Encounter with a Geometer, Part I

Marcel Berger

Editor's Note. *Mikhael Gromov is one of the leading mathematicians of our time. He is a professor of mathematics at the Institut des Hautes Études Scientifiques (IHÉS), and, as an article elsewhere in this issue of the* Notices *reports, he recently won the prestigious Balzan Prize.*

The present article, with Part I in this issue and Part II in the next, discusses Gromov's mathematics and its impact from the point of view of the author, Marcel Berger. It is partially based on three interviews with Gromov by Berger, and it first appeared in French in the Gazette des Mathématiciens *in 1998, issues 76 and 77. It was translated into English by Ilan Vardi and adapted by the author. The resulting article is reproduced here with the permission of the* Gazette *and the author.*

The aim of this article is to communicate the work of Mikhael Gromov (MG) and its influence in almost all branches of contemporary mathematics and, with a leap of faith, of future mathematics. Because of its length, the article will appear in two parts. It is not meant to be a technical report, and, in order to make it accessible to a wide audience, I have made some difficult choices by highlighting only a few of the many subjects studied by MG. In this way I can be more leisurely in my exposition and give full definitions, results, and even occasional hints of proofs.

I wrote this article because I believe that MG's work is greatly underrated. I will not analyze the reasons for this phenomenon, although a general idea of why this is so will become clear from a reading of the text. Apart from Riemannian geometry, notable exceptions to this phenomenon occur in two fields: hyperbolic groups and symplectic geometry via pseudoholomorphic curves; these areas will be discussed in the first two sections below. In any case, MG expressed his own view to me:

> The readers of my papers look only at corollaries, sometimes also at the technical tools of the proofs, but almost always never study them deeply enough in order to understand the underlying thought.

Adrien Douady said of his own reading of *Homological Algebra* by Cartan and Eilenberg that he read it "until the pages fell off." The works of MG should be read in that same way.

The exposition is classical: algebra, analysis, geometry. One could object to the use of the word "geometer" in the title, since we will be discussing algebra and analysis, but it will become clear to the reader that the method MG uses to attack problems is to turn them into ones formulated in the language of geometry. I owe to Dennis Sullivan the remark, "It is incredible what MG can do just with the triangle inequality." Yet MG's results outside of Riemannian geometry are really in algebra and analysis, even though they have not received much recognition, especially in analysis.

In order to shorten the text, I have omitted essential intermediate results of varying importance and have therefore neglected to include numerous names and references. Although this practice might lead to some controversy, I hope to be forgiven for the choices.

Algebra: Geometry of Groups of Finite Type

We consider here only "discrete groups of finite type", i.e., finitely generated discrete groups. These groups are of course interesting by themselves, but they appear also as transformation groups in various situations: notably in number theory as the modular group $SL(2, \mathbb{Z})$ acting on the upper half plane or, more generally, as discrete subgroups Γ acting on homogeneous spaces G/H, yielding

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The author expresses his immense debt to Ilan Vardi and Anthony Knapp—to Vardi for translating the article into English and for improving the clarity of the mathematical exposition, and to Knapp for editing the article into its current form.

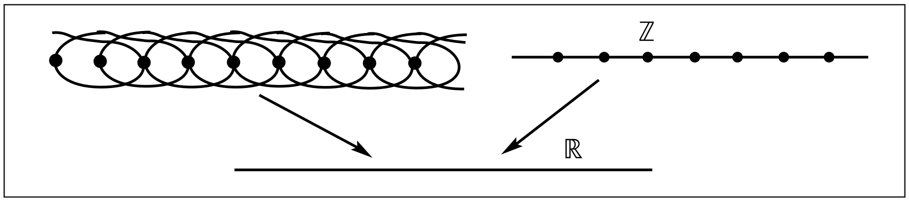

Figure 1. Seen from infinity—practically speaking, from farther and farther away—the two discrete pictures above look more and more like the continous real line.

geometric quotients $\Gamma\backslash G/H$. They appear also as the fundamental groups of compact differentiable manifolds, in particular those of negative curvature. Here I highlight two results.

The first is an affirmative answer in [3], "Groups of polynomial growth and expanding maps", to Milnor's 1968 conjecture: Every finite-type group of polynomial growth contains a subgroup of finite index that is itself a subgroup of a nilpotent Lie group. The function Growth with domain the set[1] of finitely generated groups is defined on a group L as the number Growth(L) of elements of the group that can be expressed as words of length $\leq L$ in these generators and their inverses. A change of generators does not affect growth type: polynomial, exponential, etc. In many cases the result was known. For example, suppose that G is linear. By a result of Tits, either G contains a free nonabelian subgroup and is then of exponential growth or it is solvable up to finite index. In the solvable case a 1968 theorem of J. A. Wolf shows that either a linear solvable group contains a nilpotent subgroup of finite index and is of polynomial growth or it does not and is of exponential growth.

For a general abstract group the problem is to find a Lie group, i.e., a continuous object, into which the discrete group can be embedded. MG's solution is to look at the group from infinity (Hermann Weyl said that "mathematics is the science of infinity"). For example, the group $\mathbb{Z}^d$ seen from infinity will appear as the Euclidean space $\mathbb{R}^d$, because this lattice from far away has a very small mesh that ultimately looks continuous. Even under a change of generators, this structure from infinity will remain the same. For example, take $\mathbb{Z}$, first with the trivial generators $\{\pm 1\}$, and second with the generators $\{\pm 2 \cup \pm 3\}$. The corresponding Cayley graphs are shown in Figure 1.

It is clear that the second graph, seen from infinity, is squeezed down to a line. For any abstract group Γ, one naively generalizes this as follows: first make Γ into a homogeneous metric space (Γ, d) by letting the distance $d(g, h)$ between two elements be the length of $h^{-1}g$ when expanded in terms of generators; the Cayley graph of Γ then consists in joining two points by a line if and only if their distance equals one. Looking at Γ from infinity amounts to looking at the limit of the sequence of metric spaces $(\Gamma, \epsilon^{-1}d)$ as ϵ goes to zero. This limit must be precisely defined, and this can be achieved by considering everything in nothing less than the set of all separable complete metric spaces. On this, MG defines a metric, denoted by d_{G-H} and now called the *Gromov-Hausdorff metric*. We postpone its definition to the beginning of Part II. Very delicate technical computation is required in order to show that there is a convergent subsequence in $(\Gamma, \epsilon^{-1}d)$, that the limit is a Lie group, and that G operates suitably on it. Finally, one succeeds in assembling in a single entity all the structure at infinity, and this entity is continuous. An important remark is in order: in the above proof MG uses the solution of Hilbert's Fifth Problem to obtain a Lie group. This problem has a fascinating history; it was solved in 1955 by Gleason, Montgomery, and Zippin, but there had been practically no applications until this result. The problem was to show that a group without arbitrarily small nontrivial subgroups is necessarily of Lie type. MG used the following corollary: The isometry group of a locally compact, connected, locally connected metric space of finite dimension is a Lie group with only a finite number of connected components. To navigate through the technical details of the proof, the interested reader may look at the informative expository text of Tits in the Séminaire Bourbaki for 1980–81.

