Million-dollar Mathematics Prizes Announced

On May 24, 2000, at a special event at the Collège de France in Paris, the Clay Mathematics Institute (CMI) drew worldwide attention by announcing that it would award prizes of $1 million each for solutions to seven mathematics problems. The CMI's aim in establishing the prizes is to celebrate the new millennium and to increase the visibility of mathematics among the general public.

The mood was celebratory as an audience of about 500 people listened to lectures throughout the afternoon and quaffed champagne at the closing reception. That morning a press conference attracted about thirty journalists, and the story of the "Millennium Prize Problems" was widely reported. It appeared on the front pages of *Le Monde*, *Figaro*, and the *International Herald Tribune*, as well as in other major European newspapers. In the U.S. an Associated Press wire story was carried by many papers, and the *Washington Post* published a piece by a Paris correspondent. Articles also appeared in weeklies such as *Science* and *Nature*, and the story was broadcast on a number of radio programs around the world, including National Public Radio in the U.S.

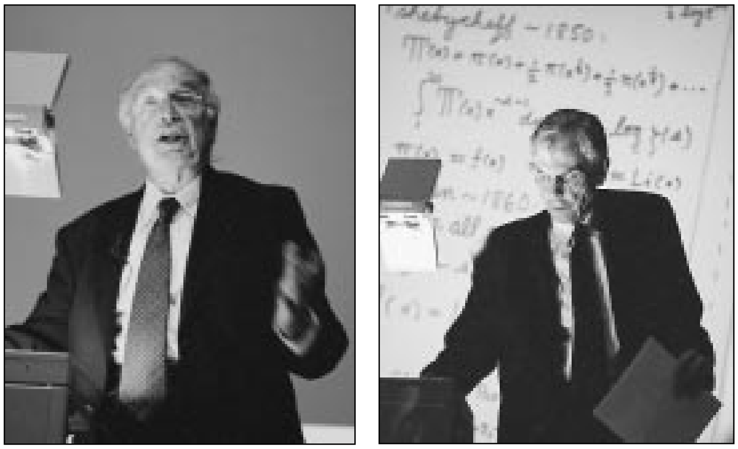
The list of prize problems (see sidebar) will be immediately recognizable to mathematicians as containing some of the great unsolved mathematical problems of the last century. As some of the questions at the press conference made clear, it can be difficult for nonmathematicians to understand what the problems are about, much less to appreciate why they are important. But the $7 million prize fund provided a powerful symbol. And it commanded attention: the day after the announcement, visitors eager to read the descriptions of the prize problems posted on the CMI Web site quickly overwhelmed the site's capacity. The AMS, whose site has a much larger capacity, offered to host the CMI pages temporarily, and soon the AMS site was almost overwhelmed.

The Clay Mathematics Institute was founded in 1998 by Boston businessman Landon T. Clay and is headed by Arthur Jaffe of Harvard University. The CMI has moved swiftly to establish itself as an international supporter of mathematics, funding individual mathematicians, summer programs for high school students, workshops, conferences, and other activities. Although based in Cambridge, Massachusetts, the CMI chose Paris as the locale for the announcement of the prize problems in order to celebrate the 100th anniversary of David Hilbert's famous lecture at the International Congress of Mathematicians in Paris in 1900. In connection with that lecture, Hilbert presented his list of twenty-three outstanding problems, which have since had a good deal of influence on mathematics.[1]

But there is an important difference between Hilbert's problems and the Millennium Prize Problems. "Hilbert was trying to guide mathematics by his problems," noted Andrew Wiles of Princeton University, who is a member of the CMI Scientific Advisory Board. "We're trying to record great unsolved problems." Such a selection of problems inevitably ignores large parts of mathematics. "There are big programs in mathematics that are important but where it is very hard to isolate one problem that captures the program," Wiles explained. "In some areas we were disappointed that we couldn't pick something appropriate. But we wouldn't be surprised if the solutions to these problems involved those areas."

