{m\geq 1}$, where K_m consists of the integral matrices $g \in G$ such that $g \equiv 1 \bmod (\varpi^m)$. We say that (π, V) is *admissible* if

- every vector $v \in V$ is fixed by K_m for some m, and
- the subspace of vectors fixed by each K_m is finite-dimensional.

From now on (π, V) will denote an *irreducible* admissible representation. We note that to each irreducible unitary representation (π', V') there is attached an admissible representation (π, V) in a natural way: V is the subspace of vectors $v \in V'$ fixed by K_m for some m, and π is the restriction of π' to V. Furthermore, the space V of an irreducible admissible representation is either one-dimensional or infinite-dimensional. Indeed, if $\dim V < \infty$, then K_m is contained in $\ker \pi$ for some m, and since $\ker \pi$ is normal, it must contain $SL_n(F)$. The *contragredient* of (π, V) is the representation π^* of G on the admissible dual space V^*, that is, the space of linear functionals on V that are fixed by K_m for some m. A *matrix coefficient* of (π, V) is a function on G of the form $f(g) = \langle \pi(g)v, w\rangle$ with $v \in V$ and $w \in V^*$; here $\langle v, w\rangle$ is the bilinear pairing on $V \times V^*$. The representation π is called *supercuspidal* if the support of every matrix coefficient is compact modulo the center Z of G.

The *Local Langlands Conjecture* asserts that for all n, there exists a natural bijection

$$\sigma_n : C_n \to \mathcal{G}_n,$$

where C_n is the set of equivalence classes of supercuspidal representations of $G = GL_n(F)$. In the special case $n = 1$, we have $C_1 = \mathrm{Hom}(F^*, \mathbb{C}^*)$. Of course, to give this conjecture some content, we must specify what is meant by "natural". To do so, we exploit the fact it is possible to attach so-called L and ϵ factors to objects in either C_n or $\mathcal{G}_n$, although the procedure is quite different in the two cases.

L and ϵ Factors

L factors generalize the individual factors of the Euler product representation of the Riemann zeta function $\zeta(s)$, which is represented for $\mathrm{Re}(s) > 1$ by the Euler product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

There are two fundamentally different ways to make the generalization, both of which play a key role in our story. Our goal is to define more general L factors and the closely related ϵ factors in the local case, but our description will be clearer if we begin with the global case. The global factors arose first historically, and it was an important theoretical step to define their local counterparts. For the moment then F will denote a number field.

First of all, we can follow Hecke and attach an L function $L(s, \chi)$ to each continuous homomorphism $\chi : \mathbb{A}_F^*/F^* \to \mathbb{C}^*$. Such a homomorphism is called a *Hecke character*. A Hecke character χ gives rise, by restriction, to a homomorphism $\chi_v : F_v^* \to \mathbb{C}^*$ for all places v of F, and the *Hecke L function* of χ is defined as an infinite product

$$L(s, \chi) = \prod_v L(s, \chi_v)$$

of *local factors* $L(s, \chi_v)$.

If v is finite, we say that χ_v is *unramified* if the restriction of χ_v to $\mathcal{O}_v^*$ is trivial, where $\mathcal{O}_v^*$ is the group of units in the ring $\mathcal{O}_v$ of integers in F_v. In this case, we set $L(s, \chi_v) = (1 - \chi(\varpi_v)q_v^{-s})^{-1}$, where ϖ_v is any prime element in $\mathcal{O}_v$ and $q_v = |\mathcal{O}_v/(\varpi_v)|$. If v is finite but χ_v is ramified, then $L(s, \chi_v) = 1$. If v is infinite, $L(s, \chi_v)$ is defined in terms of a certain finite product of gamma functions. The continuity of χ implies that χ_v is unramified for all but finitely many v, and thus $L(s, \chi)$ is an Euler product. The product converges absolutely and uniformly for $\mathrm{Re}(s)$ sufficiently large, and it thus defines an analytic function in a suitable right half plane. The Riemann zeta function is the Hecke L function corresponding to the trivial Hecke character of $\mathbb{A}_\mathbb{Q}^*/\mathbb{Q}^*$.[8] The Dirichlet L functions that arise in the proof that there are infinitely many primes in arithmetic progressions are also Hecke L functions, apart from a contribution from a finite product arising from the infinite places.