This might be the right time to make the following remark, communicated to me by Hermann Karcher. Almost all of MG's big results have three features. At the start is a very simple, even naive, idea, so simple that it does not seem possible to do anything serious with it! The second feature is that the development to the conclusion is very elegant and technical, sometimes using subtle computations but always introducing new tools. The third feature is that MG in the course of his proof introduces one or more new invariants, i.e., new concepts. These are often as naive as the starting idea, but they play a basic role in the proof and most often have the paradoxical property that they are impossible to compute explicitly, even for the simplest spaces. In many cases it is also impossible to decide whether they vanish or not; we will meet many of these below.

We return to the Gromov-Hausdorff distance between metric spaces, which has become a basic tool of Riemannian geometry. It was introduced in full detail in Gromov's address at the 1978 International Congress of Mathematicians[2] and then used heavily in his 1981 preliminary French version of [13]. Today it is of primary importance when looking at a Riemannian manifold (M, g) from infinity, i.e., looking at the sequence $(M, \epsilon^{-1}g)$ with ϵ going to zero.

[1] *If isomorphic groups are identified, then the set of finitely generated groups is a set.*

[2] *The address was entitled "Synthetic geometry of Riemannian manifolds".*

The second algebraic topic is the concept of *hyperbolic group*. It was invented in [9], "Hyperbolic groups", which gives many of the properties of these groups as well as the spectacular probabilistic statement: *almost all groups are hyperbolic*. This long paper triggered an immense wave of research and results. Apart from work in Riemannian geometry, this and the symplectic geometry paper [7], "Pseudoholomorphic curves in symplectic manifolds", are, in my opinion, the only Gromov works to achieve full recognition. For the paper [7] Gromov received the AMS Steele Prize in 1997.

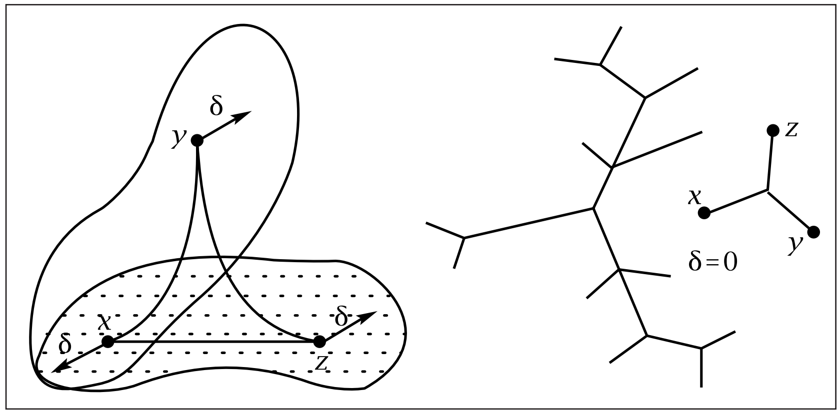

Figure 2. A triangle is δ-*thin* when any point of any of its sides is universally within δ of the union of the two other sides. In hyperbolic geometry, there is a δ so that all triangles are δ-thin, but in Euclidean geometry this is not the case. Any tree is trivially thin for $\delta = 0$.

The first naive notion to come into play for hyperbolic groups is that of *geodesic space*, sometimes also called *length space*; this is a metric space where any pair of points of distance a can be joined by at least one *segment* of length a, i.e., an isometric map of the interval $[0, a]$ into the space with ends at the two given points. The basic example is a complete Riemannian manifold (the Hopf-Rinow theorem). At the other end of the spectrum is the Cayley graph of a group in which one lets every edge be isometric to $[0, 1]$.[3]

The second notion is to isolate for our group a concept defined in terms of the metric that is invariant under change of generators and that therefore will respect the various growth types. Such a concept is that of *quasi-isometry*. This concept was introduced explicitly by Margulis in 1969 for discrete groups and by Mostow in 1973 for the general case. The general definition is quite involved: Two metric spaces X and Y are quasi-isometric if there exist two maps $f : X \to Y$ and $g : Y \to X$ and two constants $\lambda > 0$ and $C \geq 0$ such that

$$\begin{aligned} d_Y(f(x), f(y)) &\leq \lambda d_X(x, y) + C \\ &\qquad \text{for every } x, y \in X, \\ d_X(g(x'), g(y')) &\leq \lambda d_Y(x', y') + C \\ &\qquad \text{for every } x', y' \in Y, \\ d_X(g(f(x)), x) &\leq C \\ &\qquad \text{for every } x \in X, \\ d_Y(f(g(x')), x') &\leq C \\ &\qquad \text{for every } x' \in Y. \end{aligned}$$

In other words, f and g are Lipschitz at large distances, and f and g are almost inverses. One can check that for Cayley graphs a change of generators yields quasi-isometric metric spaces. Here are three more typical examples of quasi-isometry:

- a group and any subgroup of finite index,
- $\mathbb{R}$ and $\mathbb{Z}$,
- the universal cover of a compact Riemannian manifold of negative curvature (a continuous object) and its fundamental group (a discrete object).

MG calls a *hyperbolic metric space* a geodesic space for which there exists a constant δ such that all triangles of the space are δ-*thin*; i.e., any point on an edge of the triangle has a maximum distance to the union of the two other edges that is $\leq \delta$. In other words, for every triangle $[x, y] \cup [y, z] \cup [z, x]$ with vertices x, y, z and every $u \in [y, z]$, one should have

$$d(u, [x, y] \cup [z, x]) = \inf_{t \in [x,y] \cup [z,x]} d(t, u) \leq \delta.$$

It is not hard to check that this notion is stable under quasi-isometry and that Euclidean spaces are not hyperbolic. The *classical* hyperbolic spaces of hyperbolic geometry in the ordinary sense, e.g., the simply connected complete spaces of constant negative curvature, are indeed hyperbolic. Another basic example is the fundamental group of a compact Riemannian manifold of negative curvature (or the manifold itself) and, more generally, any polyhedron of negative curvature. Trees are an interesting case as they are hyperbolic for the constant $\delta = 0$. A group is said to be *hyperbolic* if its Cayley graph is hyperbolic. Evidently the Cayley graph depends on the chosen system of generators, but the property of being hyperbolic does not, since the differences are only by quasi-isometries.

We avoid any detailed exposition of [9], "Hyperbolic groups", and instead will give a brief overview. What is the structure of a hyperbolic group? MG shows that such a group is always finitely related; i.e., it can be defined by giving only a finite number of relations among its generators. This is a corollary of the existence, for any hyperbolic group, of a polyhedron of finite dimension that is contractible and on which the group acts simplicially with compact quotient. The contractibility of this polyhedron, which is indispensable, is due to Rips.

