A Treasure Trove of Geometry and Analysis: The Hyperquadric

François Treves

The mathematical landscape is peppered with simple-looking objects whose analysis uncovers structures of surprising richness. This article takes a look at one of them, the classical *hyperquadric*.

Transfer to $\mathbb{R}^4$ the standard paraboloid of revolution (around the vertical axis) in three-dimensional space $\mathbb{R}^3$ (coordinates x, y, t); place it in the hyperplane on which the fourth coordinate, s, vanishes; and from there let it slide along the s-axis, all the way from $s = -\infty$ to $s = +\infty$. This is our hyperquadric $\mathcal{Q}$, defined in $\mathbb{R}^4$ by the equation $t = x^2 + y^2$. We shall denote by $\mathcal{Q}^+$ the convex side of $\mathcal{Q}$, i.e., the set $t > x^2 + y^2$, and by $\mathcal{Q}^-$ the concave side, $t < x^2 + y^2$. We look at $\mathcal{Q}$ from the viewpoint of complex analysis, identifying $\mathbb{R}^4$ to $\mathbb{C}^2$ with the coordinates $z = x + iy$, $w = s + it$; the equation defining $\mathcal{Q}$ becomes $\mathfrak{Im}\, w = |z|^2$.

Complex Equivalence

Circa 1900 (see [15]) Henri Poincaré began to build the theory of functions of several complex variables (SCV in what follows). One of the first questions he confronted was that of the local equivalence under *biholomorphisms* (i.e., one-to-one, onto, holomorphic maps whose inverses are holomorphic) of *real* hypersurfaces in complex space. In more modern language, the problem is that of describing the biholomorphic invariants of a real hypersurface in $\mathbb{C}^n$, $n \geq 2$.

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For simplicity suppose $n = 2$. Let the hypersurface S be defined near one of its points, taken to be the origin O, by a real equation $\rho(x, y, s, t) = 0$. We can approach the equivalence problem as we would in the one-variable case: is there a holomorphic change of variables, $z = f(z', w')$ and $w = g(z', w')$, that reduces the Taylor expansion of the defining equation to its simplest possible expression? When $n = 1$ (in which case there is no w, only z) and ρ is real-analytic (otherwise one must deal with formal power series), we can determine recursively the Taylor expansion of $f(z')$ (which does not contain any power of $\bar{z}'$) so as to reduce the equation of S near O to $y' = 0$. In other words, we can transform an arc of the real-analytic curve S containing O into an open interval of the real axis. If there are two or more complex variables, an attempt of this kind runs immediately into trouble: there are terms (or groups of terms) in the Taylor expansion of ρ that simply refuse to be eliminated.

The simplest test cases are the unit sphere $\mathbb{S}^3$ and the hyperquadric $\mathcal{Q}$. The same approach as in the one-variable situation to mapping the hyperquadric $\mathcal{Q}$ in a neighborhood of the origin succeeds only exceptionally: it blatantly fails when the target is the hyperplane $t = \mathfrak{Im}\, w = 0$; it succeeds when the target hypersurface is the 3-sphere $\mathbb{S}^3$: $|z|^2 + |w|^2 = 1$ (near, say, the south pole). Actually the biholomorphism of the open unit ball $\mathfrak{B}$ onto $\mathcal{Q}^+$ given by

$$(1) \qquad z = \frac{z'}{w' - i}, \quad w = \frac{1 - iw'}{w' - i},$$

maps $\mathbb{S}^3 \setminus \{(0, i)\}$ onto $\mathcal{Q}$ and extends as a biholomorphism of an open neighborhood of $\mathbb{S}^3 \setminus \{(0, i)\}$ onto one of $\mathcal{Q}$.

Of course, there are smooth one-to-one mappings of $\mathbb{S}^3 \setminus \{(0,i)\}$ onto the hyperplane $\mathbb{R}^3$, for example the stereographic projection, but these maps cannot be extended holomorphically to neighborhoods of $\mathbb{S}^3 \setminus \{(0,i)\}$ in $\mathbb{C}^2$. Holomorphy carries with it a number of limitations.

The Most Basic Invariant: The Levi Form

The first obstruction to transforming biholomorphically the hyperquadric $\mathcal{Q}$ into the hyperplane $t = 0$ lies in the "complex curvature" of $\mathcal{Q}$. This is a notion subtler than the classical curvature in Riemannian geometry. But just as the latter can be described in terms of the eigenvalues of the real Hessian matrix (when looking at the hypersurface as a local graph over its tangent space), the "complex curvature" can be described in terms of the eigenvalues of the complex Hessian.

For precision some notation is needed, first of all for the Cauchy-Riemann vector fields in $\mathbb{C}^n$ ($n \geq 2$) and their complex conjugates,

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$
$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$

for $j = 1, \ldots, n$. A *holomorphic function* of $z = (z_1, \ldots, z_n)$ in a *domain* (i.e., a connected open subset) $\mathcal{D}$ of $\mathbb{C}^n$ is simply a function $h \in C^1(\mathcal{D})$ such that $\frac{\partial h}{\partial \bar{z}_j} \equiv 0$ for every $j = 1, \ldots, n$. Let $S \subset \mathcal{D}$ be a smooth hypersurface: S is defined by an equation $\rho(x, y) = \rho(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$, where $\rho \in C^\infty(\mathcal{D})$ is real-valued and the differential $d\rho$ of ρ does not vanish at any point of S. The vectors of the kind

$$L = \sum_{j=1}^{n} a_j \frac{\partial}{\partial \bar{z}_j}, \quad a_j \in \mathbb{C},$$

that are tangent to the hypersurface S at one of its points, $z_\circ$, form an $(n-1)$-dimensional complex linear space, often denoted by $T^{0,1}_{z_\circ}(S)$, defined by the condition $(L\rho)(z_\circ) = 0$. The smooth vector fields that lie in $T^{0,1}_z(S)$ at every point $z \in S$ are called the *tangential Cauchy-Riemann vector fields* on S.

Definition 1. *The Levi form of S at $z_\circ$ is the quadratic form*

$$T^{0,1}_{z_\circ}(S) \ni Z = \sum_{j=1}^{n} a_j \frac{\partial}{\partial \bar{z}_j}$$
$$\to \mathcal{L}_{z_\circ}(Z) = (\partial\bar{\partial}\rho)_{z_\circ}(Z) = \sum_{j,k=1}^{n} a_j \bar{a}_k \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_k}(z_\circ).$$

This definition depends on the choice of the complex coordinates z_j and of the "defining function" ρ. A change of the coordinates replaces the *Levi matrix*, i.e., the self-adjoint matrix associated to the Levi form $\mathcal{L}_{z_\circ}$ by a conjugate self-adjoint matrix. Multiplication of ρ by a smooth real-valued, nowhere vanishing function results in multiplying $\mathcal{L}_{z_\circ}$ by a nonzero real number. In all rigor one ought to define the Levi matrix as a conjugacy class of self-adjoint matrices, defined up to multiplication by a nonzero (or positive, if one wishes to distinguish the sides of S) real number.

An equivalent, often used, definition of the Levi matrix is the following. In an open neighborhood U of $z_\circ \in S$ there is a simultaneous basis of the (complexified) tangent space to S consisting of $n-1$ smooth vector fields $L_1, \ldots, L_{n-1}$ spanning $T^{0,1}_z(S)$ at every point $z \in S \cap U$, of their complex conjugates $\bar{L}_1, \ldots, \bar{L}_{n-1}$ ($\bar{L}_j$ is obtained by replacing each coefficient in L_j by its complex conjugate), and of one real vector field ϑ. It follows that to each pair $j, k = 1, \ldots, n-1$, there is a complex number c_{jk} such that, *at the point* $z_\circ$,

(2)
$$\frac{1}{2i}[L_j, \bar{L}_k] = \frac{1}{2i}(L_j\bar{L}_k - \bar{L}_k L_j)$$
$$\equiv c_{jk}\vartheta \text{ mod Span}(L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}).$$

Up to similarity and scalar multiplication the *self-adjoint* $(n-1) \times (n-1)$ matrix $\{c_{jk}\}$ is the same as the matrix associated to the quadratic form $\mathcal{L}_{z_\circ}$.

