Pictures and Proofs

Bill Casselman

A cognitive life in which all truth can be simply "seen" would be the life of … an angel.
—C. S. Lewis, in The Discarded Image

百聞は一見にしかず *—Japanese proverb*

One look is better than a hundred hearings is what the proverb says in a somewhat literal translation. More colloquially in English, *A picture is worth a thousand words.* The idea is commonly asserted in many cultures. The important role played by visualization and illustration in mathematics in particular is widely recognized and has apparently been so since the very beginnings of the subject. It is also poorly understood, even in its simplest aspects.

Of course there are many facets to the relationship. The one that is probably both the most fascinating and the most elusive is the role of internal visualization in the heuristic stages of mathematical development, but this is an almost unlimited topic I will leave untouched here. Nor am I going to explore, as did a recent article of Richard Palais in the *Notices*, how computers can make possible feats of visualization never before even imagined. Instead, I want to explore the much more down-to-earth topic of how pictures are used, and should be used, in mathematical exposition. There are a number of points that might be made:

- The importance of good illustrations is underestimated.
- The application of even a few very simple ideas would greatly improve the overall quality of mathematical illustration.
- Computers can make a few relatively unostentatious but nonetheless significant improvements in quality.
- The techniques needed to create good illustrations can be of computational and mathematical interest in their own right.

One of the principal problems, of course, is that while a good picture may be worth more than a thousand (English) words, it may also require a great deal more trouble to produce than text. Technology has an effect on this cost, as measured by the effort expended, and has had such an effect since early days. Even now, when computers have made so much so easy for us, it is still rarely trivial to produce a good mathematical illustration. Nonetheless, it is arguable that the lack of quality of the illustrations used commonly in mathematics is largely a matter of habit and convention rather than innate obstacles.

Why Are Good Mathematical Illustrations Important?

In an ideal world figures would return with interest what has been invested in them. And what sort of return? Clarity, even transparency. In reading and writing mathematics, as Yuri Manin mentioned in his 1990 ICM talk "Mathematics As Metaphor", it is important to distinguish the knowledge of mathematical truth from the understanding of mathematics. What does this have to do with mathematical graphics? In spite of disclaimers and for better or worse, pictures—even if only internalized ones—often play a crucial role in logical demonstration. But as tools for understanding they are indispensable.

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I find it intriguing that medieval philosophers from Boethius to Thomas Acquinas, like Manin, were concerned with distinguishing *ratio*, by which they meant a carefully assembled chain of reasoning, from *intelligentia*, in which something was comprehended all at once. It is very likely that the clearest examples they had in mind were taken from their limited acquaintance with Euclid. The ultimate in *intelligentia* was the way in which the Deity was able to comprehend all of the world, past and future, in one glance. This degree of understanding is not something we can hope to achieve, but the nearest we can come to *intelligentia* is probably through pictures.

A simple example is the formula

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}$$

for a converging geometric series with $|r| < 1$. The usual rigorous argument is quite satisfactory, answering just about all questions that might arise about the convergence. But it is still an example of *ratio*. Can this formula be visualized? Thanks to Zeno's paradoxes, a very large number of people are familiar with the simple picture in Figure 1 that goes with the case $r = 1/2$:

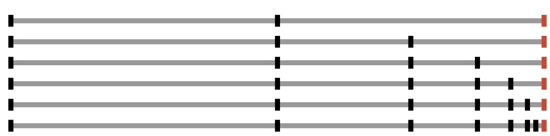

Figure 1. Zeno's paradox: summing a geometric series with common ratio $r = 1/2$.

A few years ago an undergraduate student in a course of mine came up with a way to illustrate the general case. Figure 2 shows a brief excerpt from what was a kind of animation: These are pretty good and elicit a pleased response from most who see the illustration.

But these pictures are still a bit abstract, in that the eye has to digest several pieces of algebraic information simultaneously, and I find that a few extra pictures to handle some explicit cases like $r = 1/3$, $r = 1/4$, etc., improve the response noticeably. If r is set equal to $2/3$, for example, Figure 3 is what we get. The shaded triangle and the large one with the sum as base are similar, and with the grid in place it is now visibly evident that one is three times as large as the other, making the base equal to 3.