[3] *We shall use this metric on a Cayley graph throughout this article.*

We now give the reader a sense of the great power of the hyperbolicity notion by discussing many conditions that are related to it. First is the viewpoint of "complexity and algorithms" for discrete groups: If a group has a finite presentation and is of "small cancellation" $C'(1/6)$, then it is hyperbolic. *Small cancellation* means that proportionately little cancellation is possible in the product of two defining relations. More precisely, in a group defined by generators and relations, the relations being certain words in the generators, let R^* be the set of all cyclic permutations of these relations. The hypothesis $C'(1/6)$ is that the set R^* can contain two distinct words of the form uv and uv' of the same length only if the length of u is $< 1/6$ of the common length of uv and uv'. An important consequence of MG's results is that the decision problem for discrete groups is only a local one.[4]

The second concept is that of isoperimetric constant. For Euclidean spaces $\mathbb{R}^d$ the *isoperimetric inequality* says that there exists a constant $k(d)$ such that for all bounded domains D the boundary ∂D always satisfies

$$\mathrm{Vol}(\partial D) \geq k(d) \cdot (\mathrm{Vol}(D))^{(d-1)/d}.$$

This can be expressed by saying that $\mathbb{R}^d$ satisfies an isoperimetric inequality for maximal dimension domains with exponent $(d-1)/d$. One gets this type of inequality also for the volume of compact closed submanifolds N of a given dimension s and the minimum volume of the $(s+1)$-submanifolds whose boundary is N. For Euclidean spaces, the exponent is equal to $s/(s+1)$. For classical hyperbolic spaces there is again an isoperimetric inequality in maximum dimension, but the exponent now equals 1 because the volume of balls and of spheres grows exponentially and with the same factor as a function of the radius. For discrete objects, the volume of a domain is understood to be the number of points it contains, and the same for its boundary.

What MG shows concerning isoperimetric behavior is twofold. First, if a group Γ with finite presentation enjoys a *linear isoperimetric inequality in dimension* 2, then it is hyperbolic. To define this notion precisely, one considers any finite polyhedron X whose fundamental group is isomorphic to Γ and calls its universal cover X^*. Consider in X^* a simplicial loop γ contained in the 1-skeleton of X^*; simple connectivity implies that there exist simplicial disks whose boundary is γ. Call the area of a disk the number of its 2-simplices, and let $A(\gamma)$ be the minimal area of disks with boundary γ. The isoperimetric inequality one wants is: for every loop (of length $L(\gamma)$), one always has $A(\gamma) \leq CL(\gamma)$ for some constant C.

The second way to use isoperimetric inequalities is to prove that amenable groups[5] can be defined as those enjoying an isoperimetric inequality in maximal dimension with an exponent equal to zero. One can also define this concept on Cayley graphs: volume will be the number of points contained in the set, while the boundary of a set X will be the set of points that are not in X but at distance 1 from X. Thus amenable groups are not "too big". MG proves that a nonelementary hyperbolic group, if not amenable, possesses a normal subgroup that is free and nonabelian. For those groups MG goes on by defining on them a type of "geodesic flow", a dynamical object but only topological at first, because there is not initially an invariant measure. These flows have the same properties of ergodicity, mixing, etc., as the geodesic flows on manifolds with negative curvature. In this and a number of other unexpected situations in mathematics, dynamics is occasionally invading mathematical objects in which time is not present in the definition and yet the object can be endowed with a dynamical system structure, discrete or continuous.

Hyperbolic groups, like manifolds of negative curvature, admit natural compactifications, so have a boundary "at infinity" (sometimes a sphere, but not always). This compactification and the "sphere" at infinity are to be distinguished from the limit of $(\Gamma, \epsilon^{-1}d)$ as above. I make only the rather vague assertion that, seen from infinity, hyperbolic groups "look like trees".

In [9], "Hyperbolic groups", MG wrote the heuristic statement, "A group chosen at random is hyperbolic." This statement was completely proved by Ol'shanskii in 1992. It should be compared with MG's heuristic vision that "almost all polyhedral geometries have negative curvature." For geometries this certainly is true in dimension two, but for higher dimension more study is required, and the assertion remains an open problem.

In [11], "Asymptotic invariants of infinite groups", MG considerably refined the result about randomness. The first thing is to define precisely how a group is chosen at random, and the next is to define, for any group under study, a suitable notion of the ratio d of the number of relations to the number of generators. The result is this: For a random group, if $d < 1/2$, then the group is hyperbolic, but if $d > 1/2$, then the group is trivial. MG calls such a result an example of "phase transition". It fits very well with his philosophy concerning the three states: solid, liquid, and gas. He

[4] *For the various algorithmic notions of simplification, the reader may consult Strebel's appendix in the 1990 book* Sur les Groupes Hyperboliques d'après Mikhael Gromov *by Ghys and de la Harpe, which is an excellent detailed presentation of many of MG's results.*

[5] *A group G is* amenable *if whenever G acts continuously on a compact metrizable space X, X has a G-invariant probability measure.*

compares it to that for conics: ellipses, parabolas, and hyperbolas. For groups, the elliptic ones are the finite groups, the hyperbolic groups are the hyperbolic groups in the above sense, and the parabolic groups are yet to be defined. The trichotomy arises also with partial differential equations (PDEs): elliptic, parabolic, and hyperbolic. We shall discuss MG's contribution in this area in the next section.

Number theorists will appreciate the following property of hyperbolic groups. Fix a system of generators for such a group, and for any integer n let $\sigma(n)$ denote the number of elements of the group of length equal to n. Then the formal series $\zeta(t) = \sum \sigma(n)t^n$ is a rational function of the variable t.

It is now time to talk about [11], "Asymptotic invariants of infinite groups", which is a new development growing out of [9], "Hyperbolic groups". It studies asymptotic properties of infinite groups. Briefly, asymptotic properties of groups refers to properties that are stable under passage to subgroups of finite index. The same kind of thought applies to Riemannian manifolds considered only up to finite coverings. This is the first text by the author of a very new kind of mathematical writing. A later paper of this type is [12], "Positive curvature, macroscopic dimension, spectral gaps and higher signatures". These works amount to more of a research program than a standard paper, although they still contain new notions and results. The article [11] contains in particular two remarkable geometric notions, both due to MG: that of *filling* and that of *simplicial volume*. We shall return to these a little later in the present article. My belief is that these works will influence many generations of mathematicians.

The above classification of groups and the probabilistic result do not address two essential examples of groups. These are discrete uniform subgroups of Lie groups associated with symmetric spaces of rank at least two and the fundamental groups of compact manifolds of nonpositive curvature. Right notions, such as "parabolic group" or "semi-hyperbolic group", are still awaiting their creator.

For more details (and solid classical mathematical writing) the 1990 book of Ghys and de la Harpe is invaluable, even if it contains only a part of MG's work on the subject.