Easily verified fact. *A biholomorphism φ of an open neighborhood U of $z_\circ \in S$ onto an open neighborhood $V \subset \mathbb{C}^n$ of $\varphi(z_\circ)$ transforms the Levi form of S at $z_\circ$ into the Levi form of the hypersurface $\varphi(U \cap S)$ at $\varphi(z_\circ)$.*

When $S = \mathcal{Q}$ (and thus $n = 2$), it is customary to take $L_1 = \frac{\partial}{\partial \bar{z}} - 2iz\frac{\partial}{\partial \bar{w}}$ and

$$\vartheta = \frac{\partial}{\partial s} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}} = \frac{1}{2i}[L_1, \bar{L}_1];$$

when $S = \mathbb{S}^3$, $L_1 = w\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial \bar{w}}$ and

$$\vartheta = \frac{1}{2i}[L_1, \bar{L}_1]$$
$$= \frac{1}{2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{1}{2}\left(s\frac{\partial}{\partial t} - t\frac{\partial}{\partial s}\right).$$

For both $\mathcal{Q}$ and $\mathbb{S}^3$, $\mathcal{L}(Z) = 1$ at every point. This shows that no biholomorphism of a ball $\mathcal{B}$ with center at a point of $\mathcal{Q}$ onto an open subset of $\mathbb{C}^2$ can map $\mathcal{Q} \cap \mathcal{B}$ into a hyperplane, since the Levi form of a hyperplane (the zero set of a linear function) vanishes at every point of the hyperplane.

Pseudoconvexity and the Continuation of Holomorphic Functions

The continuation properties of analytic functions of two or more complex variables differ radically from those of analytic functions in the plane. That much has been known since the early stages of SCV theory, first of all through the *Hartog phenomenon*: every function, defined and holomorphic in a domain of $\mathbb{C}^n$ ($n \geq 2$) whose complement is bounded, extends as an entire holomorphic function in the whole of $\mathbb{C}^n$. We are now going to encounter new, and finer, manifestations of this difference through the spectral properties of the Levi matrix.

Take $S = \partial\Omega$, the boundary of an open set $\Omega \subset \mathbb{C}^n$ defined by the inequality $\rho < 0$. Suppose ρ is twice continuously differentiable and $d\rho(z) \neq 0$ for every $z \in \partial\Omega$. The set Ω is said to be *strongly pseudoconvex* at a point $z_\circ \in \partial\Omega$ if the Levi form at $z_\circ$, $\mathcal{L}_{z_\circ}$, is *positive-definite*. The selection of the sign of ρ allows us to speak of the sign of the eigenvalues of $\mathcal{L}_{z_\circ}$. This is very much like looking at the convex side of a "convex" surface—or at the opposite side, which in the complex set-up is called *pseudoconcave*. And indeed it is easily checked that if Ω is strongly pseudoconvex at $z_\circ \in \partial\Omega$, there is a local change of complex variables such that, in the new coordinates, $\partial\Omega$ is strictly convex (in the real sense) in a neighborhood of $z_\circ$ with convex side Ω. We have seen that the hyperquadric $\mathcal{Q}$ as well as the unit sphere $\mathbb{S}^3$ are strongly pseudoconvex. In a sense they are the prototypes of all strongly pseudoconvex hypersurfaces (which does not mean that every such hypersurface is locally equivalent to the sphere—see below).

The concept of a strongly pseudoconvex domain can be weakened to that of a *weakly pseudoconvex* domain, in which the Levi form at every point of its (reasonably smooth) boundary is required to be merely positive semidefinite, and more generally to the definition of a *pseudoconvex* domain as the union of an increasing sequence of strongly pseudoconvex subdomains (without any boundary assumption). Of course, these concepts, weak or strong pseudoconvexity, as well as pseudoconvexity, are biholomorphic invariants.

A pioneering observation of E. E. Levi (see [12]) was that if a domain $\Omega \subset \mathbb{C}^n$ is smoothly bounded and strongly pseudoconvex at every point of its boundary, then Ω is *holomorphically convex*. The latter means that Ω is the union of an increasing sequence of compact subsets K_n endowed with the following property: to each $z \in \Omega \setminus K_n$ there is a holomorphic function h in Ω such that $|h(z)| > \max_{K_n}|h|$. Holomorphic convexity is also equivalent to the fact that Ω is a *domain of holomorphy*: there exists a holomorphic function f in Ω that cannot be extended holomorphically across any portion of the boundary $\partial\Omega$. One of the triumphs of SCV theory was the proof by H. Grauert (1958), based on the sheaf-theoretical foundations laid out earlier by K. Oka, H. Cartan, and J.-P. Serre, that *domain of holomorphy = pseudoconvex domain*. In $\mathbb{C}^1$ every domain is a domain of holomorphy.

In the case of the hyperquadric $\mathcal{Q}$ the (non)extension properties are not difficult to check:

i) For suitably chosen complex numbers a and b, the vanishing of $\Re e\,\lambda$, where

$$\lambda(z, w) = a(z - z_\circ) + b(w - w_\circ),$$

defines the tangent hyperplane to $\mathcal{Q}$ at $(z_\circ, w_\circ)$. It follows that λ does not vanish at any point of $\mathcal{Q}^+$, as $\mathcal{Q}^+$ lies entirely on one side of any hyperplane tangent to $\mathcal{Q}$. But $1/\lambda$ cannot be continued analytically across $\mathcal{Q}$ in a full neighborhood of $(z_\circ, w_\circ)$.

ii) Any holomorphic function h in the intersection of $\mathcal{Q}^-$ with an open ball $\mathcal{B}$ centered at the origin O can be extended as a holomorphic function $\tilde{h}$ in a full neighborhood of O. Indeed the hyperquadric $\mathcal{Q}_\varepsilon$ defined by the equation $t = |z|^2 - \varepsilon$, $\varepsilon > 0$, lies inside $\mathcal{Q}^-$; the restriction of h to any "horizontal" circle $|z| = r$ and $w = w_\circ$ contained in $\mathcal{B} \cap \mathcal{Q}^-$ (thus $|w_\circ|^2 + r^2$ must be suitably small) can be extended analytically to the interior of this circle in the plane $w = w_\circ$, thanks to the fact that the "negative" Fourier coefficients of h are all equal to zero. As $w_\circ$ ranges over a neighborhood of O, these extensions make up $\tilde{h}$.

Property (ii) can be generalized. If the domain Ω is defined by a real equation $\rho < 0$ with $\rho \in C^2$ and $d\rho$ nowhere vanishing on the boundary $\partial\Omega$ and if the Levi form at a point $z_\circ \in \partial\Omega$ has at least one eigenvalue < 0, then every holomorphic function on the side of $\partial\Omega$ opposite to Ω extends analytically to an open ball centered at $z_\circ$.

CR Functions, Distributions, and Hyperfunctions

As shown in [13] a more precise version of the preceding result can be proved. Rather than looking at holomorphic functions defined in the side opposite to Ω, we must look at CR functions on $\partial\Omega$.