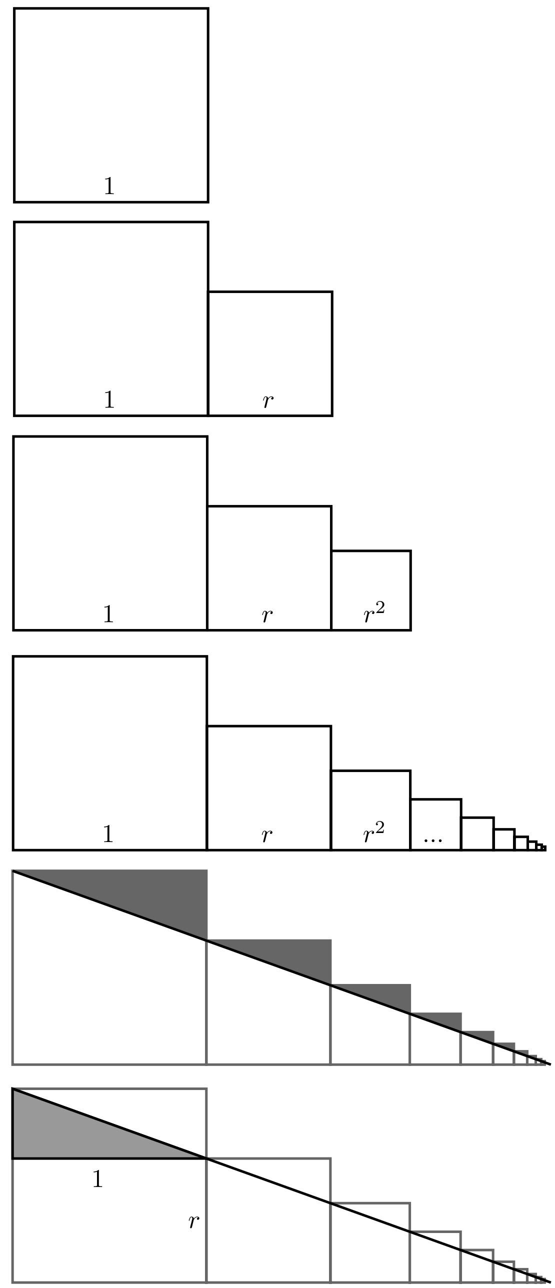

Figure 2. Animation for summing a geometric series with common ratio r.

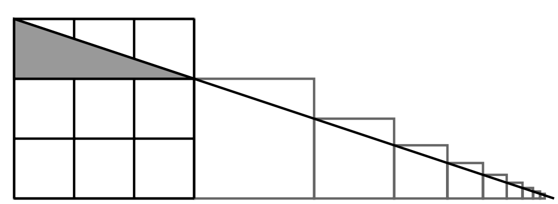

Figure 3. Pictorial argument for summing a geometric series with common ratio $r = 2/3$.

What Makes a Good Mathematical Illustration?

Primarily because I do not really understand exactly how pictures work towards mathematical understanding, I can offer only some tentative suggestions. Some of them I have taken from Edward Tufte's books:

- *Reduce visual clutter and—what is not quite the same thing—eliminate distraction.* Put in only what the diagram really needs to make its point. Tone down components that just add context.
- *Highlight components that are central to the current discussion.* If necessary, repeat a diagram several times, but with different components emphasized. This is a variant of what Tufte calls *small multiples*.
- *The figures themselves should tell a story.* Coordination between text and illustration is surprisingly tricky, and ideally the two should be as independent of each other as possible. Movies handle this problem with audio, but that approach is not an easy option yet for mathematics, nor is it easy to think about how to deal with it even if it were. Most of us are still restricted to making silent films.
- *It is rare for there to be too many illustrations in a mathematics paper.* Keep in mind that illustrations can serve a number of distinct purposes—for example, I find that I can often tell better what a paper with plentiful illustrations is about by skimming diagrams rather than by reading text. Pictures can often be read rapidly, and adding more should always be taken as a serious option.
- *In drawing figures, think out how the material would be presented in spoken discourse.* Draw pictures that follow the same narrative, even if this means a fair amount of repetition. Computers can help in dealing with this sort of repetition.
- *Ask constantly whether the figures really convey the point they are meant to.* Redo them if necessary. Figures should be redrawn, as text is rewritten, until they are right.
- *Use imagination.* Sometimes very small and subtle changes in a figure will have an enormous impact. Experimenting will help.
- *Do not depend often on pictures to make a point.* People's interpretations of figures vary unpredictably, as indeed thousands of psychological experiments show. Always try out a figure on a number of people just to see how it goes over.

Let me try to explain a few of these points with an elementary example. It might be objected that it is too elementary to be of much use, but I find that it is unusual enough to generate heated discussion, even with rather sophisticated mathematicians.

The traditional figure for Pythagoras's Theorem from Euclid is Figure 4. It matches perfectly with the text in Euclid, and it or some slight variant is still used frequently in explaining Euclid's proof of Pythagoras's Theorem. It is less frequently admired. I think it is fair to say that the traditional way in which Euclid's proof is presented and the figure that accompanies it disguise its charms.

As I suggested earlier by the remarks about coordination of pictures and text, the scheme that Euclid himself uses is rarely the best way to explain his proofs, except for historical purposes. In a course incorporating both computer graphics and geometry that I give frequently to third-year university students, I usually start off with a discussion of shears, particularly about how they preserve area. Superficially simple as this topic is, it can lead to many graphical images and can also lead to very subtle points about Euclid's concept of area and Hilbert's improvements of Euclid's logical structure. Then follows a sequence of figures again suggesting animation (Figure 5). It is measurably much better than following Euclid, in the sense that as far as I can tell no student has *ever* forgotten it, once explained. Here is certainly a place where a computer made it very simple to make redundant calculations trivial and where the difficulty of making repetitions by hand would have been inhibitive.

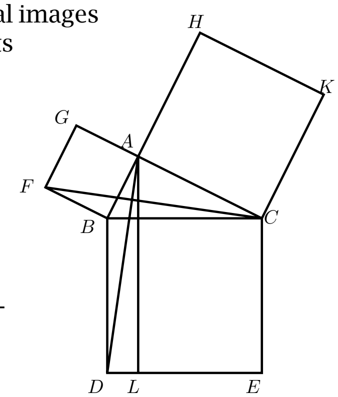

Figure 4. Traditional picture for proving Pythagoras's Theorem.