Analysis: The h-Principle and Applications

Having roots in his dissertation and articles in the 1970s, the *h-principle* ("h" stands for homotopy) is the hardest part of MG's work to popularize. It was developed fully in his long book [8], *Partial Differential Relations*. But it was only in 1998 in a book by Spring that parts of [8] were first treated in detail in book form. Gromov's work [8] is visionary but very dense, and it tackles many subjects. It is written with the language of "jets", "transversality", "sheaves"—language that is not familiar to every analyst. In addition, the writing is extremely dense and hard to master. This explains why quite regularly in analysis journals results appear that are a small part of some statement of [8]. To describe the essence of the book, that of the h-principle, it is best to quote from the preface:

> The classical theory of partial differential equations is rooted in physics, where equations (are assumed to) describe the laws of nature. Law abiding functions, which satisfy such an equation, are very rare in the space of all admissible functions (regardless of a particular topology in a function space).
>
> Moreover, some additional (like initial or boundary) conditions often insure the uniqueness of solutions. The existence of these is usually established with some a priori estimates which locate a possible solution in a given function space.
>
> We deal in this book with a completely different class of partial differential equations (and more general relations) which arise in differential geometry rather than in physics. Our equations are, for the most part, undetermined (or, at least, behave like those) and their solutions are rather dense in the space of functions.
>
> We solve and classify solutions of these equations by means of direct (and not so direct) geometric constructions.

I add to this a portion of my interview with MG:

> This book is the cornerstone for building a geometric theory of PDEs (the space of solutions, etc.). The book is practically ignored because it is too conceptual. However, it is so universal! I maintain that most underdetermined PDEs are soft, while the rigid ones are exceptional. In my book one can get almost anything, even fractals. Therefore there is a certain robustness; one knows that robustness is essential in physics, because "one sees only what is robust".
>
> The h-principle: it is indeed hard to believe; analysis experts do not believe in it, and as a result prove from time to time parts of what is already in the book. Hard to believe, because it contradicts mathematical intuition and physical intuition at the same time. For

> example, for a C^1 equation, the solutions are dense in C^0.
>
> But there is still much to do: find a measure for the space of solutions (something like Markov fields), transform the h-principle into measure theory. In fact, to be able to say something statistical about solutions, one must find a measure.

The h-principle is too long to formulate in detail, and moreover it has to be defined and proved explicitly for each specific problem. But what it says, roughly, is that for the equations under consideration there are no obstructions to constructing the space of solutions other than those of topological and transversal nature found in a suitable set of jets.[6] Here, as mentioned above and as will be mentioned again later, the initial idea is simple and geometric, and beyond that is a long technical and subtle path.

Without further definitions, here are some applications of the philosophy of the h-principle taken either from the book itself or from subsequent texts.

• First, MG made almost complete progress in the quest for the optimal dimension $N(d)$ of the isometric embedding problem of abstract Riemannian manifolds M^d of a given dimension into some $\mathbb{R}^{N(d)}$. Nash's famous theorem yielded such an isometric embedding, but for dimensions of magnitude like d^3, which is too large.

• Another contribution is the extension of Oka's principle for Stein manifolds.[7] Oka's principle says that every continuous section of a holomorphic vector bundle is homotopic to a holomorphic one. In MG's way of looking at it, this is nothing but the h-principle for the Cauchy-Riemann equations. Like every application of the h-principle, this has to be stated precisely in the context with which one is working.

• In Riemannian geometry Lohkamp, in his series of papers starting in 1992, was inspired by the h-principle and proved it in the cases he worked on. His results illustrate in a spectacular way the deep nature of the h-principle, as outlined by MG above. The problem is to decide whether the hypothesis of negative Ricci curvature for a Riemannian manifold has strong topological consequences. For sectional curvature, negativity has a drastic consequence: the manifold's universal cover gotten by the exponential map is diffeomorphic to $\mathbb{R}^d$. For Ricci curvature, which will be discussed in Part II, positivity has many implications, but one still does not know all the topological consequences of this condition. As for the negativity of Ricci curvature, Lohkamp's result is: "negative Ricci curvature amounts to nothing". More precisely, any manifold admits such a metric, but also on such a manifold the space of all metrics with negative Ricci curvature is contractible as a topological space. There are more surprises in store: on any manifold each Riemannian metric can be approximated by negative Ricci curvature metrics. Of course, this can hold only in the C^0-topology, since we can start with positive curvature, which is a C^2-object, whereas the metric is a C^0 object. Again, using suitably adapted h-principles, Lokhamp very recently proved that on any compact manifold there exists a dense set of Riemannian metrics whose geodesic flow is Bernoulli, hence is mixing, ergodic, etc. Previously, for the sphere S^2, individual ergodic metrics had been constructed very painfully. Lohkamp's results illustrate the power of the h-principle.

Practically at the same time that he was publishing the book [8], *Partial Differential Relations*, MG completely revolutionized symplectic geometry in his article [7], "Pseudoholomorphic curves in symplectic manifolds". This tidal wave, which is not yet finished, is the subject of the 1994 book *Pseudoholomorphic Curves in Symplectic Geometry* by Audin and Lafontaine. We shall borrow here from the introduction, and we refer the reader to the book for a detailed exposition.

Symplectic geometry (sometimes called symplectic topology by those who use the word geometry exclusively in situations where there is a metric) is the study of manifolds M^{2n} endowed with a *symplectic form*, namely, a differential 2-form ω that is closed ($d\omega = 0$) and nondegenerate (one also says of maximal rank), i.e., is such that the maximal exterior product form $\bigwedge^n \omega$ is everywhere nonzero. This product is a volume form and hence defines a canonical measure on the manifold. We will speak of symplectic manifolds and symplectic structures, since there is an obvious notion of symplectic diffeomorphism (= *symplectomorphism*). Symplectic structures are found in the mathematical world in a number of different settings, but two are especially important. The first is that of Kähler manifolds, in which the symplectic form is nothing but the Kähler form. The second is that of Hamiltonian mechanics, where the form under study is the "Liouville form" on the cotangent bundle (= "phase space") of the manifold where mechanics is studied.

As a consequence of a theorem of Darboux, one knows that, locally, every symplectic form ω is symplectomorphic to the canonical symplectic structure on $\mathbb{R}^{2n}$ given by $\sum_{i=1}^{n} dx_i \wedge dx_{i+n}$. It

[6] *A* jet *is a system of values of Taylor coefficients through a certain order.*

[7] *MG waited a long time to publish his result in 1992 in a joint paper with Eliashberg, "Embeddings of Stein manifolds of dimension n into the affine space of dimension $3n/2 + 1$".*

follows that there is no local invariant for symplectic geometry. In dimension two a symplectic form is just a smooth orientable measure, so it is legitimate to ask whether in dimension four or more there are more symplectic *global* invariants besides ones that come only from the measure. What freedom do we have? MG's contributions to this question are various.

A Lagrangian submanifold is one where the form ω vanishes when restricted to it. In his 1986 book *Partial Differential Relations*, MG used the h-principle and addressed the question of freedom by constructing an immense number of Lagrangian immersed submanifolds. On the other hand, he exhibited a certain rigidity in symplectic geometry; this is the essence of the 1985 paper [7], "Pseudoholomorphic curves in symplectic manifolds", where MG proved a conjecture of Arnold: in $\mathbb{R}^{2n}$ there exists no compact Lagrangian *embedded* submanifold. Now, among the many results and new tools introduced in [7], I choose a typical one and treat it in detail so that the reader might appreciate the full depth of MG's ideas, both creative and technical.