Hecke proved that his L functions $L(s, \chi)$ can be meromorphically continued to the entire complex plane and that they satisfy a *global* functional equation of the form

$$L(s, \chi) = \epsilon(s, \chi)L(1 - s, \chi^{-1}),$$

where $\epsilon(s, \chi)$ is a factor of the form $A \cdot B^s$ for some constants A and B. This generalizes the functional equation of the Riemann zeta function. An important method for proving Hecke's results using Fourier analysis on local fields and adele rings was developed in 1950 by J. Tate in his well-known Ph.D. thesis and also by K. Iwasawa. The great virtue of this method is that it provides a local interpretation of the local factors and, equally important, a factorization of $\epsilon(s, \chi)$ into local ϵ factors as described below.

What emerges clearly in the work of Langlands and Jacquet-Langlands is that the nonabelian

[8] *More precisely, it is the product of the Riemann zeta function and a factor coming from $v = \infty$. This is the gamma factor that enters the usual functional equation for the zeta function.*

generalization of a Hecke character is a *cuspidal* representation [BK], [CO], [G], [L1] of the adelic group $GL_n(\mathbb{A}_F)$ or, more generally, of the adelic points of any reductive group over F. Just as a Hecke character χ has local components χ_v, a cuspidal representation π has local components π_v that are irreducible admissible representations of $GL_n(F_v)$. In this light, Tate's thesis concerns the case $n = 1$, and one can then ask if there exists a generalization of it to $n > 1$. In the late 1950s and early 1960s the analogous problem for the multiplicative group of a division algebra in place of GL_n was considered by Godement and Tamagawa. Shimura also considered the quaternion algebra case in connection with the analytic continuation of the zeta-functions attached to arithmetic quotients of the upper half-plane (*Shimura curves*). A full representation-theoretic treatment of the problem for GL_n containing, in particular, a detailed local theory was given by Godement and Jacquet.

We shall focus only on the local part of the theory that is relevant to the statement of the Local Langlands Correspondence. Let F be a p-adic field. The Godement-Jacquet theory attaches local L and ϵ factors to any irreducible admissible representation (π, V) of $GL_n(F)$ in the following way. One introduces the so-called *zeta integrals*:

$$Z(s, f, \Phi) = \int_{GL_n(F)} f(g)\Phi(g)|\det(g)|^s \, dg$$

where dg is a Haar measure on $GL_n(F)$, f is a matrix coefficient of π, and Φ is a locally constant function of compact support on the vector space $M_n(F)$ of n-by-n matrices over F. Although the integral converges for $\mathrm{Re}(s)$ sufficiently large, it can be shown that $Z(s, f, \Phi)$ extends to a rational function in q^{-s}. Furthermore, there is a unique zeta integral of the form $L(s, \pi) = P(q^{-s})^{-1}$, where P is a polynomial of degree at most n with $P(0) = 1$, such that $Z(s, f, \Phi)/L(s, \pi)$ is entire for all Φ and f. Finally, the choice of a nontrivial additive character $\psi : F \to \mathbb{C}^*$ plays a role: one proves that for each choice of ψ there is a unique factor $\epsilon(s, \pi, \psi)$ such that the following *local functional equation* holds for all f and Φ:

$$\frac{Z(s, f, \Phi)}{L(s, \pi)} = \epsilon(s, \pi, \psi) \frac{Z(1 - s, f^*, \hat{\Phi})}{L(1 - s, \pi^*)}.$$

Here $f^*(g) = f(g^{-1})$, $\hat{\Phi}$ is the Fourier transform of Φ relative to the character ψ of F, and π^* is the contragredient representation of π. Furthermore, $\epsilon(s, \pi, \psi)$ is of the form $\epsilon q^{\alpha s}$, where ϵ and α are constants.