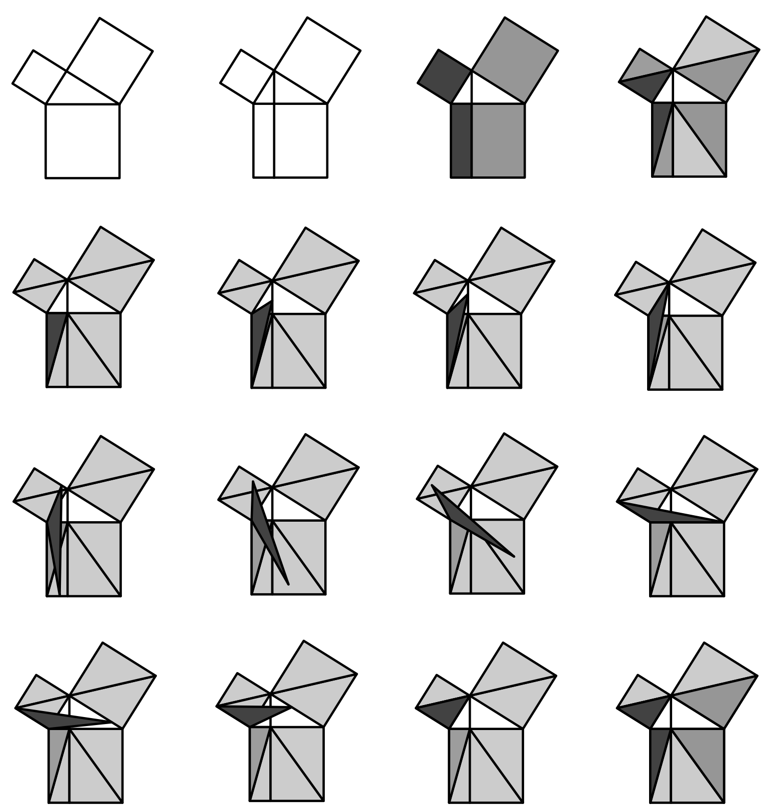

Figure 5. Animation for proving Pythagoras's Theorem.

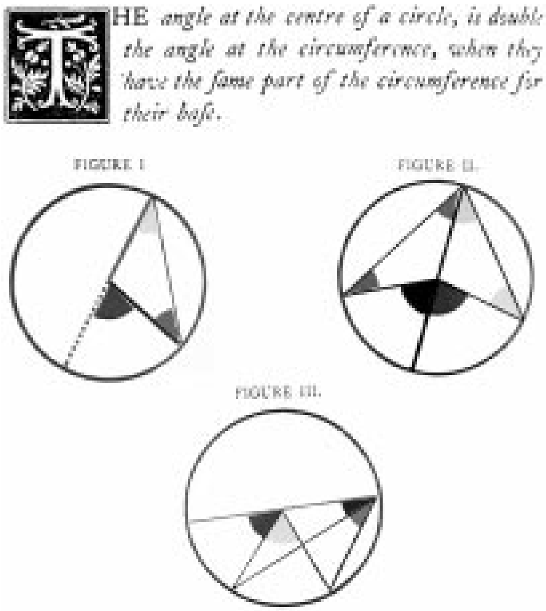

Figure 6. Byrne's pictorial proof that an angle from a point on a circle is half the arc cut off by the sides.

The idea of rewriting Euclid's *Elements* in pictures seems to have occurred first to the nineteenth-century Englishman Oliver Byrne. His version of the first six books was published in 1847 in a book well known to bibliophiles, if not to mathematicians, for its striking use of color. The book is not an unalloyed success, but even its failures are interesting, and for students it has proved to be a fruitful source of projects. The curmudgeonly David Eugene Smith says of it (footnote on p. 329 of his edition of Augustus De Morgan's *A Budget of Paradoxes*, volume I), "There is some merit in speaking of the red triangle instead of the triangle ABC, but not enough to give the method any standing." This is not quite a fair appraisal. Byrne's technique is by no means flawless, and his understanding in particular of the mathematics in Euclid's difficult Book V very weak, but he does more than just refer to colors instead of labels. He uses diagrams effectively in the lines of text themselves, and many of his figures manage to convey information, sometimes a whole proof, by careful use of color. It is at any rate an interesting counterbalance to the conventional methods of exposition, and my guess is that the basic idea could be used well in mathematics classes at all levels.

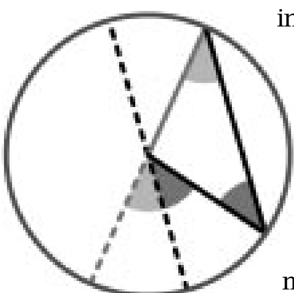

Figure 7. Modification of first diagram in Figure 6.

One can get some idea of how Byrne's book reads in the excerpt from his proof of III.20 reproduced in Figure 6. There is text in Byrne's book to accompany this, but III.20 is a good example of where text is almost unnecessary. Almost, but not quite. Perhaps a few more figures would have made it easier, setting up the initial data more clearly. These data are a point lying on a circle, together with an arc of that circle not containing the point. A sort of triangle with the point as one vertex and the arc as the opposite side is then examined. Byrne's three figures correspond to the three cases where the center might lie relative to that triangle. I myself prefer the slight modification of the Bryne's Figure I that appears in Figure 7, which seems to be more self-contained.

The textual excerpt illustrates well the style of the press, which was that of Charles Whittingham, well known in nineteenth-century England for its attempt to revive fine press work. It is said that the cost of Byrne's book was so high and the demand for it so low that it brought about Whittingham's bankruptcy in 1853. Who among publishers would risk the family business on an edition of Euclid?