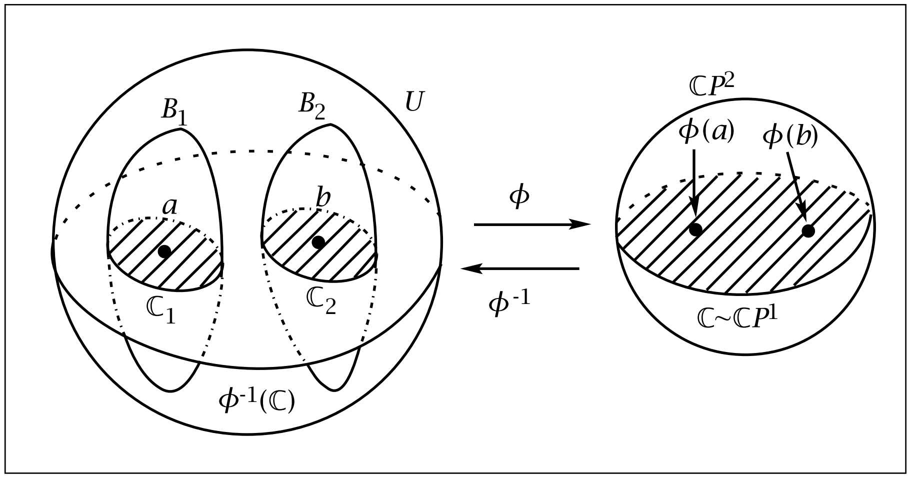

Figure 3. In the complex projective space, for any reasonable symplectic structure, pseudoholomorphic curves mimic exactly the projective geometry axioms for the projective lines.

We will consider the problem of packing balls via symplectic diffeomorphisms and try to see whether there are obstructions beyond their measure. MG has many results about this problem, and I choose the simplest one, both to state and to prove. Let U be an open set in $\mathbb{R}^{2n}$ (endowed with the standard symplectic structure) that contains two Euclidean disjoint balls $B_1(a, r_1)$ and $B_2(b, r_2)$ of respective radii r_1 and r_2 and respective centers a and b. The result is this: if there is a symplectic embedding ψ of U into a ball of radius R of $\mathbb{R}^{2n}$ (still endowed with the standard symplectic structure), then one necessarily has $r_1^2 + r_2^2 \leq R^2$. For $n \geq 4$, this condition is stronger than the measure condition; in dimension 2, it is exactly the measure condition. We cannot resist mentioning the less symmetric but more striking result: with their induced standard symplectic structures in $\mathbb{R}^{2n}$, a ball and a half ball (a ball cut in two parts through its center) having the same volume are never symplectomorphic as soon as $n > 1$.

We now sketch the proof of the two-ball packing problem, which uses a good deal of the techniques introduced in the 1985 paper. We need some definitions. An *almost complex structure* on a manifold (necessarily of even dimension) is an automorphism J with square minus the identity ($J^2 = -Id$) on its tangent bundle. As soon as the dimension is larger than two, such a structure is typically not integrable; i.e., there is no holomorphic complex structure whose J will be multiplication by $i = \sqrt{-1}$. Every symplectic structure admits some (in fact many) almost complex structures J that are *compatible*, i.e., have $\omega(x, Jx) > 0$ for every nonzero tangent vector. In this situation a *pseudoholomorphic curve* is a submanifold of dimension two whose tangent bundle is stable under J. In the holomorphic case this is nothing but a complex curve, but the interest is in the noncomplex case. The condition for a two-dimensional submanifold to be a pseudoholomorphic curve is a PDE. Motivated by the h-principle, MG showed that there are numerous global pseudoholomorphic curves.

In particular, MG studied the case of the complex projective space $\mathbb{C}P^n$ endowed with its standard symplectic structure. MG considered *any* compatible almost complex structure J, and in particular he was interested in the case when J is not the standard complex structure on $\mathbb{C}P^n$. For $\mathbb{C}P^2$ the h-principle reflects the fact in projective geometry that through any pair of points there passes a unique projective line. Indeed, MG shows that through any pair of points there passes a pseudoholomorphic curve (for any J) and there is uniqueness. For $\mathbb{C}P^n$, $n \geq 3$, MG shows existence, but uniqueness no longer holds. According to Gromov this is the best example illustrating that the h-principle can be, despite its general softness, sometimes soft and sometimes rigid. More recently, in 1996, Donaldson used the h-principle to study the similarity between pseudoholomorphicity and genuine holomorphicity.

Returning to our open set U in $\mathbb{R}^{2n}$ and the two balls B_1 and B_2 packed symplectically into the ball B of radius R in $\mathbb{R}^{2n}$, we now explain the proof of the ball-packing result in $\mathbb{R}^4$; things are the same in higher dimensions. We easily construct a symplectic embedding ζ of B into the complement $\mathbb{C}P^2 \backslash \mathbb{C}P^1$, where $\mathbb{C}P^2$ has its canonical symplectic structure ω but the structure is normalized in such a way that the total volume equals $\pi^2 R^4/2$, the volume of the ball B of radius R in $\mathbb{R}^4$. Let

$\phi = \zeta \circ \psi : U \to \mathbb{C}P^2$ be the symplectic embedding obtained by the composition of ψ and ζ, where ψ is the given symplectic embedding. We now push the canonical almost complex structure[8] J_0 of $\mathbb{R}^4$ into a partial almost complex structure $\phi(J_0)$ on $\mathbb{C}P^2$. It is easy to extend $\phi(J_0)$ into a global and compatible almost complex structure J on $\mathbb{C}P^2$. The object yielding the result is the pseudoholomorphic curve C in $\mathbb{C}P^2$ that is homologous to $\mathbb{C}P^1$ and passes through the two points $\phi(a)$ and $\phi(b)$. The total area of C equals πR^2 since it is $\int_C \omega$. This integral is

$$> \int_{\phi(B_1)} \omega + \int_{\phi(B_2)} \omega.$$

Consider the two surfaces $C_1 = \phi^{-1}(\phi(B_1) \cap C)$ and $C_2 = \phi^{-1}(\phi(B_2) \cap C)$ in $\mathbb{R}^4$. Then

$$\text{Area}(C_1) = \int_{C_1} \phi^*(\omega)$$

and

$$\text{Area}(C_2) = \int_{C_2} \phi^*(\omega).$$

Moreover, both the surfaces C_1 and C_2 are pseudoholomorphic curves for the standard J_0 structure of $\mathbb{R}^4$, which comes from a complex structure on $\mathbb{R}^4$. So C_1 and C_2 are in fact holomorphic, hence minimal surfaces in $\mathbb{R}^4$. By construction, C_1 contains the center a of B_1. Thus the monotonicity principle for minimal surfaces implies that the area of C_1 is larger than or equal to πr_1^2. The same applies to C_2, and hence $\pi r_1^2 + \pi r_2^2 \le \pi R^2$.