Definition 2. *A function f defined and of class C^1 in an open subset U of S is called a Cauchy-Riemann (CR) function if it satisfies the tangential Cauchy-Riemann equations, i.e., if $Lf \equiv 0$ in U for every tangential Cauchy-Riemann vector field L on S.*

The restriction to $U \subset S$ of any holomorphic function in an open subset of $\mathbb{C}^n$ containing U is a CR function, but there are CR functions that are not of this kind: for instance,

$$\exp(-(w/i)^{-1/2}) = \exp\left(-\frac{1}{\sqrt{x^2 + y^2 - is}}\right)$$

if $\sqrt{\cdot}$ denotes the main branch of the square root.

Theorem 1. *Let $S = \{z \in \mathbb{C}^n;\ \rho(z) = 0\}$ with $\rho \in C^2$ real-valued and* grad ρ *nowhere vanishing on S. If the Levi form at a point $z_\circ \in S$ has at least one eigenvalue < 0, every CR function f in a neighborhood of $z_\circ$ in S is the boundary value of a holomorphic function in an open set*

$\{z;\ \rho(z) < 0,\ |z - z_\circ| < \varepsilon\}$ *for* $\varepsilon > 0$ *sufficiently small.*

It follows from the approximation formula in [4] that *for a function f on a real hypersurface in* $\mathbb{C}^n$ *to be CR, it is necessary and sufficient that locally f be the uniform limit of a sequence of holomorphic polynomials* P_ν. The hypothesis on the Levi form in Theorem 1 ensures that in a suitably small compact set $K = \{z;\ |z - z_\circ| \le \varepsilon,\ \rho(z) \le 0\}$ the absolute values $|P_\nu(z)|$ reach their maxima on the "edge" $K \cap S$ and therefore converge to a holomorphic function in the interior $\{z;\ |z - z_\circ| < \varepsilon,\ \rho(z) < 0\}$, thereby extending f. That the functions defined and holomorphic on the side $\rho > 0$ of S extend to the opposite side, Ω, is seen by applying Theorem 1 to their restrictions to hypersurfaces parallel and arbitrarily close to S near $z_\circ$.

CR distributions on a real C^∞ hypersurface S of complex space can also be defined. It suffices to interpret the tangential Cauchy-Riemann equations in the weak sense. Locally CR distributions are sums of derivatives of CR functions.

Even *CR hyperfunctions* can be defined. Near a point $z_\circ \in S$, a CR hyperfunction is an equivalence classes of pairs (f, g) consisting of a holomorphic function f defined on one side of S and a holomorphic function g on the opposite side. The equivalence relation is very simple: $(f, g) \approx (f_1, g_1)$ if there is a holomorphic function h in a full neighborhood of $z_\circ$ (in the ambient complex space) that is equal to $f - f_1$ and to $g - g_1$ in their respective sides. The hyperfunction corresponding to (f, g) must be thought of as the "jump" from f to g—more precisely, as the difference between the boundary value of f and that of g. And indeed these boundary values can be defined rigorously by integration on hypersurfaces parallel to S and converging to S. If S satisfies the hypothesis in Theorem 1, each CR hyperfunction can be represented by a pair $(f, 0)$ with f holomorphic in the side $\rho < 0$. This is the convex side $\mathcal{Q}^+$ for $\mathcal{Q}$: CR hyperfunctions on $\mathcal{Q}$ are the "boundary values" of holomorphic functions in $\mathcal{Q}^+$. Among them CR distributions are distinguished as the boundary values of the holomorphic functions whose absolute values are bounded by some power $(1/\mathrm{dist}(z, \mathcal{Q}))^k$ as $z \in \mathcal{Q}^+$ approaches an arbitrary point $z_\circ \in \mathcal{Q}$.

Polynomial approximation is also valid for CR distributions (see [4]), as well as for CR hyperfunctions (the latter proved by P. D. Cordaro in 1999).

CR Manifolds, CR Invariance

The tangential Cauchy-Riemann vector fields define what is nowadays called a *CR structure* on the hypersurface S. Actually the concept of a *CR manifold* $\mathcal{M}$ can be defined independently of any embedding in Euclidean space $\mathbb{C}^n$ or even in a complex-analytic manifold. On a sufficiently small open subset U of $\mathcal{M}$ the structure is defined by ν smooth vector fields with complex coefficients, $L_1, \dots, L_\nu$ (a local basis of the tangential Cauchy-Riemann vector fields), that have the following two properties, valid at every point of U:

i) the vector fields $L_1, \dots, L_\nu$ and their complex conjugates, $\bar{L}_1, \dots, \bar{L}_\nu$, are linearly independent;
ii) the commutation brackets $L_j L_k - L_k L_j$ are linear combinations of the $L_1, \dots, L_\nu$.

The local definitions form a coherent system (i.e., they define a vector bundle): if $(U', L'_1, \dots, L'_{\nu'})$ is another "local chart" such that $U \cap U' \neq \emptyset$, then $\nu = \nu'$ and

$$L'_k = \sum_{j=1}^{\nu} g_k^j L_j, \quad k = 1, \dots, \nu,$$

for some invertible $\nu \times \nu$ complex matrix $(g_k^j(z))_{1 \le j,k \le \nu}$ whose entries are smooth, complex-valued functions in $U \cap U'$. Here we are solely interested in CR manifolds of the hypersurface type, meaning that $\dim \mathcal{M} = 2\nu + 1$. In this case there is a basis of the complexified tangent space to $\mathcal{M}$ at an arbitrary point z of U (possibly contracted) consisting of $L_1, \dots, L_\nu, \bar{L}_1, \dots, \bar{L}_\nu$, and of a single additional vector field ϑ. One can then define the Levi form (or rather the conjugacy class of any one of its nonzero scalar multiples) by (2), just as in the embedded case. A (connected) CR manifold $\mathcal{M}$ is strongly pseudoconvex if at every point of $\mathcal{M}$ all the eigenvalues of the Levi form are different from zero and have the same sign (in the nonembedded case there are no "sides" to distinguish).

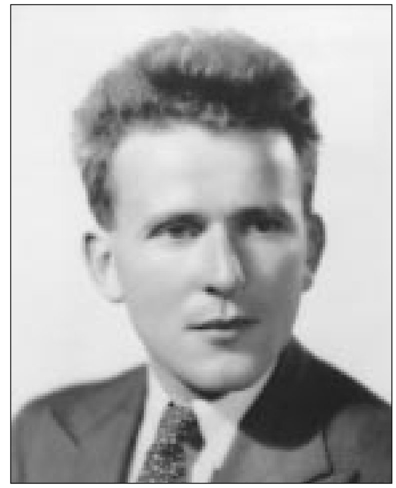
A theorem of L. Boutet de Monvel states that every compact, strongly pseudoconvex CR manifold of hypersurface type, of dimension 5 or greater, can be embedded in complex space $\mathbb{C}^N$ for suitably large N (where it inherits its CR structure from the ambient complex structure). There are strongly pseudoconvex 3-dimensional CR manifolds that are not so embeddable. The simplest example is due to H. Rossi (1965): the unit sphere $\mathbb{S}^3$ on which the tangential Cauchy-Riemann vector field is taken to be

$$w \frac{\partial}{\partial \bar{z}} - z \frac{\partial}{\partial \bar{w}} + \varepsilon \left(\bar{w} \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial w} \right), \quad 0 < \varepsilon < 1.$$

This CR structure is locally embeddable (since this is true of all *real-analytic* CR structures). In 1972 L. Nirenberg found strongly pseudoconvex 3-dimensional CR manifolds that are not even locally embeddable. In 1982 M. Kuranishi proved that all strongly pseudoconvex CR manifolds of dimension at least 9 are locally embeddable; in 1987 T. Akahori was able to extend this result to the seventh dimension. Whether all strongly pseudoconvex 5-dimensional CR manifolds are

E. E. Levi introduced the concept of pseudoconvexity.

É. Cartan solved the equivalence problem in two complex variables.