There seems no doubt that Byrne was extremely eccentric, in a country and an age renowned for eccentricity. He is certainly one of the backyard dwellers of mathematics and perhaps rightly considered a crank by his contemporaries, as De Morgan's comments about him in *A Budget of Paradoxes* tell us convincingly. It would be interesting to know De Morgan's opinion of Byrne's Euclid, but unfortunately I am not aware that he recorded one.

A Gallery of Blunders

One of the reasons that mathematicians do not seem to deal well with illustrations is that they tend to see what an illustration is trying to say rather than what it actually says. This distinction is widely considered a virtue. If the accuracy of an illustration is criticized, for example, it will often be said in defense that in mathematics a picture is intended only to convey a rough idea and need not be exact. Pictures are deceptive, it is claimed, and will seduce one to confound attractive pictures with sound logical reasoning. What Littlewood has to say about this is one of my favorite quotations:

> A heavy warning used to be given that pictures are not rigourous; this has never had its bluff called and has permanently frightened its victims. (*Littlewood's Miscellany*, 1986, p. 54)

He goes on to say that "pictorial arguments, while not so purely conventional, can be quite legitimate." He then gives a few intriguing examples of such arguments. Some mathematicians will even say that there is merit to bad graphics, since mathematics is supposed to teach one how to reason, how not to trust one's intuition. This contention borders on nonsense and is often an excuse for laziness or incompetence. There may be a role for sloppy pictures in training one in how to deal with faulty intuition, but it is too subtle a matter to deal with at the start. My experience leads me to the completely contrary assertion: *good graphics has appeal to a wide range of people*, even those who might otherwise feel negatively about the interest and importance of mathematics overall.

It might seem that what I have stated so far is so obvious that it requires no emphasis. Judging from what I see in the literature, this is not the case.

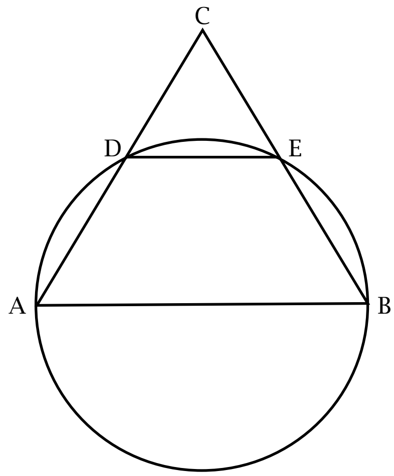

Figure 8. Intentionally deceptive picture from a document for secondary school teachers.

Blunders in pictures are ubiquitous, and awkward use of illustrations is even more common. Errors are at best severely distracting and at worst lead to serious confusion. For some reason that I do not completely understand, errors in illustrations seem to be more acceptable than errors in text. There is a paradox implicit here—errors in illustrations are at the same time both more and less visible. Part of the reason for this is that there are several sources of graphics errors or, at any rate, there are many different causes.

Intentional Deception

Nonetheless, there is certainly a kernel of truth to the claim that one must be careful about interpreting pictures, and indeed in some circumstances there might be some point to an inaccurate or deceptive illustration. In a recent document from my local provincial government trying to explain to secondary school teachers a few things about the geometry curriculum, the following problem occurs:

> A circle, which has as its diameter side AB of the equilateral triangle ABC, intersects the other two sides of the triangle at D and E. If the diameter of the circle is 16 cm, find the area of the quadrilateral ABDE.

It goes with the picture in Figure 8.

The specifications imply that the length of AD be equal to that of DC. Thus the large triangle ABC is made up of four copies of the smaller triangle CDE, and the answer is straightforward to calculate. The picture, however, is not accurately drawn. Although it is asserted that AB is a diameter, the drawing does not quite suggest this, so that it is not visibly apparent that the length of AD is equal to that of DC. On the contrary, it is visibly apparent that it is not! I suspect that for most students only careful consideration and perhaps a slight tendency to skepticism will bring them to the right answer, using reason to counteract what their eyes are telling them. This is not necessarily a bad thing. The illusion depends on the fact that locating the height of a horizontal diameter is not a well-conditioned problem, and this is a good trick to use occasionally to force students to reason rather than guess. There are also a few other subtle manipulations that help make the deception work.

Inattention

There is good evidence that mathematicians are frequently blinded by intellectual expectations. Many years ago Branko Grünbaum pointed out to the Mathematical Association of America (MAA) that the image in use as logo (Figure 9), which was intended to be a representation of a regular icosahedron, was in fact mathematically impossible. In the figure the lines indicated are parallel in three dimensions. If rendered by orthogonal projection from 3D into 2D, they will remain parallel; and if rendered in perspective, they will all intersect in a single point "at infinity". *Mathematics was not used to draw this figure.*

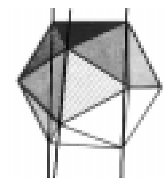

Figure 9. MAA logo, a regular icosahedron, before an error was pointed out.

The MAA treated this embarrassing discovery rather well: Grünbaum published an article in one of the MAA journals (*Mathematics Magazine*, January 1985) about this along with even more inaccurate mathematical illustrations in the literature, most of which could be explained if not excused by the difficulties of drawing in three dimensions. Doris Schattschneider, then editor of the journal, wrote an appendix to his article in which she traced some of the history of the MAA logo and its error, and she added that the new image would "become the master for all new renderings of the MAA logo." She undoubtedly meant what she said, but the gremlin turned out to be too strong to control, as the pair of images in Figure 10, taken from the cover of the *Monthly*, shows. The old logo crept back onto the cover! This was not apparently caught until sometime in 1998, but it was not until very recently that it was corrected, as Figure 11 shows.