Introducing Quantitative Methods in Algebraic Topology

A fundamental but immense program is to study quantitative (as opposed to qualitative) algebraic topology and differential topology. An example is constructing a quantitative theory of homotopy. A typical example of the need, arising from a question of Thom, is: if catastrophe theory is not predictive, it is precisely because this theory does not take into account a metric or a measure. Another aspect of the need is the common definition of algebraic topology as the study of properties invariant under "any" deformation. It has been MG's constant concern to tackle such questions. We describe some of MG's major contributions to quantization of algebraic topology.

In [2], "Homotopical effects of dilatation", two subjects are tackled. The first is the complexity of the space of all maps $f : X \to Y$ from one manifold into another. To study this setting, one works with manifolds that are compact and have finite fundamental group and puts on them an arbitrary Riemannian metric. As was true in the section above on algebra, it is important to realize that, for the current purpose, the metric that is used is quite irrelevant. This is because two metrics on a compact manifold are always quasi-isometric. The *dilatation* of f is defined to be the supremum of the ratios

$$\frac{d_Y(f(x), f(x'))}{d_X(x, x')}$$

over all the pairs (x, x') with $x \neq x'$. The main result of the article is this: *for Y simply connected, the number of homotopically distinct maps grows polynomially with respect to dilatation.*

In fact, [2] also has a completely new approach to Morse theory. Briefly, Morse theory considers differentiable functions $f : M \to \mathbb{R}$ on a given compact manifold M and expresses the existence of critical points (those where the derivative of f vanishes) as a function of the topological complexity of M. The precise statement needs the notion of *index* of a critical point, which is nothing but the number of negative eigenvalues in the second differential of f. Morse proved that the number of critical points of index equal to i is always $\geq$ the i-th Betti number $b_i(M)$. But in many cases, typically in the study of periodic geodesics, Morse's result leaves *untouched* the values $f(x)$ at these critical points. Are they very close to each other, very sparse, or what? Moreover, Morse theory imposes a technical restriction: all the critical points have to be nondegenerate; i.e., the quadratic form given by the second differential of the function has to be everywhere nondegenerate.

Some of the results of [2] have been treated in more detail in the book [13] *Metric Structures for Riemannian and Non-Riemannian Manifolds*, which is an enlarged translation of the French 1981 version, *Structures Métriques pour les Variétés Riemanniennes*, known to all Riemannian geometers as the "little green book". The idea is, as always with MG, naive at first: get metric control over algebraic topology operations. But complete proofs are hard. MG introduced new intermediate invariants, which are surprisingly subtle. We discuss a typical example.

In the pioneering work [6] "Volume and bounded cohomology", MG introduced the *simplicial volume* of a compact manifold M: consider all possible ways to write the fundamental class $[M]$ in the top degree of the homology of M as a linear combination $[M] = \sum_i a_i \sigma_i$ of singular simplexes σ_i with *real* coefficients a_i. By definition, the simplicial volume $\|M\|$ of M is the infimum of the sum $\sum_i |a_i|$. For a first example, take the sphere S^2, where a single simplex will suffice. Map one side of the standard 2-simplex to a great circle from pole

[8] *The canonical almost complex structure refers to the one coming from regarding $\mathbb{R}^4$ as $\mathbb{C}^2$.*

to pole, and map the other two sides to another such great circle. The angle between these great circles at a pole can be increased through any multiple of 2π that we please, and in particular this singular simplex can represent $n[M]$ with n as large as we please. The $a_1 = 1/n$, and we see that the simplicial volume of S^2 therefore vanishes. In fact, this happens any time a manifold admits maps into itself of degree larger than 1, since for any map $f : M \to M'$ of (topological) degree d, one has $\|M\| \geq d\|M'\|$. More generally, $\|M\| = 0$ as soon as the fundamental group $\pi_1(M)$ is "not too large"—more precisely, as soon as $\pi_1(M)$ is amenable.[9] This is one of the results of [6], and it shows that the simplicial volume is an invariant of the geometry at infinity of a manifold (actually of its universal cover).

Another good example is that of a compact manifold of negative curvature. The geometric vision here is exactly the opposite of the spherical case. Namely, when one stretches a simplex, its volume does not grow, since one easily proves that the volume is bounded above. Thus $\|M\| > 0$. A classical principle of good mathematics is that after introducing a new object, one should use it to prove something deep. In [6] MG uses the simplicial volume to re-prove a part of the Mostow rigidity theorem, one of the most spectacular results in geometry. This part of the Mostow theorem says that two discrete compact quotients of a hyperbolic space in dimension > 2 are isometric if they have isomorphic fundamental groups. Gromov's result is in fact a variable-curvature version of this part of Mostow's theorem.

The simplicial volume is also interesting for discrete groups, as was shown in [11], "Asymptotic invariants of infinite groups". More generally, it applies to various asymptotic properties of manifolds. It strongly inspired the 1995 results of Besson, Courtois, and Gallot as follows: Define a *space form of rank one* to be any compact quotient of a noncompact Riemannian globally symmetric space of rank 1 (i.e., a "hyperbolic space" over $\mathbb{R}$, $\mathbb{C}$, the quaternions, or the Cayley numbers) by the action of a torsion-free discrete subgroup.[10] Their results are that the standard Riemannian structure minimizes all the usual invariants: entropy, growth of lengths of periodic geodesics, asymptotic volume of balls, etc.

Nevertheless, positivity of simplicial volume is incredibly difficult to establish. Simplicial volume has the following functorial properties: there are universal constants c and c' depending only on the dimensions such that

$$c\|M\|\ \|M'\| \geq \|M \times M'\| \geq c'\|M\|\ \|M'\|$$

and, in dimension 2,

$$\|M\#M'\| = \|M\| + \|M'\|.$$

Here # denotes the connected sum operation. In particular, products of manifolds of negative curvature have positive simplicial volume. Such manifolds have curvature that is only nonpositive. The other natural type of nonpositive-curvature manifold is a *space form of rank at least* 2, i.e., a compact quotient of a noncompact Riemannian globally symmetric space of rank at least 2, but it is still unknown whether these manifolds have positive simplicial volume. It remains almost a total mystery whether manifolds with nonvanishing simplicial volume are numerous or scarce. The tie with groups is subtle: if $\pi_1(M)$ is hyperbolic, then it does not necessarily follow that $\|M\| > 0$.

In [6] Gromov introduced a second topological invariant for compact differentiable manifolds. The *minimal volume* of M, written MinVol(M), is the lower bound of the volume over all Riemannian metrics one can put on M that have their sectional curvature in $[-1, 1]$. One can think of it as measuring the extent to which a metric on M is bumpy. It could in principle be used to find the least bumpy metric on a given M, but this would not be robust, since sectional curvature is not robust. The minimal volume is less subtle than the simplicial volume. It has recently become clear because of results of Cheeger and Rong in 1996 that most compact manifolds have zero minimal volume. Except for surfaces, where the Gauss-Bonnet theorem gives the answer, and for space forms of rank 1, where the Besson-Courtois-Gallot result applies, not a single explicit minimal volume is known, even for the simplest geometric spaces: not for the standard projective spaces over $\mathbb{R}$ or $\mathbb{C}$, or the quaternions, or the Cayley numbers, and not even for the even-dimensional spheres starting with S^4.