H. Lewy gave the first example of a nonsolvable linear PDE.

locally embeddable is at present an unanswered question.

With the "CR category" introduced, it is natural to raise the question of which CR manifolds are "equivalent", i.e., CR isomorphic. A *CR isomorphism* $\varphi : \mathcal{M} \to \mathcal{M}'$ between CR manifolds is a diffeomorphism (i.e., φ is one-to-one onto and C^∞, and its inverse is C^∞) transforming every Cauchy-Riemann vector field on $\mathcal{M}$ into one on $\mathcal{M}'$. Special cases are the restrictions on a hypersurface $S \subset \mathbb{C}^n$ of biholomorphisms of neighborhoods of S. The Levi form is a CR invariant, not just a biholomorphic invariant.

The equivalence problem for strictly pseudoconvex 3-dimensional manifolds was completely solved in [5]. É. Cartan reasoned within the framework of his exterior differential calculus and sought a geometric/group-theoretical interpretation of the CR structure of a 3-dimensional manifold S. Such a structure can be defined locally by two linearly independent differential forms, α and θ, such that $\alpha \wedge \bar\alpha \wedge \theta \neq 0$ and $\theta = \bar\theta$ (the forms α and θ annihilate the tangential Cauchy-Riemann vector fields). The real-valued, smooth function g in the exterior derivative

$$d\theta = 2ig\alpha \wedge \bar\alpha + A\alpha \wedge \theta + \bar A\bar\alpha \wedge \theta$$

is another embodiment of the Levi form. Under the assumption that $g \neq 0$ at every point of S, the exterior product $\theta \wedge d\theta$ spans $\Lambda^3 T^* S$; θ is a *contact form*. Replacing θ by $g^{-1}\theta$ allows us to posit that $d\theta \equiv 2i\alpha \wedge \bar\alpha \bmod \theta$.

On the hyperquadric $\mathcal{Q}$ it is natural to take $\alpha = dz$ and $\theta = ds - i(\bar z\, dz - z\, d\bar z)$. If $L = \frac{\partial}{\partial \bar z} - iz\frac{\partial}{\partial s}$, then $\langle \theta, L\rangle = \langle \theta, \bar L\rangle$ vanishes identically. The real tangent vectors to $\mathcal{Q}$ annihilated by θ form a plane distribution, spanned by $X = \frac{1}{2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial s}$ and $Y = \frac{1}{2}\frac{\partial}{\partial y} - x\frac{\partial}{\partial s}$. This plane distribution is not integrable, since $[X, Y] = \frac{\partial}{\partial s}$. Each plane carries a natural complex structure, defined by the one-form α (i.e., by the coordinate z).

Cartan's approach is to associate to the CR structure of S the fiber bundle with base S and fiber the manifold of all forms $\Omega_1 = \lambda(\alpha + \mu\theta)$ and $\Theta = |\lambda|^2\theta$ satisfying $\Omega_1 \wedge \overline{\Omega}_1 \wedge \Theta \neq 0$ and $d\Theta \equiv 2i\Omega_1 \wedge \overline{\Omega}_1 \bmod \Theta$. Thus the real coordinates in the fibers are $\Re e\,\lambda$, $\Im m\,\lambda$, $\Re e\,\mu$, $\Im m\,\mu$;Cartan introduces an additional real coordinate ρ, and thus the resulting bundle $\mathcal{B}$ has real dimension 8. The forms α and θ are lifted to $\mathcal{B}$, and three more one-forms on $\mathcal{B}$—denoted $\Omega_2, \Omega_3, \Omega_4$—are adjoined to Ω_1 and Θ to define a "connection" on $\mathcal{B}$. For instance, Ω_2 is defined by the requirement that $d\Theta = 2i\Omega_1 \wedge \overline{\Omega}_1 + \Theta \wedge (\Omega_2 + \overline{\Omega}_2)$; and $d\Omega_3 = -\Omega_1 \wedge \Omega_4 - \overline{\Omega}_2 \wedge \Omega_3 - R(z)\Theta \wedge \overline{\Omega}_1$, where the "curvature" $R(z)$ is a real-valued smooth function in S. The essential property of the Cartan connection is that the CR automorphisms of the base S are in one-to-one correspondence through lifting with the fiber-preserving automorphisms of $\mathcal{B}$ that preserve each one of the forms Θ and Ω_j for $j = 1, 2, 3, 4$. It is not too difficult to prove then that a 3-dimensional strongly pseudoconvex CR manifold is locally CR equivalent to the hyperquadric $\mathcal{Q}$ if and only if R vanishes identically.

In the 1960s J. Moser solved the same problem by a completely different method. These results were extended (with the appropriate modifications) to three or more complex variables by N. Tanaka (1962, 1978) and by S. Chern and J. Moser (1974)—always under the hypothesis that the Levi matrix is nowhere singular. For more details as well as bibliographical information on this topic, see [10].

The CR equivalence when the Levi form is merely semidefinite is currently the object of active investigation. However, it would be unrealistic to expect sweeping results, freed from severely restrictive hypotheses on the location and nature of the degeneracy of the Levi form.

The Hyperquadric as a Realization of the Heisenberg Group

The boundary of a half-space—a hyperplane—carries an abelian group law defined by the addition of the vectors issuing from a common origin. Remarkably, the hyperquadric $\mathcal{Q}$ also carries a natural group structure, but one that is not commutative (although barely so), reflecting once again its pseudoconvexity, i.e., the nonvanishing of the Levi form. In complex coordinates the group law is given by

$$(z,w)\cdot(z',w') = (z+z', w+w'+2iz\bar{z}').$$

Indeed, if $\Im m\, w = |z|^2$ and $\Im m\, w' = |z'|^2$, then $\Im m(w+w'+2iz\bar{z}') = |z+z'|^2$. This is the law of the *Heisenberg group* $\mathbf{H}_1$, well known for its role in quantum mechanics and in many areas of mathematics, such as symplectic geometry and algebraic number theory. It is convenient to think of $\mathbf{H}_1$ as the group of unipotent, upper-triangular 3×3 matrices $\exp A(z,s)$, with

$$A(z,s) = \begin{pmatrix} 0 & z & \frac{1}{2i}s \\ 0 & 0 & \bar{z} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } z\in\mathbb{C},\ s\in\mathbb{R}.$$

The matrices $A(z,s)$ form a three-dimensional *real* vector space. In fact, they form a Lie algebra $\mathfrak{h}_1$ for the commutation bracket:

$$\begin{aligned}&[A(z,s),A(z',s')]\\ &\quad = A(z,s)A(z',s') - A(z',s')A(z,s)\\ &\quad = A(0, 2i(z\bar{z}'-z'\bar{z})).\end{aligned}$$

The isomorphism $\mathbf{H}_1 \cong \mathcal{Q}$ is given by

$$\exp A(z,s) = \begin{pmatrix} 1 & z & \frac{w}{2i} \\ 0 & 1 & \bar{z} \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with } w = s+i|z|^2.$$

It respects the group law, since $\exp A(z,s)\exp A(z',s')$ equals

$$\exp A(z+z', s+s'+2\Im m(z\bar{z}')).$$

There is a corresponding Lie algebra isomorphism of $\mathfrak{h}_1$ onto the real span of the vector fields on $\mathcal{Q}$ of the form

$$D = \zeta L + \bar{\zeta}\bar{L} + \frac{\sigma}{2i}[L,\bar{L}]$$

with $L = \frac{\partial}{\partial\bar{z}} - 2iz\frac{\partial}{\partial\bar{w}}$: if $D' = \zeta' L + \bar{\zeta}'\bar{L} + \frac{\sigma'}{2i}[L,\bar{L}]$, then $[D,D'] = (\zeta\bar{\zeta}'-\zeta'\bar{\zeta})[L,\bar{L}]$ is exactly the image of $[A(\zeta,\sigma), A(\zeta',\sigma')]$.

