Small Worlds

Reviewed by Carson C. Chow

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Duncan J. Watts
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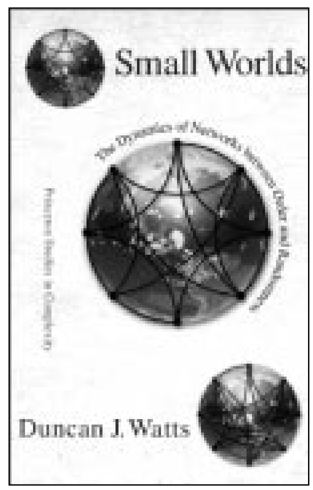
The phenomenon that any person is only six handshakes, or "six degrees of separation", away from any other person on the planet is the essential feature of the "small worlds" referred to in the title of this book. The term "small worlds" also provides an apt self-referential metaphor for this book, which shows the way many diverse areas of research—from graph theory to epidemiology to coupled oscillators—have some surprising connections.

Almost everyone has had the experience of finding that he or she shares a mutual acquaintance with a total stranger. This casual observation was put to the test by Stanley Milgram in the late 1960s at Harvard University [7]. He conducted an experiment in which he asked a random group of people in Kansas and Nebraska to send letters to a particular individual whom they did not know in Boston, Massachusetts. Each person was instructed to send the letters to a person whom they knew on a first name basis and who they felt might have a greater chance of knowing the intended recipient. The new letterholder was then asked to continue the chain. Amazingly enough, some letters actually reached the person in Boston, and the median chain length was six. Milgram coined the term "six degrees of separation", and the concept has permeated the popular consciousness ever since. *Six Degrees of Separation* is the title of a play by John Guare and a subsequent film adaptation. In the parlor game "Six Degrees of Kevin Bacon" participants are given the name of a movie actor and must link that actor to Kevin Bacon through mutual appearances in films. In the lounges of academia, how many mathematicians have not made some attempt to compute their "Erdős number", the number of connections through coauthorship to Paul Erdős? As of January 2000, 507 privileged people have an Erdős number of one [4].



Small Worlds stems from Watts's Ph.D. dissertation in the Department of Theoretical and Applied Mechanics at Cornell University. Part of the inspiration for the research came from an attempt to understand the influences of network connectivity on the synchronization of biological oscillators, in particular the singing of crickets. The other part was a curiosity about how to prove that there are only six degrees of separation between any two people. Along the way to resolving these questions, Watts took a peripatetic journey with forays

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into a wide range of fields. He has collated these results into a stimulating and enjoyable book. It is divided into two parts. The first part addresses the question of what types of networks or graphs possess the small-world property and includes a primer on graph theory. The second half asks what the "dynamical" consequences are of such a network. On the whole the book provides mostly conceptual insights and new avenues of exploration rather than hard mathematical results. I believe it could be of interest to a diverse scientific audience, including biologists, computer scientists, economists, mathematicians, physicists, psychologists, and social scientists, if only as an introduction to concepts in graph theory and various complex dynamical systems (the specific topics are outlined below). Although no advanced mathematics is presented and Watts derives or defines most of the technical terms and concepts used, there are probably too many formulas in the book for it to be palatable to the general public.

The first thought that came to mind when I read the original paper by Watts and Strogatz [10], [3] which introduced small-world networks was, Should I be surprised at all that there are just six degrees of freedom between me and the pope? After all, if each person has a hundred friends, then through exponential growth five links will span a range of ten billion people. The catch is that our social connections are not independent but highly interconnected: many of our friends know each other. This idea is made more precise by considering a graph with undirected edges. Watts defines two measures of interest. The first is the characteristic path length L defined as the median of the means of the shortest path length connecting each vertex to all other vertices. The second measure is the clustering coefficient γ, which is the average fraction of vertices adjacent to any vertex which are also adjacent to each other. A graph exhibiting the small-world property is one with a high level of clustering and a small characteristic path length. It is not so obvious what type of graph satisfies both constraints.