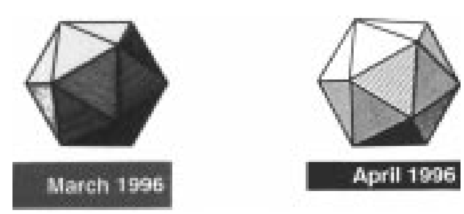

Figure 10. Reintroduction of error in MAA logo on the cover of the *American Mathematical Monthly* in 1996, before and after.

I was sorry to see that this time the logo was corrected with less fanfare. One can speculate that the modification in 1996 was made for legitimate aesthetic reasons, since

Figure 11. Correction of error in MAA logo on the cover of the *Monthly* in 2000, before and after.

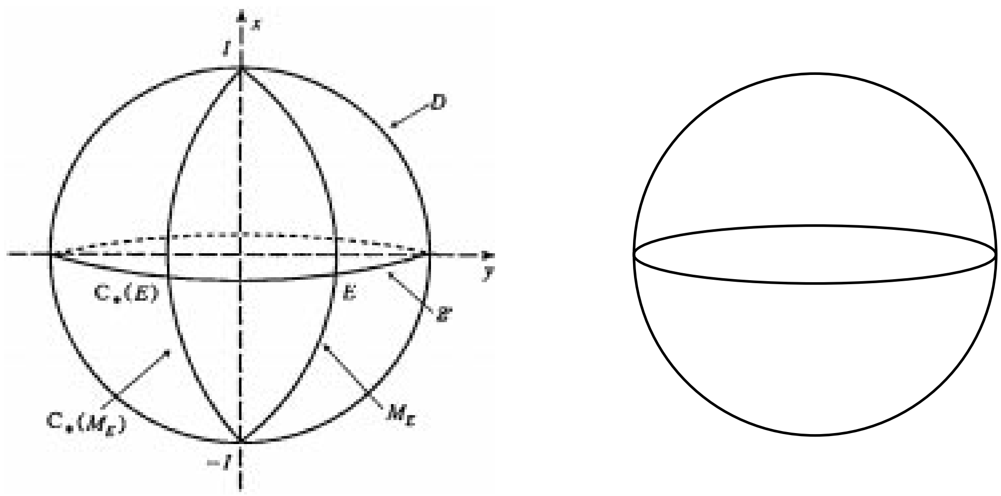

Figure 12. "Elliptical sections" of a sphere, rendered incorrectly and then correctly.

the icosahedron of early 1996 is in black, and the replacement—which is exactly the one Grünbaum complained about—is in blue, which seems to be a sort of signature color for the MAA. The earlier image is also arguably a bit too dark in the shadows, and if mathematically correct, it is poorly designed otherwise. But should it not have been a simple task for a mathematician to design a correct replacement? The good news is that now, at last, there would *seem* to be no conceivable reason for regression.

There are other ironies in this tale. Anyone paging forward from the cover of March 1996 to look at the one for April will turn past the left diagram in Figure 12, showing a sphere and several "elliptical" sections. A correct elliptical section is shown for comparison on the right. It is true that the awkward nonelliptical curves do not ruin the picture on the left, but to many they would surely be distracting. And in any event the "true" section is certainly more … well, *pleasant*.

Since this blunder is in one of the categories that Grünbaum mentions in his article, Doris Schattschneider got it exactly right in the quotation at the end of her appendix to that article: "Plus ça change, plus c'est la même chose."

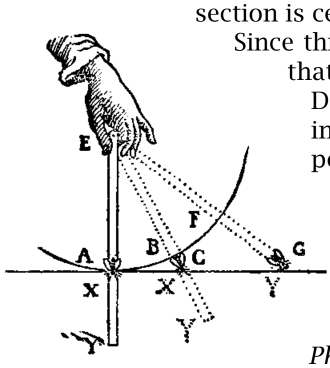

Figure 13. Picture of "flying off on a tangent", originally in Descartes's *Principles of Philosophy*.

Laziness

In one of his books on information graphics Edward Tufte includes a picture from Descartes's *Principles of Philosophy* that does a fine job of suggesting what it means to "fly off on a tangent". This picture is reproduced in Figure 13.

This might lead one to think that Descartes was exceptionally careful about his use of pictures, and indeed there is much evidence from the *Principles* as well as the technical essays accompanying the *Discourse on Method* that this was true. The third of these essays, by the way, is *La Géometrie*, in which was published for the first time a description of (x, y) coordinate systems and the equations of algebraic curves of degree greater than two. Some of the illustrations, particularly in the first essay, *La Dioptique*, are deservedly famous for their deft combination of art and mathematics. It happens that we know something, but not a great deal, about how they were produced.

The first mention of the figures is in a letter to Mersenne, where Descartes is discussing the possibility of having the *Discourse* published in Paris with Mersenne's help. One of the difficulties, he says, is that the figures are drawn by his own hand and hence are very bad. Mersenne would have to "draw the intelligence from the text" in order to interpret them for the engraver, because otherwise they would be impossible to understand. Interestingly enough, there do not seem to be extant any interesting autograph drawings by Descartes's own hand, so we cannot verify his self-deprecation. This contrasts with the situation for other mathematicians of the seventeenth century, such as Harriot or Newton, who were both careful and talented draftsmen and whose own sketches have survived in abundance.