One cannot expect to find a group law on a randomly chosen hypersurface $S\subset\mathbb{C}^n$, since a group law implies a certain amount of "rigidity". But one can try to "approximate" locally the hypersurface S by a sequence of hypersurfaces S_k with $k=1,2,\ldots$, each equipped with a group structure. A natural approach is to approximate (near a central point, say the origin) a special basis $X_1+iX_n,\ldots,X_{n-1}+iX_{2n-2}$ of the tangential Cauchy-Riemann vector fields of S by an equal number of vector fields with carefully chosen polynomial coefficients. The success of this approach is predicated on a property of the (real) vector fields $X_1,\ldots,X_{2n-2}$, variously known as *finite type* (in SCV theory) or *Hörmander's condition* (in partial differential equations): the condition is simply that freezing at an arbitrary point of S the Lie algebra $\mathfrak{g}$ generated by the X_j yields the total tangent space to S. When S is strongly pseudoconvex, the whole tangent space to S is spanned by the vector fields X_j and by their *first* brackets $[X_j,X_k]$. A trivial example of infinite type is the hyperplane $t=0$. A nontrivial example is the hypersurface $S_\star\subset\mathbb{C}^2$ defined by the equation $t=\exp(-1/|z|^2)$: $S_\star$ is strongly pseudoconvex off the origin in the unit ball. Geometrically the meaning of finite type is rather subtle. It does imply *accessibility* in the language of optimal control, that is, the possibility of joining any two points $z_\circ$ and $z_\star$ by a piecewise smooth curve $\gamma\subset S$ (S is connected) whose smooth arcs are integral curves of one of the vector fields X_j (with j depending on the arc). But there is more to it: after all, the hypersurface $S_\star$ above has the same "connecting" property. With finite type one can associate a metric on S that gives a different weight to different directions but is bounded from below by a power of the Euclidean distance at small distances. In the local coordinates x, y, $s=\Re e\, w$ on the hyperquadric, the new metric would be something like $(|z-z'|^2+|s-s'|)^{1/2}$ in accordance with the homogeneity properties of the Heisenberg group.

On a hypersurface S of finite type we can select the basis $X_1+iX_n,\ldots,X_{n-1}+iX_{2n-2}$ and the approximating vector fields $X_1^{(k)},\ldots,X_{2n-2}^{(k)}$ (with polynomial coefficients) in such a way that the Lie algebra generated by $X_1^{(k)},\ldots,X_{2n-2}^{(k)}$, i.e., the linear span of the iterated Lie brackets

$$\left[X_i^{(k)},X_j^{(k)}\right] = X_i^{(k)}X_j^{(k)} - X_j^{(k)}X_i^{(k)},$$
$$\left[X_{i_2}^{(k)},\left[X_{i_1}^{(k)},X_{i_0}^{(k)}\right]\right],\ldots,$$

is *nilpotent*, which is to say that

$$\left[X_{i_r}^{(k)},\cdots\left[X_{i_2}^{(k)},\left[X_{i_1}^{(k)},X_{i_0}^{(k)}\right]\right]\cdots\right]\equiv 0$$

for r sufficiently large. The approximating nilpotent groups provide us with many symmetries that can be exploited to refine the analytic tools needed for a function theory in a domain bounded by S or on S itself. This approach was initiated in [16] and has been exploited by E. M. Stein and his collaborators.

The Convex Side of the Hyperquadric as a Homogeneous Space

If we compactify at infinity the convex side $\mathcal{Q}^+$ of our hyperquadric $\mathcal{Q}$, the map (1) extends as a homeomorphism from the closed unit ball $\overline{\mathfrak{B}}$ onto the one-point compactification $\bar{\mathcal{Q}}^+$. The right way to do this is by embedding $\mathbb{C}^2$ into $\mathbb{CP}^3$, say by setting $z_0 = 1$, $z_1 = z$, $z_2 = w$. Then let $\mathbf{U}(1,2)$ denote the "unitary" group of complex (nonsingular) 3×3 matrices preserving the quadratic form $|z_0|^2 - |z_1|^2 - |z_2|^2$. Through the maps

$$(z, w) \to \left(\frac{a_{10} + a_{11}z + a_{12}w}{a_{00} + a_{01}z + a_{02}w}, \frac{a_{20} + a_{21}z + a_{22}w}{a_{00} + a_{01}z + a_{02}w}\right)$$

for $(a_{jk})_{j,k=0,1,2} \in \mathbf{U}(1,2)$, $\mathbf{U}(1,2)$ acts transitively on $\overline{\mathfrak{B}}$. The isotropy of a point is equivalent to the action of the matrices

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & A \end{pmatrix}, \quad 0 \le \theta < 2\pi,\ A \in \mathbf{U}(2),$$

and thus $\bar{\mathcal{Q}}^+ \cong \overline{\mathfrak{B}} \cong \mathbf{U}(1,2)/(\mathbf{U}(1) \times \mathbf{U}(2))$.

Nonsolvability: Hans Lewy's Example

We can use the local coordinates x, y, s on the hyperquadric $\mathcal{Q}$, in which case the expression of the vector field $\frac{\partial}{\partial \bar{z}} - 2iz\frac{\partial}{\partial \bar{w}}$ becomes, *along* $\mathcal{Q}$,

$$L_\circ = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - i(x + iy)\frac{\partial}{\partial s}.$$

This is the famed Hans Lewy example [14]. *For any pair* $V \subset U$ *of open subsets of* $\mathbb{R}^3$ *and most functions* $f \in C^\infty(U)$, *the Lewy equation* $L_\circ u = f$ *does not have any* (classical, distribution, hyperfunction) *solution in* V. Here "most" means all functions except those in some set of first Baire category. To construct a right-hand side f such that $L_\circ u = f$ is not solvable in a neighborhood of the origin is fairly simple: in polar coordinates (r, θ) in the (x, y)-plane the Lewy equation yields

$$\frac{1}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right)(zu) - ir\frac{\partial}{\partial s}(zu) = rf.$$

Select a C^∞ function f independent of θ with $f > 0$ in the "ball"

$$\mathfrak{B} = \mathfrak{B}_\varepsilon(r_\circ, s_\circ) : (r^2 - r_\circ^2)^2 + (s - s_\circ)^2 < \varepsilon(r_\circ^4 + s_\circ^2),$$

where $0 < \varepsilon < 1$, and with $f \equiv 0$ everywhere else. Integration with respect to θ over $(0, 2\pi)$ yields

$$\frac{1}{2}\frac{\partial v}{\partial r} - ir\frac{\partial v}{\partial s} = rf, \tag{3}$$

where $v(r, s) = \frac{r}{2\pi}\int_0^{2\pi} u(r, \theta, s)e^{i\theta}\, d\theta$. In the notation $\zeta = s + ir^2$, (3) reads $\partial v/\partial\bar{\zeta} = \frac{i}{2}f(r, s)$. If a solution v of (3) were to exist, it would be a holomorphic function of ζ in $\mathbb{R}^2 \setminus \mathfrak{B}_\varepsilon(r_\circ, s_\circ)$. But $v(0, s) \equiv 0$ implies $v \equiv 0$ in $\mathbb{R}^2 \setminus \mathfrak{B}_\varepsilon(r_\circ, s_\circ)$, contradicting the fact that

$$\oint_{\partial\mathfrak{B}} v\, d\zeta = -\int\int_{\mathfrak{B}} f\,\frac{d\zeta \wedge d\bar{\zeta}}{2i} \neq 0.$$

By taking disjoint disks $\mathfrak{B}_{\varepsilon_j}(r_j, s_j)$ converging to $(0, s_\circ)$, we can preclude the existence of v in *any* neighborhood of $(0, s_\circ)$. The same construction can be carried out about any point $(z_\circ, w_\circ) \in \mathcal{Q}$.