There is a basic link between these two volume invariants, the main inequality of [6]: there is a constant $c(d)$ such that, for any Riemannian manifold (M^d, g) whose Ricci curvature[11] satisfies Ricci $> -(d-1)$, the simplicial volume satisfies

$$\mathrm{Vol}(g) > c(d)\|M\|.$$

A sufficient condition for Ricci $> -(d-1)$ is that the sectional curvature satisfy $K > -1$. Consequently[12] MinVol(M) $> c(d)\|M\|$. Thus simplicial volume fits quite well with Ricci curvature; it is an important open problem to see whether it also fits well with scalar curvature. The proof of the main inequality above is quite difficult, and to my knowledge it has never been written up in detail. Moreover, it uses a new technique called "diffusion

[9] *See footnote 5.*

[10] *We take space forms to have sectional curvature normalized to be in the interval* $[-4, -1]$.

[11] *This notion is defined in Part II.*

[12] *This corollary is much weaker than the initial statement.*

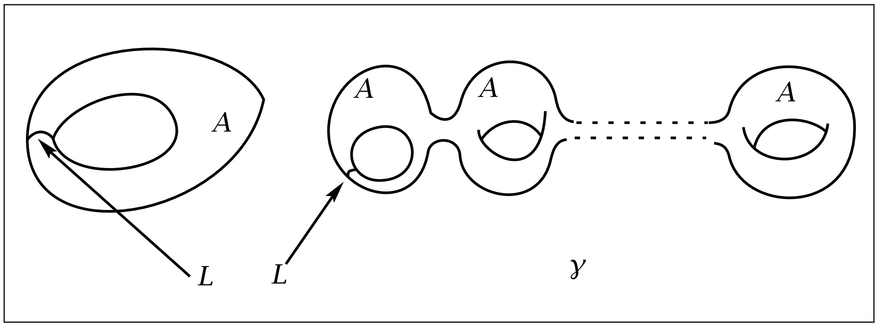

Figure 4. It is very intuitive that, for surfaces with one or more handles (say the genus is positive) as in the picture, the ratio between the area and the squared length of the shortest noncontractible curves is larger and larger when the number of holes grows. But in fact this turns out to be extremely difficult to prove.

of cycles"; this technique is used in the "Filling paper", which we consider next.

The preceding results were mainly concerned with "topology and metric". The next result concerns "topology, metric, and measure". It first appeared in the huge work [5] "Filling Riemannian manifolds", which is called the "Filling paper". This text is a seminal work, and many of its results have been used, but to my knowledge nobody has gone completely through it. In particular, the content of the last four of the nine sections has never been elaborated in complete detail. We will now explain in detail the first topic of the Filling paper, since it is extremely simple to visualize and formulate. It starts with a result of Loewner concerning the torus: We consider a two-dimensional torus with an arbitrary Riemannian metric.[13] We define the *systole* of this torus to be the infimum of the lengths of the noncontractible closed curves on this torus. This minimum is classically achieved by a periodic geodesic, but this is not the point. Let us call this systole L. Then Loewner proved that the total area A of our torus always satisfies the inequality $A/L^2 \geq \sqrt{3}/2$. Moreover, equality is attained if and only if the torus is flat and regular hexagonal (i.e., corresponds to the lowest point of the classical modular domain). Loewner never published this result, since he did not consider it deep enough.

The mathematical mind will immediately ask for generalizations—first to surfaces of higher genus, then to higher-dimensional manifolds, and then to higher-dimensional submanifolds. As Thom pointed out to me in Strasbourg in 1960, Loewner's result is the simplest case of the general problem of finding universal relations between norms of various homology classes when putting metrics on manifolds, these relations being valid for any metric. For example, for surfaces of genus γ, one would expect an inequality of the form $A/L^2 \geq c(\gamma)$ with a constant $c(\gamma)$ depending on the genus, and one would expect the constant to grow with the genus. Before the Filling paper nobody could obtain any result of this form, apart from independent work of Accola and Blatter in 1960 that for surfaces yields a constant going to zero tremendously fast with the genus.

The Filling paper solves these two problems. The one for surfaces of genus γ yields a constant growing like $\gamma/\log^2\gamma$, and unpublished examples of MG show that one cannot do better. The second problem, the volume as a function of the systole of curves for higher-dimensional manifolds, is solved for so-called "essential manifolds". A universal inequality $\mathrm{Vol}/(\mathrm{Systole})^d > c > 0$ is hopeless for general manifolds; just take the product of a circle with any other compact manifold. What is needed is a condition expressing the geometric fact that the noncontractible curves generate the fundamental class (which yields the volume). Essential manifolds constitute a large class, containing all tori, all real projective spaces, and all compact manifolds whose universal covering is diffeomorphic to $\mathbb{R}^n$ (e.g., space forms and other manifolds with nonpositive curvature). MG's result for surfaces uses the technique of "diffusion of cycles" mentioned above. The situation for surfaces is exceptional, since the result for surfaces is optimal yet is obtained by "elementary" geometry. At the time, the uniformization theorem and other techniques of theory of one complex variable were yielding very poor results.

For higher dimensions one needs new ideas and invariants. The first thing that MG does in the Filling paper is to embed the Riemannian manifold under consideration in the infinite-dimensional Banach space $C^0(M)$ of all continuous functions on M^d by the map f given by distance functions to points, $f(x) = d(x, \cdot)$. The basic property of this map is that it is an isometric embedding of M^d into $C^0(M)$ with the supremum norm. The isometry condition is stronger than the usual one for isometric embeddings into Euclidean spaces: the induced metric is not the Riemannian one based on length of curves, but the induced one in the elementary sense:

$$d(x,y) = d(f(x), f(y)) = \sup_{z\in M} |d(x,z) - d(y,z)|.$$

Such a strong isometric embedding is impossible in finite dimensions, but it is precisely the one needed for the proof.

In $C^0(M)$, an infinite-dimensional setting, MG proves an isoperimetric inequality for the volume of the manifolds N^{d+1} that *fill* the image $f(M^d)$; i.e., the boundary of N^{d+1} is precisely $f(M^d)$. The

[13] *This can be taken as the one induced by $\mathbb{R}^3$ when we have a "real" torus. On the other hand, if the torus is flat (that is to say, locally Euclidean), the torus has to be treated as an abstract manifold or else embedded in some higher-dimensional Euclidean space.*

infimum of these volumes is called the *filling volume* of M^d. Another new invariant is needed: the *filling radius* of M^d, the smallest number ρ such that if an N^{d+1} has M^d as boundary, then it cannot be contracted to $f(M^d)$ inside the ρ-neighborhood of $f(M^d)$. Now MG proves (this is the relatively easy part) that for essential manifolds the systole is $\leq 6\rho$. The proof of the isoperimetric inequality in infinite dimensions is hard, and MG considers it as belonging to "geometric surgery".