The second generalization of the Riemann zeta function is due to E. Artin, who defined an L function $L(s, \rho)$ for any (continuous) finite-dimensional representation of the Galois group

$$\rho : \mathrm{Gal}(\bar{F}/F) \to GL_n(\mathbb{C})$$

of a number field F. Every such ρ factors through $\mathrm{Gal}(K/F)$ for some finite Galois extension K/F. For each finite place v, the choice of an embedding $\bar{F} \to \bar{F}_v$ induces an embedding $\mathrm{Gal}(\bar{F}_v/F_v) \to \mathrm{Gal}(\bar{F}/F)$, uniquely defined up to conjugation. The pullback ρ_v of ρ to $\mathrm{Gal}(\bar{F}_v/F_v)$ is thus well defined up to equivalence, and we may define a local factor

$$L(s, \rho_v) = \det(I - q_v^{-s}\rho_v(\Phi_v)|V^{I_v})^{-1}.$$

Here V^{I_v} denotes the subspace of invariants under the inertia subgroup I_v of F_v, and Φ_v is a Frobenius element. For infinite v the local factor is a finite product of gamma functions, as in the case of Hecke L functions. The *Artin L function* is defined as an infinite product $L(s, \rho) = \prod_v L(s, \rho_v)$.

The infinite product defines $L(s, \rho)$ as a meromorphic function for $\mathrm{Re}(s) > 1$. R. Brauer proved that $L(s, \rho)$ extends meromorphically to the complex plane and satisfies a functional equation of the form $L(s, \rho) = \epsilon(s, \rho)L(1 - s, \hat{\rho})$, where $\hat{\rho}$ is the contragredient of ρ. The existence of a factorization $\epsilon(s, \rho)$ as a product of local factors was established by Langlands using local methods (unpublished, but now available on the Web site in footnote 4). Deligne published a shorter proof using global methods. The factorization, which depends on the choice of a nontrivial additive character $\psi : \mathbb{A}_F/F \to \mathbb{C}^*$, has the form $\epsilon(s, \rho) = \prod_v \epsilon(s, \psi_v, \rho_v)$ in analogy with the factorization provided by the theory of Tate and Godement-Jacquet. The local factor $\epsilon(s, \psi_v, \rho_v)$ depends only on the restriction ψ_v of ψ to F_v and is characterized by suitable compatibility requirements.

The Local Langlands Correspondence

Once the local factors are known to exist, it is tempting to define the map σ_n by requiring that $L(s, \pi) = L(s, \sigma_n(\pi))$ and $\epsilon(s, \pi) = \epsilon(s, \sigma_n(\pi))$ for all supercuspidal π. Unfortunately, this does not determine σ_n uniquely. Henniart proved, however, that σ_n can be characterized by matching a more general type of local factors. Thus we introduce the final ingredient needed in order to state the correspondence precisely: the local factors attached to *pairs*. Again let F be a p-adic field.

If ρ and ρ' are representations of $\mathrm{Gal}(\bar{F}/F)$ of dimensions n and n' respectively, the results quoted in the previous paragraph allow us to attach local factors $L(s, \rho \otimes \rho')$ and $\epsilon(s, \psi, \rho \otimes \rho')$ to the tensor product $\rho \otimes \rho'$, which is a representation of dimension nn'. The analogue of this construction for admissible representations was carried out by Jacquet, Piatetski-Shapiro, and Shalika. They defined local factors $L(s, \pi \times \pi')$ and $\epsilon(s, \psi, \pi \times \pi')$ for any pair π and π' of irreducible admissible representations of $GL_n(F)$ and $GL_{n'}(F)$ respectively. These reduce to the Godement-Jacquet factors when π' is the trivial representation of $GL_1(F)$.

The center Z of $GL_n(F)$ consists of scalar matrices and thus is isomorphic to F^*. If π is an irreducible representation of $GL_n(F)$, then there is a character ω_π of Z, called the *central character* of π, such that $\pi(z) = \omega_\pi(z)I$ for all $z \in Z$. We regard ω_π as a character of F^*.

Main Theorem. There exists a unique family of bijections

$$\sigma_n : \mathcal{C}_n \to \mathcal{G}_n$$

for $n \geq 1$ satisfying the following conditions:

(a) σ_1 is the correspondence (2) of abelian local class field theory.

(b) For all pairs $\pi \in \mathcal{C}_n$ and $\pi' \in \mathcal{C}_{n'}$, we have

$$L(s, \pi \times \pi') = L(s, \sigma_n(\pi) \otimes \sigma_{n'}(\pi'))$$

and

$$\epsilon(s, \pi \times \pi') = \epsilon(s, \sigma_n(\pi) \otimes \sigma_{n'}(\pi')).$$

(c) For all π, $\sigma_1(\omega_\pi) = \det(\sigma_n(\pi))$.