The Lewy example has been the starting point of two major developments: the investigation of the local solvability of linear partial differential equations (PDEs) and the construction of a cohomology theory on CR manifolds.

The Local Solvability of Linear PDEs with Simple Real Characteristics

Until 1956 a widely held belief of analysts working in the field of linear PDEs was that all these equations, provided they had reasonably smooth coefficients, could be solved at least locally. It is true of the classical *types*—elliptic, hyperbolic, parabolic—and it had been known since the findings of L. Ehrenpreis and B. Malgrange in the early 1950s that all linear PDEs with constant coefficients could be solved globally in any *convex* open subset of Euclidean space (in the classical sense as well as in all the reasonable distribution spaces). In the early 1960s, in searching for the roots of the Lewy example, L. Hörmander demonstrated the importance of the Levi form for general linear PDEs in a wider context than just SCV. Let $L = X + iY$ be a smooth, nowhere vanishing complex vector field in an open subset Ω of $\mathbb{R}^n$, X and Y being real vector fields. For the equation $Lu = f$ to be locally solvable (say at every point of Ω), it is necessary that, *at each point* of Ω,

$$[X, Y] = \frac{1}{2i}[L, \bar{L}] \in \operatorname{Span}(L, \bar{L}) = \operatorname{Span}(X, Y).$$

As shown by the vector field $\frac{\partial}{\partial x} + ix^2\frac{\partial}{\partial y}$ in the plane, this is *not* the same as saying that $[X, Y]$ is a smooth linear combination of X and Y, which is the Frobenius property. Hörmander's necessary condition was refined in 1962 by L. Nirenberg and the author, as Condition (P) (a name that has stuck): it requires that X keep a fixed (oriented) direction along any integral curve of Y and that the same be true of Y along any integral curve of X. They proved that Condition (P) was sufficient for the local solvability of the equation $Lu = f$ and that it was necessary under a finite-type hypothesis.

Whereas in studying a vector field L it is possible to restrict one's attention to the integral curves of its real and imaginary parts, in order to generalize the above results to a higher-order linear partial differential operator $P(x, D)$, the analysis must be lifted to phase space $\Omega \times \mathbb{R}^n$ (i.e., to the *cotangent bundle* of the base manifold, here Ω). This lifting and the increased analytic procedures that go with it are what is known as *microlocalization*. Those increased analytic procedures start from the concepts and methods of Hamiltonian mechanics and of the symplectic geometry on which it is based and proceed to *quantization* (which made its appearance in the "old" quantum theory): to a C^∞ function f in $\Omega \times \mathbb{R}^n$ whose growth as $|\xi| \to +\infty$ is suitably controlled, quantization associates an operator, usually called *pseudodifferential*, of which then f is the *symbol*. In local coordinates the correspondence *symbol→operator* is mediated by Fourier-like integrals. Indeed, this "microlocal analysis" can be thought of as the extension of Fourier analysis to smooth manifolds.

Quantization associates to a *polynomial* of degree m with respect to ξ, $P(x, \xi)$, whose coefficients are C^∞ functions of x in Ω, a *differential operator* $P(x, D)$. The principal symbol $P_m(x, \xi)$ of $P(x, D)$, that is, the homogeneous part of degree m of $P(x, D)$, is invariant under changes of coordinates in the base Ω. The set of points $(x, \xi) \in \Omega \times \mathbb{R}^n$ such that $P_m(x, \xi) = 0$ and $\xi \neq 0$ form the *characteristic set* of the operator $P(x, D)$. It is at those characteristic points that the obstructions to existence, as well as to the regularity, of the solutions of the equation $P(x, D)u = f$ originate. The hypothesis that the real characteristics of $P(x, D)$ are simple, in the sense that, if $\xi \neq 0$,

$$P_m(x, \xi) = 0 \Longrightarrow \sum_{j=1}^{n} \left| \frac{\partial P_m}{\partial \xi_j}(x, \xi) \right| \neq 0,$$

allows one to disregard the lower-order terms $P(x, D) - P_m(x, D)$. In this framework the formulation of Condition (P) is that *$\Re e\, P_m$ does not change sign along the integral curves of the Hamiltonian vector field of $\Im m\, P_m$ and that the same is true after multiplying P_m by $\sqrt{-1}$*. The Hamiltonian vector field of $f \in C^1(\Omega \times \mathbb{R}^n)$ is the vector field

$$H_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

The freedom afforded by microlocalization to use (symplectic) changes of variables (x, ξ), and not just x, is made concrete by associating invertible Fourier integral operators F to those changes, and through similarities $F^{-1}P(x, D)F$ reduces the study to that of simplified model forms. At the beginning of the 1970s rather sophisticated versions of pseudodifferential calculus were used to establish that Condition (P) is indeed necessary and sufficient for the local solvability of a linear PDE with simple real characteristics; the sufficiency was proved by R. Beals and C. Fefferman, and the necessity by R. Moyer. Thus the solvability theory of linear PDEs has turned out to be one of the earliest testing grounds of microlocal analysis.

Unfortunately, little progress has been made since 1980 in our understanding of the solvability of linear PDEs with real characteristics of higher multiplicity, and very little is known about *systems* of PDEs, especially overdetermined systems. A question asked by D. Spencer in the 1960s is still without an answer: whether an *elliptic* (!) system of *three* linear PDEs with C^∞ (but nonanalytic) coefficients in *two* unknown functions is locally solvable.

Tangential CR Cohomology

Since the 1950s it has become customary to rephrase the solvability properties of some important systems of differential operators (e.g., the exterior derivative, the Cauchy-Riemann operator $\bar{\partial}$, etc.) in terms of cohomology spaces. The same can be done with the *tangential Cauchy-Riemann differential complex*.

It is easy enough to define this complex on an abstract CR manifold S, but it will suffice here to consider the case of a C^∞ hypersurface S embedded in a complex manifold $\mathcal{M}$ (of complex dimension $n + 1$). One can cover S with holomorphic coordinate charts of $\mathcal{M}$, written $(\tilde{V}, z_1, \ldots, z_n, w)$ with $z_j = x_j + iy_j$ and $w = s + it$, such that $U = \tilde{V} \cap S = \{(z, w) \in \tilde{V};\ t = F(x, y, s)\}$; the real-valued function F is defined and smooth in $\tilde{V}$, and x_j, y_k $(j, k = 1, \ldots, n)$ and s are real coordinates in U. A simple definition of the tangential CR operator can be given using differential forms $\varphi \wedge dz \wedge dw$, where $dz = dz_1 \wedge \cdots \wedge dz_n$, $dw = ds + idF$, and

$$\varphi = \sum_{|J|=q} f_J(x, y, s) d\bar{z}_J;$$

the coefficients f_J are complex-valued C^∞ functions in U. We have used multi-index notation: $J = (j_1, \ldots, j_q)$ with $1 \leq j_1 < \cdots < j_q \leq n$, $|J| = j_1 + \cdots + j_n$, and $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. The forms $\varphi \wedge dz \wedge dw$ are often called $(n + 1, q)$-forms. If d stands for the exterior derivative on S, then

$$(4)\quad d\varphi \wedge dz \wedge dw = \sum_{k=1}^{n} L_k f_J \wedge d\bar{z}_k \wedge d\bar{z}_J \wedge dz \wedge dw.$$

The vector fields $L_1, \ldots, L_n$ form a basis of the tangential Cauchy-Riemann vector fields in U. The form (4) is denoted by $\bar{\partial}_b^{(q)}(\varphi \wedge dz \wedge dw)$ or by $\left(\bar{\partial}_b^{(q)} \varphi\right) \wedge dz \wedge dw$. We denote by $C^\infty(U, \Lambda^{n+1,q})$

the linear space of differential forms $\varphi \wedge dz \wedge dw$.