In the end the book was published in Leiden by the relatively small press of Jan Maire, with the assistance of the elder Huygens. Late in 1635 Huygens proposed to Descartes that the figures for the book be woodcuts rather than engravings, because this would make it possible to insert them alongside the text where they are discussed rather than all on a few separate sheets at the end of the book, as was commonly done at that time. This thoughtful remark suggests that it is to Huygens that we owe what Tufte found admirable about Descartes's diagrams. When he refers to a figure that has been displayed earlier, it is repeated. This is a simple technique that could be used effectively far more often than it is, since having to turn back and forth between two pages in order to follow an argument is very annoying.

The woodcuts were eventually based on drawings done by the younger Frans van Schooten, who eventually played a large role in Descartes's life. Perhaps they were even done by van Schooten himself, who was an artist as well as a mathematician. Van Schooten also did the figures for the later *Principia Philosophiae*, and Descartes records at least once his satisfaction with the work done by him, but in spite of this his attitude towards van Schooten seems to have been a somewhat grudging gratitude.

We can deduce fairly that the figures are important to Descartes. And yet a few errors in the figures escaped him. The most interesting one is that in Figure 14, which accompanies an attempt to explain refraction in familiar terms—here the trajectory of a tennis ball with a somewhat artifical assumption about change in speed. It is clear from the text that the distance HF is supposed to be twice AH, and the fact that it is not has a qualitative and confusing effect on the diagram. It is easy to imagine Descartes's embarassment when Mersenne forwarded to him a letter from the English philosopher Hobbes, well known in that era for his well-meaning but often bumbling interest in mathematics, pointing out that the diagram contradicted the text! In his response Descartes blames the error on the printer, but this excuse does not ring true in light of what we know about his relations with van Schooten.

Descartes's exposition in the technical essays that accompanied the *Discourse* was notorious in his time for its lack of perspicacity. This was especially true of *La Géometrie*, the most innovative of the three, which played an extremely important role in the development of mathematics later in the century. Its importance, however, probably comes largely from the second Latin edition, which was translated by the same Frans van Schooten and in which van Schooten added a good deal of his own material and corrected a few of the graphical errors. It was this edition that became a popular textbook and in particular was apparently Newton's principal introduction to geometry. It is unfortunate that we do not know as much about the talented van Schooten as we would like.

False Economy

When he was only sixteen, Pascal discovered a number of new and remarkable facts about conics. He announced his discoveries in what seems to be a kind of poster entitled "Essai pour les Coniques", a large single sheet on which the results were summarized. This poster is one of the rarest of mathematical documents, extant now in exactly two copies. Space on this sheet, even though it was quite large, was severely limited, and as a result Pascal achieved what I believe to be a kind of record—on the diagram in Figure 15 are superimposed the diagrams for five separate propositions! The editors of the most recent *Pléiade* edition of Pascal's works apparently enjoyed separating this single figure into its separate parts.

I have brought up the graphical errors of Descartes and Pascal partly because of intrinsic historical interest. It might be objected that these errors are not relevant to modern times, but in fact I could just as easily have found the same faults in more recent publications. One of my reasons for using Descartes and Pascal was simply not to step on live toes. And also, I suppose, to reassure those who make similar ones that they are in the best of company.

Final Remarks

There are other obstacles to good figures as well, such as the urge simply to follow tradition or the difficulty of surmounting technical obstacles. In this article I shall not really deal with the second, but we shall see later a historically interesting example of the first, involved in an early printing of Euclid.

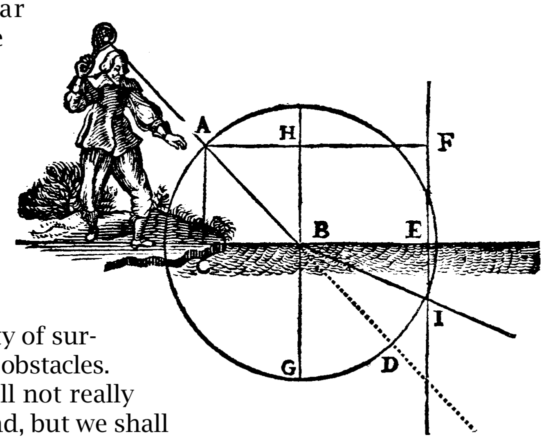

Figure 14. A diagram of Descartes that contradicted the accompanying text.

More about the History of Mathematical Illustration

It is not uninteresting to inquire about the origins of many of our current practices in mathematical graphics. Many are very old.

The Oldest Figure

The very earliest mathematical pictures we know about are on Babylonian clay tablets. Unfortunately for us, in Euclid's time mathematics was written on papyrus, which does not weather as well as clay. One of the very earliest fragments of Euclid that has been found, and therefore one of the earliest mathematical pictures involved in mathematical reasoning, is a small piece of papyrus from the famous Oxyrhynchus rubbish piles, dated about 300 A.D. It contains both the statement and the figure of II.5 from the *Elements* and is interesting for a number of reasons. I reproduce in Figure 16 a copy of the photograph that was published in Arthur Hunt's survey article of 1914.

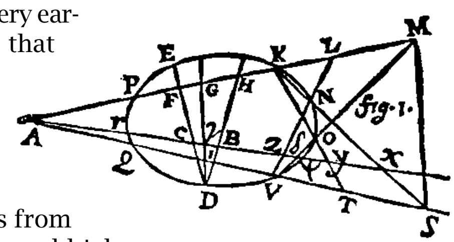

Figure 15. False economy: superimposed diagrams by Pascal.