As mentioned above, the uniqueness assertion had been established earlier by Henniart (1993). The articles [HT] and [H] establish existence in full generality, relying heavily on global results and methods coming from the theory of Shimura varieties and automorphic forms. Prior to that, Harris had proved the local correspondence for $n < p$ and introduced a key technique of "non-Galois automorphic induction" used in both [HT] and [H]. The proof by Henniart is the shorter of the two. However, Harris and Taylor carry out a deep study of certain Shimura varieties at primes of bad reduction, thus generalizing an approach invented by Deligne for the case of modular curves. This represents important progress independent of its application to the local correspondence. Work of Arthur, Berkovitch, Carayol, Clozel, Deligne, Drinfeld, Kottwitz, Langlands, and others also plays a fundamental role.

The problem is to define a representation $\sigma_n(\pi)$ of the Weil group given a supercuspidal representation π. The strategy is to embed the local problem in a global one. Roughly speaking, one chooses a suitable number field F' having F as one of its localizations and attempts to construct $\sigma_n(\pi)$ by restricting certain representations of the global Galois group of F' to the local Weil group $W(\bar{F}/F)$. The global representations are constructed geometrically. In the case $n = 2$ and $F' = \mathbb{Q}$, they arise from the theory of modular curves, which yields a correspondence between cuspidal representations of $GL_2(\mathbb{A}_\mathbb{Q})$ of a certain type and irreducible representations of the Galois group of $\mathbb{Q}$ occurring in the ℓ-adic cohomology spaces of the modular curves. Since, as is easily shown, every supercuspidal representation π occurs as the local component of (infinitely many) cuspidal representations of the appropriate type, the modular curves provide natural candidates for $\sigma_2(\pi)$. The difficulty lies in showing that all of these candidates coincide and satisfy the conditions of the Main Theorem.

One would like to employ the same strategy in the case $n > 2$. However, there are no analogues of the modular curves for GL_n in this case. The way around this, according to a suggestion first made by H. Carayol, is to exploit the fact that the group GL_n over a p-adic field can be obtained as the localization at a p-adic place of certain "twisted" unitary groups over a suitable totally real number field F'. The unitary groups do give rise to a family of "Shimura varieties" that are analogous to the family of modular curves, and one obtains candidates for the representations $\sigma_n(\pi)$ as in the case $n = 2$.

Carrying out this strategy is a difficult task. We refer to [C] for an excellent technical overview of the strategy of the proofs. The articles [C] and [Ku] describe also how the Main Theorem can be extended to a correspondence between the set of equivalence classes of all irreducible admissible representations and the set of finite-dimensional representations of the so-called Weil-Deligne group.

Many cases of the Local Langlands Correspondence had been established previously. We mention only the case $n = 2$ which, of course, was handled first. Jacquet and Langlands verified the local correspondence for $GL_2(F)$ for local fields of residual characteristic different from 2. The case $n = 2$ and residual characteristic 2 was settled using local methods in full generality by P. Kutzko in 1980, following work of J. Tunnell, who used global results to treat $\mathbb{Q}_2$ itself and fields of residual characteristic 2 containing a cube root of unity. We also mention that the full Local Langlands Correspondence for GL_n over local fields of characteristic p (fields of Laurent series in one variable over a finite field) was proved by Laumon, Rapoport, and Stuhler in 1991.

Although the Main Theorem establishes an intimate relation between two dissimilar kinds of objects, neither the statement nor its proof provides a direct method for construction of either type of object explicitly. A local approach to the theory of admissible representations of $GL_n(F)$ has been developed by C. Bushnell and Kutzko. Although this approach does not yet provide a proof of the local correspondence, it does provide important insight into the internal structure of the representations themselves.

As mentioned above, the global reciprocity law is a statement relating Galois representations and cuspidal representations [CL]. It seems out of reach at present in the number-field case. However, for global fields of positive characteristic, Drinfeld established the global correspondence for GL_2 using his theory of shtukas. In an exciting recent breakthrough, Laurent Lafforgue succeeded in generalizing Drinfeld's approach and established the "global Langlands correspondence" in full generality for GL_n over a global field of positive characteristic.

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