Watts introduces three models of relational graphs that possess the small-world property. Each of the models has a parameter that interpolates between a graph with a high degree of clustering (e.g., a lattice with nearest-neighbor coupling) and a random graph with a small characteristic length. For example, his β model begins with a cycle of n vertices where each vertex is connected to k of its nearest neighbors. The graph is then "rewired" by randomly assigning with probability β each edge of a given vertex to any other vertex. For $\beta = 0$ (the original cycle), in the limit of large n and k with $n > k$, $\gamma \sim 3/4$ and $L \sim n/(2k)$. This represents a world where people live in close-knit communities and the only outside people they know live in neighboring communities. For $\beta = 1$, the graph is essentially random with $L \sim \ln n / \ln k$ and $\gamma \sim k/n$. For $n > k$ there is very little clustering. In the random world a person is only a few degrees of separation from anyone else, but no one would know anyone but the host at a dinner party. As β is increased from zero to one, the graph must make a transition from a clustered world with large L to a random world with small L. Watts discovered that in this transition, L drops extremely quickly to the random graph level for β well below 1, while γ falls at a much slower rate. There is a range of β for which the graphs possess the small-world property. This range generally holds for β much smaller than 1, which implies that it may be difficult to tell whether or not a graph has the small-world property by simply examining the links to a particular node. In other words, people could not tell whether they lived in a small world simply by examining the list of their friends. They would have to calculate the characteristic path length for the entire population. Watts's other models and their variations exhibit similar properties demonstrating robustness of the small-world effect.

The intuitive explanation for the small-world effect is that the random rewiring introduces "shortcuts" between self-contained communities and it only takes a small number of shortcuts to make a large world small. In the close-knit community world, a few world travelers could link together distant villages. Watts does not have a general proof that only a small number of shortcuts makes a large world small, but he does give some theoretical backing to this intuition by introducing simpler heuristic models that allow analytical estimates of L and γ as functions of the number of shortcuts. The results match the simulation results of the relational graph models quite well. Watts then examines three real-world networks—the collaboration graph of film actors (also called the Kevin Bacon graph), the power grid of the western United States, and the neural connections of the worm *C. elegans*. All three graphs possess the small-world property in that L is similar to that of a random graph but γ is much larger. However, only the Kevin Bacon graph matched closely with the theoretical predictions of the relational graphs. That the others did not match closely might reflect the difficulty of fitting all small-world networks into one simple category. Indeed, the neural network architecture of *C. elegans* is keyed directly to function, so presumably the wiring pattern is not merely random.

The second half of the book deals with some consequences of living in a small world, with chapters on disease spreading, global computation in cellular automata, game theory, and coupled oscillators. This half of the book has much less analysis and is much more exploratory in nature than the first half. Given the properties of small-world graphs discussed in the first half, the results in the second half are sometimes unsurprising. However, there are enough unresolved questions to keep the reader interested. Watts's work has already spurred a flurry of research activity. The chapters are well written and provide good introductions to these topics for the uninitiated.

The most interesting chapters are on global computation in cellular automata and games like "Prisoner's Dilemma" on small-world networks. A cellular automaton (CA) consists of a lattice of n cells where each cell is assigned a state. The state may occupy a finite number of discrete levels and is updated at discrete intervals of time according to a transition rule that depends only on the state of the cell and its neighbors. An example of a global computation is the density classification problem for a CA with two states, "on" and "off". A transition rule must be found so that the CA turns on all of its cells (after some finite time) if initially more than half the cells are on, and off otherwise. On a one-dimensional lattice it is nontrivial to find such rules. For example, a local majority rule does not work because the network eventually freezes into local domains of on and off states. However, if instead of a regular lattice the CA is put on a relational graph with just a small number of shortcuts, the majority rule works for almost all initial conditions.

Although it is not mentioned in Watts's book, the magnetization of the Ising model of statistical mechanics comes naturally to mind in this context. The Ising model consists of an array of "spins" that can either be up or down (on or off). The spins interact with their nearest neighbors; the energy of the interaction is lower if they are aligned and higher if they are opposite. One question is whether the minimum energy state of the entire system is a state in which all the spins are aligned (i.e., magnetized). The Ising model is similar to an equilibrium version of the density classification CA. It is known that for any dimension less than 2 (including fractional dimensions!), spontaneous magnetization is impossible at a temperature greater than zero. Fluctuations destroy the order, resulting in domains of up and down spins. Recent results show that the Ising model on a small-world network will show spontaneous magnetization below a critical temperature if the probability for shortcuts exceeds a critical level [2]. Interestingly, at high temperatures the model behaves as if it were one-dimensional, but at low temperatures it has the characteristics of a mean field (infinite-dimensional) model which allows magnetization. Perhaps new insights into the properties of disordered materials may arise from using small-world lattices.