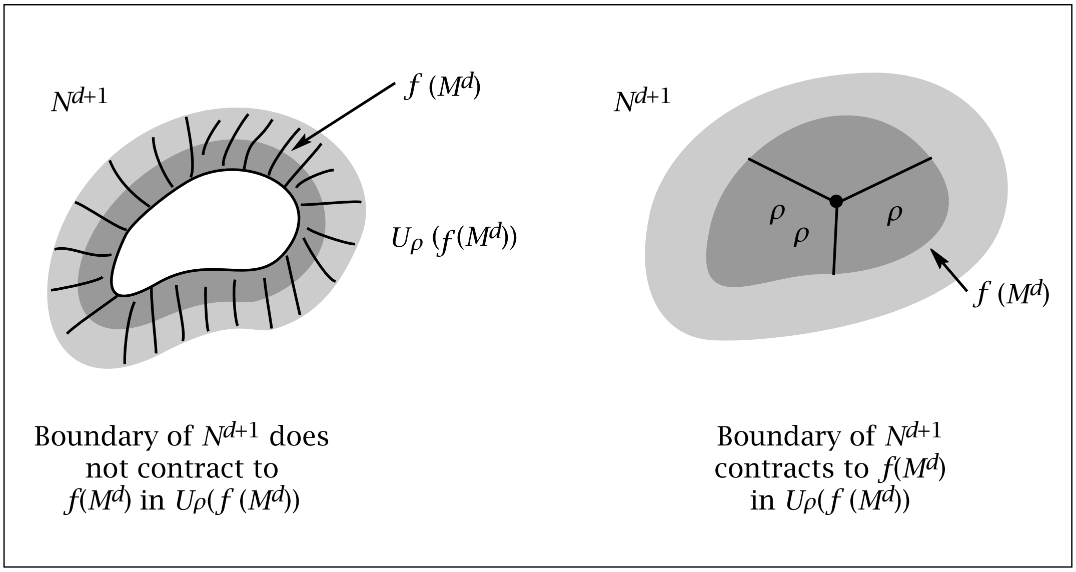

Figure 5. It is intuitive that the tubular neighborhood of radius R of a subvariety M, for small R, can be retracted to M, but that there is a limit value (called the *filling radius*) for which that property stops. Then M will bound some submanifold of one dimension higher.

Some comments are now in order. We have already mentioned the fact that MG's invariants are often impossible to compute for standard manifolds. The filling volume and the filling radius illustrate this point. The filling radius has been computed only for spheres, real projective spaces, and the complex projective plane (by M. Katz in 1983 and 1991). The filling volume is more baffling, because its exact value is still unknown for any manifold, even for the simplest one, namely, the circle S^1 (the expected value is 2π, the area of a hemisphere).

Let us continue with the Filling paper, which is 150 pages long and is written in a very succinct style. The Filling paper contains numerous other results and techniques as well as an in-depth vision of metric geometry, and not only of Riemannian manifolds but also of "Finsler manifolds".[14] *Finsler manifolds* are those for which the metric structure is given by a collection of convex symmetric bodies in the various tangent spaces of a manifold under consideration. The Riemannian case is when they all are ellipsoids. Other new invariants introduced in the Filling paper are the *visual closure* and the *visual volume*, which are awaiting other applications.

Still, the Filling paper is diffusing slowly into various fields. For example, MG proves a result that apparently seems trivial: Consider a Riemannian manifold with the property that every ball is contractible inside the ball with same center and twice the radius; then the volume of this manifold is infinite. This type of universal contractibility is now basic in Riemannian geometry and is related to the topic called *controlled topology*.[15]

The above concerned situations in which the systole was "one-dimensional". We return to Thom's program, stated above, for higher-dimensional systoles, i.e., with homology or homotopy classes in dimension > 1. Natural candidates are complex projective spaces, products of manifolds, etc. Unhappily, in a 1992 paper, "Systolic and intersystolic inequalitites", MG laid the foundations of a program that kills every possible hope for these manifolds. Starting with dimension 2, no universal systolic inequality can hold. We have here *systolic softness*, whereas there is systolic *rigidity* in the one-dimensional case. MG's program has been pursued by various authors, including Babenko, Katz, and Suciu, in recent years and has been almost completely finished. The guess is that no manifold will have any systolic rigidity. For MG's version of the history of this subject, see his "systolic reminiscences" at the end of Chapter 4 of [13], *Metric Structures for Riemannian and Non-Riemannian Manifolds.*

We turn to Gromov's contribution to surgery, a basic tool in algebraic topology. His work lies in "geometrically controlled surgery", or "quantitative surgery". The basic surgery operation works with two manifolds, M and N, of equal dimension. The trivial kind of surgery is the *connected sum*: one joins M and N by a small tube after having taken a small ball out of each. More sophisticated surgery is possible in higher dimensions. From M^d one

[14] *For these spaces, see, for example, D. Bao, S.-S. Chern, and Z. Shen,* An Introduction to Finsler Geometry, *Lecture Notes in Math., Springer-Verlag, 2000.*

[15] *See, for example, the survey of Berger "Riemannian geometry during the second half of the twentieth century", in the* Jahresbericht der DMW *in 1998 or reprinted as AMS University Lecture Notes, volume 17, in 2000 (ISBN 0-821-82052-4).*

removes a topological sphere S^k and looks at a small tubular neighborhood of it inside M^d. The boundary of this tube is $S^k \times S^{d-k-1}$. One does the same to N^d, but this time with a sphere S^{d-k-1}. Here we have again a tube, and the boundary this time is $S^{d-k-1} \times S^k$. It remains to glue the two boundaries by exchanging the order of the factors. The simplest example is two copies of $S^1 \times S^2$, with surgery around a circle in each of them. If one does not interchange the role of the parallels and the meridians on the torus $S^1 \times S^1$, the final result is still $S^1 \times S^2$. But interchanging them yields the sphere S^3 (as in the Hopf fibration of S^3).

MG used controlled surgery (with metrics) in a number of cases. The first is in a joint 1980 paper with Lawson, "The classification of simply connected manifolds of positive scalar curvature", which is the basic work about the classification of Riemannian manifolds of positive scalar curvature. The authors show that, if one starts with two Riemannian manifolds M and N, both with positive scalar curvature, and if one performs a surgery of codimension at least 3, then the manifold so obtained can be endowed with a Riemannian metric that is also of positive scalar curvature.

In the Filling paper [5] controlled surgery is used at least twice: first for the optimal systolic inequality for surfaces of high genus, second to prove the isoperimetric inequality in infinite dimensions. Controlled surgery is also a basic tool for proving systolic softness for some manifolds. One constructs local *boxes* that are systolically soft and controls the surgery done when introducing these boxes into the manifolds.

In Part II we shall discuss MG's contributions to Riemannian geometry and related subjects; some of these have been hinted at in Part I.

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