Most authors drop the factor $dz \wedge dw$ and deal solely with φ, which is called a $(0, q)$*-form*. This is all right as long as we keep in mind that φ is not a true differential form but is defined only up to forms $\omega_1 \wedge dz_1 + \cdots + \omega_n \wedge dz_n + \varpi \wedge dw$. The equivalence classes of $(0, q)$-forms represent smooth sections of a vector bundle $\Lambda^{0,q}$ over S. The $(0, 0)$-forms are functions, i.e., $\Lambda^{0,0} \cong S \times \mathbb{C}$; and $\Lambda^{0,q} = S \times \{0\}$ if $q > n$. Sections such as φ make up a linear space $C^\infty(U, \Lambda^{0,q})$, and $\bar\partial_b^{(q)}\varphi$ is in $C^\infty(U, \Lambda^{0,q+1})$. Most often the tangential CR complex in U is defined as the sequence of linear differential operators

$$\bar\partial_b^{(q)} : C^\infty(U, \Lambda^{0,q}) \to C^\infty(U, \Lambda^{0,q+1}), \quad q = 0, 1, \ldots.$$

For $q \geq 1$ the cohomology space

$$H^q_{\bar\partial_b}(U) = \frac{\left\{\varphi \in C^\infty(U, \Lambda^{0,q});\ \bar\partial_b^{(q)}\varphi = 0\right\}}{\bar\partial_b^{(q-1)} C^\infty(U, \Lambda^{0,q-1})}$$

embodies the "lack" of solvability (in U) of the equations

$$\bar\partial_b^{(q-1)}\psi = \varphi, \tag{5}$$

where φ is in $C^\infty(U, \Lambda^{0,q})$ and $\bar\partial_b^{(q)}\varphi = 0$. As for $H^0_{\bar\partial_b}(U)$ it is the space of (smooth) CR functions in $U \subset S$ (Definition 2).

All these concepts can be readily "globalized": define $C^\infty(S, \Lambda^{n+1,q})$ as the space of smooth differential forms of degree $n + 1 + q$ on S whose restrictions to any domain U of local coordinates of the kind above belong to $C^\infty(U, \Lambda^{n+1,q})$. From there ensues all that is needed.

In defining the vanishing of the local CR cohomology in degree q, at a point $z_\circ \in S$, we must allow for some contraction of the neighborhood in which the equation (5) is valid. To be precise: to each open neighborhood U of $z_\circ$ there must exist an open neighborhood $V \subset U$ such that, for every $\varphi \in C^\infty(U, \Lambda^{0,q})$ satisfying $\bar\partial_b^{(q)}\varphi = 0$ in U, the equation (5) is satisfied (in V) by some $\psi \in C^\infty(V, \Lambda^{0,q-1})$. In other words, the "restriction" of cohomology classes from U to V annihilates $H^q_{\bar\partial_b}(U)$.

A. Andreotti and C. D. Hill have shown (in [2]) that if the Levi form of S at the point $z_\circ$ is nondegenerate and has exactly ν eigenvalues of one sign and $n - \nu$ of the opposite sign, then the local CR cohomology is nonvanishing exactly in the degrees $0, \nu$, and $n - \nu$. Geometrically simple open subsets of the hyperquadric $\mathcal{Q}_{n,\nu}$ (see below) provide a model of this phenomenon. In our present state of knowledge the hypotheses on the number of positive, or negative, eigenvalues of the Levi form, leading to "vanishing theorems" for cohomology, can be somewhat—but not much—relaxed.

To get some hold on the global $\bar\partial_b$-cohomology of a hypersurface S, a frequently used approach championed by J. J. Kohn is via the CR analogue of Hodge theory. After equipping S with a Riemannian metric, one can make use of the associated inner product on $L^2(S; \Lambda^{0,q})$ to define the formal adjoint

$$\bar\partial_b^{*(q+1)} : C^\infty(S; \Lambda^{0,q+1}) \to C^\infty(S; \Lambda^{0,q})$$

of $\bar\partial_b^{(q)}$ and the self-adjoint operator, densely defined in $L^2(S; \Lambda^{0,q})$, $\Box_b^{(q)} = \bar\partial_b^{*(q+1)}\bar\partial_b^{(q)} + \bar\partial_b^{(q-1)}\bar\partial_b^{*(q)}$, whose role is analogous to that of the Laplacian in Hodge theory: suitable hypotheses on S should in principle permit the selection in each cohomology class in $H^q_{\bar\partial}(S)$ of a unique "harmonic" representative h, i.e., of a $(0, q)$-form h that solves $\Box_b^{(q)} h = 0$.

The CR Cohomology of Hyperquadrics

The cohomological obstructions that are best understood are those arising from the eigenvalues of the Levi form. A clearer picture emerges if we enlarge our collection of hyperquadrics to include the hypersurface $\mathcal{Q}_{n,\nu} \subset \mathbb{C}^{n+1}$ (complex coordinates $z_1, \ldots, z_n, w$) defined by the equation

$$\mathfrak{Im}\, w = \sum_{j=1}^{\nu} |z_j|^2 - \sum_{j=\nu+1}^{n} |z_j|^2. \tag{6}$$

The case $\nu = n$ is not precluded when there are no negative terms at the right (but the cases $\nu < \frac{n}{2}$ can be disregarded, since we can always change w into $-w$ and thus exchange ν and $n - \nu$). Here $\mathcal{Q}_{1,1} = \mathcal{Q}$.

The tangential CR vector fields are now

$$L_j = \frac{\partial}{\partial \bar z_j} - 2iz_j \frac{\partial}{\partial \bar w} \quad \text{if } j \leq \nu,$$

$$L_j = \frac{\partial}{\partial \bar z_j} + 2iz_j \frac{\partial}{\partial \bar w} \quad \text{if } j > \nu.$$

They commute. Together with their complex conjugates, $\bar L_1, \ldots, \bar L_n$, and with $\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar w} = \frac{\partial}{\partial s}$, they span the whole tangent spaces to $\mathcal{Q}_{n,\nu}$. The Levi matrix of $\mathcal{Q}_{n,\nu}$ at any one of its points can be identified to the diagonal matrix with +1 in the first ν diagonal entries and -1 in the remaining $n - \nu$ entries (but here the signs + or − are arbitrary, as there is no reason to privilege one side of the hyperquadric over the other).

Then let U be a relatively open, nonempty subset of $\mathcal{Q}_{n,\nu}$. In contrast to the usual de Rham situation, here either $H^q_{\bar\partial_b}(U) = \{0\}$ or $\dim H^q_{\bar\partial_b}(U) = +\infty$. It follows from Theorem 1 that if $1 \leq \nu < n$, $H^0_{\bar\partial_b}(U)$ can be identified to the space of holomorphic functions in a subset $\widetilde U$ of $\mathbb{C}^{n+1}$ containing U. If $\nu = n$, the elements of $H^0_{\bar\partial_b}(U)$ are the boundary values of the holomorphic functions on the convex portion of $\widetilde U$ (smooth up to the edge U).