I will not recite here exactly what Proposition II.5 is about except to say that it is essentially the geometrical equivalent of the algebraic identity

$$(a-b)(a+b) = a^2 - b^2.$$

The figure is distinguished from all other Euclid illustrations I have seen in that it is unlabeled. Of course, this can hardly be taken as evidence for a common practice in its time, much less in Euclid's, but I do believe that the convention of tying text and figures together with labels is often ill advised, and it would be interesting to know how it developed. This convention is rarely used nowadays in spoken discourse, for example, unless in an excessively formal exposition. The normal thing to do in spoken discourse is to take advantage of

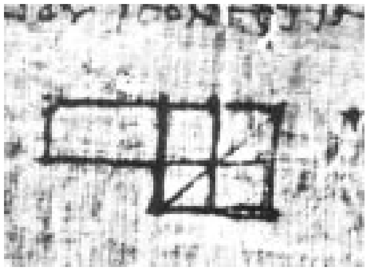

Figure 16. Photograph of a picture for II.5 of Euclid produced about 300 A.D.

the extra dimension of time to build a figure as the discourse proceeds, erasing and adding items as it goes on. Surely this was how the Greeks explained mathematics to each other as well.

The Greek tradition we mostly follow is not the only one that has come down to us. What I know of the oldest Chinese tradition leads me to think it is curiously different. I have seen reproductions of what are presumably very early illustrations to accompany Pythagoras's Theorem in which the most important text tells how to color the woodblock diagram, which is itself in black and white. It has the word meaning "red", for example, on the part of the figure that is supposed to be colored red. I have been unable to learn much about this technique.

The First Printed Mathematics Book

The first (printed) edition of Euclid's *Elements of Geometry* (in Latin, of course) was published in 1482 in Venice, which at that time was where the technology of printing was most advanced. It came from the press of the German printer Erhard Ratdolt, who had moved there from Augsburg. Ratdolt specialized in technical publishing and had apparently been thinking for a long time of bringing out an edition of Euclid, as he says in his dedicatory preface. This edition is of technical interest in the history of printing because it involved a far larger number of illustrations than any other book of its time. Ratdolt tells us that he delayed attempting to publish Euclid just because he was much afraid of botching the figures. He also tells us that he worked a long time to invent a way to reproduce the components of geometrical figures as easily as normal text, but it is not known exactly what he meant. The best guess is that he made up his figures with a small number of types of simple components by inserting it in a matrix like the one used to hold text. Without such a technique, the sheer quantity of figures would have been extremely expensive.

Copies of this edition are not uncommon, as these things go. There might be as many as two hundred copies of Ratdolt's edition still extant, from an estimated run of about five hundred. This is not something for mathematicians to be proud of, since it is often pointed out in the extensive literature on early books that the books remaining from the fifteenth century are surely those that have been preserved precisely because they were not read!

On the whole, the figures I have seen in this edition are not too different from what we are now accustomed to. There are a few curious exceptions, however. The figure that is technically the most difficult is that accompanying Proposition XII.17. This is a lemma for XII.18, which asserts that if the linear dimensions of a sphere are multiplied by a scalar c, its volume is multiplied by c^3. The analogous result is known at this point for certain polyhedra, and the proof of XII.18 (which is among several proofs in Euclid amounting to a kind of integration) requires that between two distinct concentric spheres one can find such a polyhedron. This is what XII.17 is all about. It is the only place in Euclid where a three-dimensional curved surface has to be drawn, and in a complicated context. The figure from Ratdolt's edition is reproduced as Figure 17. One's first impression upon seeing Ratdolt's figure is surely one of confusion. In attempting to explain to myself its strange quality, I thought that he was simply handicapped by technology—that he was simply unable to design a more realistic figure because his drawing tool kit was so meager. But recently I have been able to peruse a manuscript from 1398, one almost certainly in the same family as that used by Ratdolt, and the figure there is almost exactly the same as his. So his excuse would surely be that he was just following tradition and had no grounds for doing anything else. It would be interesting to know if he consulted a mathematician in the course of producing his edition, but my guess would be that he did not.

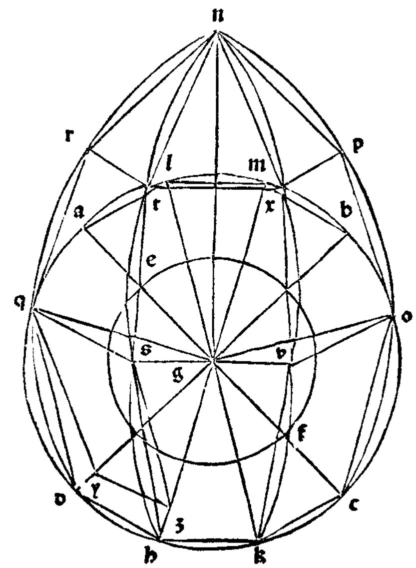

Figure 17. A three-dimensional diagram from Ratdolt's edition of Euclid, accompanying XII.17 of Euclid.