The business of formulating mathematical problems is a delicate one: sometimes a problem that

[1] *"A Sideways Look at Hilbert's Twenty-three Problems of 1900," by Ivor Grattan-Guinness, appeared in the August 2000 issue of the* Notices, *pages 752–757.*



Photos courtesy of Clay Mathematics Institute.

Sir Michael Atiyah (left) and John Tate (right) presenting the prize problems in Paris.

at first seems important turns out to be not quite the right one to pose. A certain amount of refinement has to take place until one is sure one "has the problem by the scruff of the neck," noted Michael Atiyah of the University of Edinburgh, who along with John Tate of the University of Texas at Austin presented the prize problems at the Paris event. These seven problems are probably old enough that this refinement has already taken place, but surprises are always possible. The rules for awarding the prizes are flexible enough to handle various situations, such as a simple counterexample that leads to a reformulation of a problem. Generally a prize would not be given for such a counterexample, but the CMI has left itself sufficient discretion to award a prize when a problem is judged to be "well and truly finished," as Wiles put it.

The rules for the prizes explicitly state that submissions cannot be made directly to the CMI. To be eligible for the prize, a solution must be published in a recognized journal and have been in print for two years. Following this waiting period, the CMI Scientific Advisory Board (currently composed of Jaffe, Wiles, Alain Connes of the Collège de France and the Institut des Hautes Études Scientifiques, and Edward Witten of the Institute for Advanced Study) would decide whether to consider the solution for a prize. Despite these rules the CMI is likely to see many attempted solutions filling its mailbox. In fact, the rules are probably less discouraging to amateurs than the problems themselves, which require some background in mathematics to understand.

One famous unsolved problem that is easier to state than the Millennium Prize Problems is Goldbach's Conjecture, the quarry in another million-dollar prize competition, sponsored by the publisher of the novel *Uncle Petros and Goldbach's Conjecture*, by Apostolos Doxiadis. Goldbach's Conjecture asserts that every even number greater than two can be written as the sum of two primes. Wiles said that Goldbach's Conjecture had not been suggested as a Millennium Prize Problem because the Riemann Hypothesis, which was an obvious problem to include, so dominates that area of mathematics. To collect the prize for Goldbach's Conjecture, one must submit a published solution to one of the book's publishers (Bloomsbury Publishing in the U.S. and Faber and Faber Limited in the United Kingdom) by March 2002. By contrast, there is no time limit on the prizes to be given by the CMI.

Most mathematicians are likely to be familiar with several of the problems on the list. The Riemann Hypothesis, the Poincaré Conjecture, and the Birch and Swinnerton-Dyer Conjecture are likely bets for any list of major unsolved problems. Though it hails from theoretical computer science, the question of whether P equals NP has become widely known among mathematicians as more and

Millennium Prize Problems

The *Notices* prepared the following brief summaries of the prize problems. Official descriptions of the problems are available on the CMI Web site, `http://www.claymath.org/`.

The "P versus NP" Problem: A problem is in P if it can be solved by an algorithm that runs in polynomial time (that is, the running time is at most a polynomial function of the size of the input). A problem is in NP if a proposed solution can be checked in polynomial time. Does $P = NP$?

The Riemann Hypothesis: Every nontrivial zero of the Riemann zeta function has real part equal to $1/2$.

The Poincaré Conjecture: Any closed simply connected 3-manifold is homeomorphic to the 3-sphere.

The Hodge Conjecture: On a nonsingular complex projective algebraic variety, any Hodge class is a rational linear combination of classes of algebraic cycles.

The Birch and Swinnerton-Dyer Conjecture: For every elliptic curve over the rationals, the order of vanishing of its L function at 1 is equal to the rank of the abelian group of rational points on the curve.

The Navier-Stokes Equations: Prove or disprove the existence and smoothness of solutions to the 3-dimensional Navier-Stokes equations (under reasonable boundary and initial conditions).

Yang-Mills Theory: Prove that quantum Yang-Mills fields exist and have a mass gap.