The game "Prisoner's Dilemma" involves two players (A and B) who can choose to cooperate (C) or defect (D) upon meeting. A payoff (P) is given to each player depending on the four possible outcomes: C_AC_B, C_AD_B, D_AC_B, and D_AD_B. The payoff for player A follows the ordering $P_A(D_AC_B) > P_A(C_AC_B) > P_A(D_AD_B) > P_A(C_AD_B)$. The payoff for B is the same, but with indices reversed. The dilemma arises because although the maximum payoff for both players together is obtained if both cooperate, the only rational choice is for both to defect. However, when generalized to a game with repeated interactions, the dilemma may be resolved. In a famous computer tournament that pitted multiple strategies against each other, the winning strategy was "tit-for-tat", in which a player initially cooperates and then mimics the opponent's action from the previous round [1]. Watts investigated how network topology might affect the tendency to cooperate or defect in a generalized tit-for-tat game. The game begins with an initial seed of cooperators in a sea of defectors. The players are assigned a "hardness" h $(0 < h < 1)$. At each time step they calculate the fraction of neighbors who were cooperators on the previous step, and if this exceeds h, they cooperate; otherwise they defect. Watts found that there were nontrivial dependencies on h and the number of shortcuts. Generally, the propensity for cooperation decreases as the number of shortcuts increases. It seems as if a sense of community is required to foster cooperation, and making the world more random makes for a nastier world. (Are we seeing signs of this in our world?) For relatively low values of h (easily swayed population), a small-world network supports the growth of cooperation, but for increasing h cooperation becomes much less likely. Without any shortcuts cooperation can thrive locally but will only spread very slowly to other areas initially populated by defectors. It seems that in order to have a cooperative world the populace must be amenable to change and there must be a small number of long-range connections.

The chapter on disease-spreading is somewhat less enlightening; ironically, the same can be said about the chapter on coupled oscillators, which provided some of the inspiration for this work. Watts shows that infectious diseases spread faster in networks with shorter characteristic path lengths, which is as expected. There are some quantitative curiosities in the dynamics of disease-spreading, and Watts and collaborators have since analyzed more complicated models [8]. Watts also finds that coupled phase oscillators will synchronize much more easily in a small-world network than on a regular lattice. This result was to be expected, since it is known that even small amounts of long-range coupling in a regular lattice can synchronize a network that otherwise would have supported traveling waves [5]. What would be more interesting to understand are which types of nonsynchronous dynamics are possible. This could have relevance to various firing patterns observed in the brain, for example.

In his preface Watts notes that much of the book may need to be revised by the time it is printed. Indeed, he may have written the book too soon. For example, some of his recent work with M. Newman has addressed the scaling form of L for a variant of the β model. They find that $L = (n/k)g(\beta kn)$, where g is a universal scaling function [8]. Using methods from statistical mechanics, they have shown this form applies in the limit of small β and large n. I think the book would have been much stronger if this and other recent results were included. Perhaps the book begs for a second edition, and if this comes to pass, a glossary of the technical terms would be extremely helpful.

I assume part of Watts's motivation to publish at such an early stage was to inspire more research. An obvious open question to be answered is, Under what conditions do small-world networks arise? Watts has shown that small-world graphs exist only over a subset of the parameter space of his relational graph models. Recent work has shown that if a network attempts to minimize the total "length" of edges (assuming it is embedded in a metric space) and maximize connectivity simultaneously, small-world networks can arise [6]. Another question is whether the small-world property would persist if the number and location of vertices and edges were allowed to change. An evolving network would be closer to the real social world where people constantly make new friendships, give birth, and die. Does the small-world property remain robust under such changes? If people lived forever and the population were conserved, then presumably everyone would eventually meet everyone else, creating a completely connected graph. Would we have a "large world" if the population grew at a rate faster than the rate at which people can make friends?

Watts's book has opened up a previously neglected avenue of research both in the theory of graphs and their influence on dynamics. He has found a way to classify network topology that goes beyond the narrow confines of regular lattices and random graphs. While the small-world effect is certainly interesting and likely to have applications, I think his method for interpolating between a regular lattice and a random graph may be the most important contribution. Within graph theory, exact expressions for the clustering coefficient, the characteristic path length, and other properties of relational graphs as functions of network size and interpolation parameter β await computation [9]. In the study of complex systems, one is always confronted with the issue of whether an observation or property of a system is a detail that must be studied in isolation or whether it is a general concept. People have long believed that network architecture should play a major operational role in systems such as social networks or the brain. However, there were few conceptual frameworks to address such considerations. Watts has provided us with a new way to categorize and manipulate networks. It may not be complete, but it is certainly a good start.

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