There were other traditions available to him, at least in principle. In the definitive Greek edition of J. Heiberg (1883), where most of the diagrams have a manuscript tradition behind them, the figure accompanying XII.17 in Heiberg is Figure 18. It is a much better figure than Ratdolt's, although not actually much more difficult to produce. For example, the arches seem to be made up of two

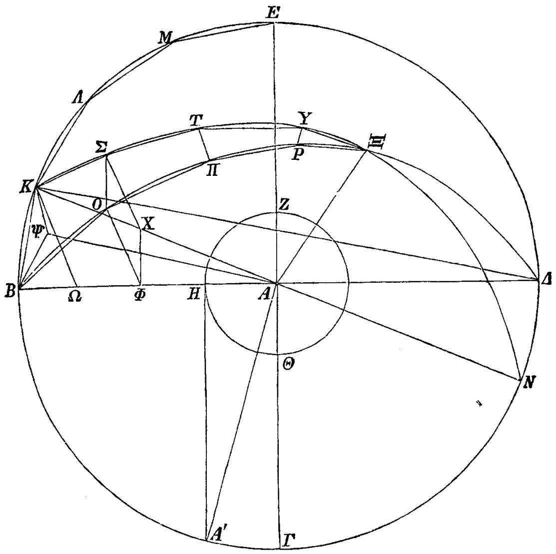

Figure 18. The same diagram as in Figure 17 but from Heiberg's edition.

circular arcs patched together, which Ratdolt might also have easily done.

It was inevitable that Ratdolt's pioneering edition of Euclid would have serious flaws, which a slightly later generation pounced on happily. Most of the flaws arose from his medieval source rather than any mistakes of his own. Nonetheless, those were truly astonishing times: how many printers can one find in the twenty-first century who are fluent in Latin and competent to produce mathematics books without professional advice? Who of them would lie awake at night thinking about how to make better diagrams so that (to translate Ratdolt) mathematical books might henceforward flow forth?

It is probably in the light of the medieval distinction between *ratio* and *intelligentia* that one should interpret Ratdolt's use of the word *intelligi* in his preface, where he says that without pictures mathematics cannot be understood.

Higher Criticism

In the eyes of some, much of my criticism so far amounts to no more than nit-picking. But I claim that small inaccuracies are more important than they might appear at first; the frustration involved in interpreting an inaccurate or badly drawn diagram might very well inhibit a large percentage of students, among others, from continuing on, whereas they would bring a mathematician only to pause. Even small errors in a drawing can be confusing, frustrating, annoying, and distracting—and an accumulation of them can be deadly. I am not completely sure why the standards for graphics are so much lower than those for text. But how many cases does one know of where an editor demanded that a picture be at least correctly drawn? Or threatened to withhold publication until extensive graphical improvement was made? I can think of several extremely recent research papers in which such graphical criticism would be legitimate.

Of course, I have dodged one important issue. It is not easy to produce good illustrations for mathematical exposition. Doing so shifts a certain amount of work involved in comprehension from the reader to the author, and not all authors are in favor of this. But it will often have the added advantage of increasing the number of readers, sometimes greatly.

What I have said here is hardly new. But I am sure that far more effort is spent extolling the virtues of understanding mathematics than in actually conveying that understanding. It is simpler, and certainly more objective, to verify a chain of reasoning step by step than it is to fit that chain into a larger context in which any overall comprehension will take place. Most of the time mathematicians probably just pretend that an accumulation of examples will lead to an overall perspective as well as some skill at assembling a chain of reasoning in the first place. And so it might, but only for a notoriously small percentage of human beings. For the rest, pictures are their best hope.

About the Images

Tufte's references to Byrne and the Chinese figure are in *Envisioning Information*, in a section on how to use color to convey information. Byrne's figures (Figures 6 and 7) are taken from his 1847 edition of Euclid. Thanks to a cooperative effort at the University of British Columbia, the whole edition has been photographed and placed on the Internet at `http://www.math.ubc.ca/people/faculty/cass/Euclid/`.

The geometry image in Figure 8 was redrawn by me from one in the "Geometry Resource Package" of September 1999.

All of the MAA images (Figures 9–12) are taken from the *American Mathematical Monthly*: March 1996, April 1996, January 2000, February 2000.

The image from Descartes in Figure 13 is from the 1902 edition of *Oeuvres de Descartes*, edited by Charles Adama and Paul Tannery. The other image from Descartes (Figure 14) has been taken from a first edition of *La Géometrie* now in the Thomas Fisher Rare Book Library at the University of Toronto, with the assistance of that library.

Pascal's drawing shown in Figure 15 was scanned from a facsimile of his poster contained in the 1906 edition of his collected works. The original is located at the Bibliothèque Nationale in Paris.

The papyrus in Figure 16 is Plate X from Arthur S. Hunt's survey article in the *Journal of Egyptian Archaeology*, volume I, 1914. The text is transliterated from an image in volume I (papyrus

29) of *The Oxyrhynchus Papyri*, a series begun by Grenfell and Hunt in 1898 and continuing through to the present. According to a recent catalogue, the papyrus fragment itself is currently located in a collection at the University of Pennsylvania.

The image in Figure 17 from Ratdolt's 1482 edition of Euclid is also from the Thomas Fisher Library.

The image in Figure 18 from Heiberg's definitive Greek version of Euclid is from a copy of the original 1883 edition.

The Japanese proverb was produced by my wife, Yuko Shibata.

All the rest of the images were produced by me directly in PostScript. The PostScript sequence for Pythagoras's Theorem was taken from a truly animated version constructed in the programming language Java by Jim Morey, who was at that time a graduate student at the University of British Columbia. This and other Java animations of Pythagoras's Theorem can be seen at the UBC Euclid site mentioned above.