A New World of Mathematics

Andrew Wiles, a member of the Clay Mathematics Institute Scientific Advisory Board, presented the following remarks during the press conference on May 24, 2000, at which the Millennium Prize Problems were announced.

The Second Millennium Prize Problems stand out, we believe, as great unresolved problems of the twentieth century. Some have their roots even earlier. These problems are not new. They are well known to the mathematical community. But we hope that, by attaching prizes to them, we will bring them home to a much wider audience and thereby bring the excitement of mathematical endeavor to a wider audience also. I personally first encountered Fermat's problem as a ten-year-old by reading a popular book about mathematics. And on the cover of this book it gave the history of the Wolfskehl Prize, a prize which had been set up fifty years earlier for the solution of this problem. We hope that these prize problems will similarly excite and inspire future generations of mathematicians and nonmathematicians alike.

Let me stress, however, that the mathematical future is by no means limited to these problems. There is a whole new world of mathematics out there, waiting to be discovered. Imagine, if you will, the Europeans in 1600. They know that across the Atlantic there is a New World. How would they have assigned prizes to aid in the discovery and development of the United States? Not a prize for inventing the airplane, not a prize for inventing the computer, not a prize for founding Chicago, not a prize for machines that would harvest vast areas of wheat. These things have become a part of America, but such things could not even have been imagined in 1600. No, they would have given a prize for solving such problems as the problem of longitude. The problem of determining longitude was a classical problem, and its resolution helped make possible the development of the New World.

We are convinced that the resolution of these prize problems will similarly open up a new world of mathematics which as yet we cannot even imagine.

more problems—some theoretical, some practical—have been linked to this fundamental question. Probably the least familiar problem on the list is the Hodge Conjecture, which lies at the frontier between algebraic geometry and topology. The conjecture, first posed by W. V. D. Hodge in 1950, posits a link between cohomology and algebraic cycles. The list contains two problems with connections to physics. One is quite straightforward to state: Prove the existence of smooth solutions to the Navier-Stokes equations, a set of nonlinear partial differential equations that describe many kinds of fluid flow. The other one echoes one of Hilbert's original problems from 1900, which called for "a mathematical treatment of the axioms of physics." The vagueness of the statement aside, the problem was clearly premature in 1900, and, as Atiyah noted, "the question is still with us. Is 2000 a better time?" Atiyah asked. "Well, we don't know." As stated by Atiyah (the official description of this problem was not finished at the time of this writing), this Millennium Prize Problem calls for establishing Yang-Mills theory as a rigorous quantum field theory.

The day-long CMI event took place in a newly constructed wing of the Collège de France. During the afternoon two Clay Mathematics Awards were presented (see "Mathematics People" in this issue of the *Notices*), and Timothy Gowers of the University of Cambridge delivered a lecture with the somewhat grandiose title "The Importance of Mathematics"—a title which, he noted, he had not chosen himself. The audience was thick with luminaries of the Parisian mathematical scene of the present and the past, including ninety-five-year-old Henri Cartan, who was sitting in the front row when Atiyah mentioned Cartan's work on sheaf cohomology in connection with the description of the Hodge Conjecture. The next day, the crowd dwindled to only a couple dozen people for lectures by four CMI Fellows (Manjul Bhargava, Dennis Gaitsgory, Laurent Lafforgue, and Terence Tao).

In remarks at the press conference Landon Clay tried to explain why he, a businessman with relatively little background in mathematics, is so interested in the field. He cited external reasons, such as low public funding for mathematics, but he also cited a personal reason. "Curiosity is part of human nature," he said. "Unfortunately, the established revealed religions no longer provide the answers that are satisfactory, and that translates into a need for certainty and truth. And that is what makes mathematics work, makes people commit their lives to it." Clay went on, "It's the desire for truth and the response to the beauty and power and elegance of mathematics that drive mathematicians." One can only hope that the Millennium Prize Problems inspire this kind of appreciation of mathematics among the general public.

—*Allyn Jackson*