For our original hyperquadric $\mathcal{Q}$ (when $n = 1$), we have $\dim H^q_{\bar\partial_b}(U) = +\infty$ for $q = 0, 1$. The fact that $\dim H^1_{\bar\partial_b}(U) = +\infty$ is Lewy's result: $H^1_{\bar\partial_b}(U)$ is the space of cosets modulo $\bar\partial_b C^\infty(U)$ of C^∞ forms $\varphi = f(x, y, s)\, d\bar z$ in U such that the equation $Lu = f$ has no solution $u \in C^\infty(U)$, L being the Lewy operator.

Not much is known about the spaces $H^q_{\bar\partial_b}(U)$ for a general open set $U \subset \mathcal{Q}_{n,\nu}$ when $n \geq 2$ and $q \geq 1$. However, if

$$U = \left\{(z, w) \in \mathcal{Q}_{n,\nu};\ \sum_{j=1}^{n} |z_j|^2 < r^2,\ a < \Re e\, w < b\right\}$$

with $0 < r \leq +\infty$ and $-\infty \leq a < b \leq +\infty$, then $H^q_{\bar\partial_b}(U) \neq \{0\}$ solely in those degrees q where the pseudoconvexity (or pseudoconcavity, as no one side of $\mathcal{Q}_{n,\nu}$ is privileged) is the "strongest": $q = \nu$ and $q = n - \nu$. In particular, $H^q_{\bar\partial_b}(\mathcal{Q}_{n,\nu}) \neq \{0\}$ if and only if $q \neq \nu, n - \nu$. These properties agree with the general results of Andreotti and Hill (see above), but without the contraction of U. The nonvanishing of $H^\nu_{\bar\partial_b}(U)$ and $H^{n-\nu}_{\bar\partial_b}(U)$ reflects the nonvanishing *singular homology* of the hyperquadric $\mathcal{Q}_{n,\nu}$ in the degrees $2\nu - 1$ and $2n - 2\nu - 1$: it is the presence of noncontractible spheres $\mathbb{S}^{2\nu-1}$ and $\mathbb{S}^{2(n-\nu)-1}$ inside $\mathcal{Q}_{n,\nu}$, evident on inspection of equation (6), that allows us to construct $\bar\partial_b$-closed forms of bidegrees $(0, \nu)$ and $(0, n - \nu)$ that are not $\bar\partial_b$-exact.

The Heisenberg invariance of $\mathcal{Q}_{n,\nu}$ has been fully exploited in [7] to construct explicitly, as Heisenberg invariant singular integral operators, the inverse of $\Box_b^{(q)}$ on the orthogonal complement of its null space as well as the orthogonal projector on its null space. Further along this road it is possible to develop variants of the classical pseudodifferential calculus based on the multiplication and reflecting the homogenities of the Heisenberg group. D. Geller (1990) has developed such a calculus, in the C^ω category, and used it to study the local existence and analyticity of solutions of the differential equations that are naturally associated with the $\bar\partial_b$ complex.

The Connection with $\bar\partial$-Cohomology

The global CR cohomology of $\mathcal{Q}_{n,\nu}$ can also be interpreted as the "jump" of the $\bar\partial$-cohomology from one side, say $\mathcal{Q}^+_{n,\nu}$, to the other, $\mathcal{Q}^-_{n,\nu}$. From the homological algebra viewpoint this is simply the Mayer-Vietoris sequence for the pair

$$(\mathcal{Q}^+_{n,\nu} \cup \mathcal{Q}_{n,\nu},\ \mathcal{Q}^-_{n,\nu} \cup \mathcal{Q}_{n,\nu}).$$

From the analysis standpoint one can define the boundary value $b[h^\pm]$ on $\mathcal{Q}_{n,\nu}$ of any $\bar\partial$-cohomology class $[h^\pm]$ in $\mathcal{Q}^\pm_{n,\nu}$—as the CR-cohomology class of the boundary value (in the classical or in the distribution sense or, more simply, in the hyperfunction sense) of any one of its representatives. Then a simple argument of [2] shows that, either locally or globally, any CR class $[\varphi]$ on $\mathcal{Q}_{n,\nu}$ is equal to a "jump" $b[h^+] - b[h^-]$.

In 1964 S. Gindikin showed how to obtain the global $\bar\partial$-cohomology of those domains in $\mathbb{CP}^N$ that can be viewed as homogeneous spaces, as the sides $\mathcal{Q}^\pm_{n,\nu}$ of $\mathcal{Q}_{n,\nu}$ can (allowing now $0 \leq \nu \leq n$). Indeed, there is a rational map, akin to (1), of the domain

$$\mathcal{B}_{n,\nu} = \left\{(z, w) \in \mathbb{C}^{n+1};\ |w|^2 + \sum_{j=1}^{\nu} |z_j|^2 - \sum_{j=\nu+1}^{n} |z_j|^2 < 1\right\}$$

onto $\mathcal{Q}^+_{n,\nu}$. The domain $\mathcal{B}_{n,\nu}$ is isomorphic to a quotient $\mathbf{U}(\nu + 1, n - \nu + 1)/\mathbf{G}$, where $\mathbf{G}$ is the isotropy subgroup of an arbitrary point of the boundary $\partial\mathcal{B}_{n,\nu}$. The ensuing symmetries can then be exploited to show that $H^q_{\bar\partial}(\mathcal{B}_{n,\nu}) \cong H^q_{\bar\partial}(\mathcal{Q}^+_{n,\nu})$ vanishes if and only if $q \neq 0$ and $q \neq n - \nu$. For example, $H^1_{\bar\partial}(\mathcal{Q}^+) = \{0\}$ whereas $H^1_{\bar\partial}(\mathcal{Q}^-) \cong H^1_{\bar\partial}(\mathcal{Q}^+_{1,0}) \neq \{0\}$. It follows from Theorem 1 that if $\nu < n$, $H^0_{\bar\partial}(\mathcal{B}_{n,\nu})$ can be identified to the space of entire functions in $\mathbb{C}^{n+1}$.

Generally speaking, there are four methods to study the $\bar\partial$-cohomology of a domain Ω in a complex manifold (i.e., to solving globally in Ω the $\bar\partial$-equations with right-hand sides that are $\bar\partial$-closed forms of a given degree):

- within the theory of coherent analytic sheaves, following Oka-Cartan;
- through PDE variational techniques (solving the so-called $\bar\partial$-Neumann problem) or through weighted (à la Carleman) estimates for the Cauchy-Riemann equations;
- by means of explicit integral kernels;
- by complete transfer of the analysis to the boundary $\partial\Omega$ based on the Calderón operator (as shown in [17], Ch. III).

The sheaf-theoretical methods were put to full use in [1]; one of the great advantages of these methods is that they also work for analytic spaces (locally, varieties with singularities). L. Hörmander exploited Carleman-type estimates for the $\bar\partial$-complex (see [9], Ch. IV). The integral kernel approach goes back to Bochner, Fantappiè, Leray, Martinelli, and many others. Today the most frequently used integral formulas for general strongly pseudoconvex domains are those of G. Henkin (see [8]). For a domain Ω in complex space the basic fact to keep in mind is that cohomological "triviality", i.e., the property that $H^q_{\bar\partial}(\Omega) = 0$ if $q \geq 1$, is equivalent to pseudoconvexity.

Solving the $\bar\partial$-Neumann problem was suggested by D. Spencer and carried out by J. J. Kohn (for an exposition see [6]). This approach requires that the boundary $\partial\Omega$ be smooth. It identifies solutions of the $\bar\partial$-equations endowed with very convenient properties, by means of the Neumann operator N, which expresses the solution of the $\bar\partial$-Neumann problem in terms of the boundary data, and of the Bergman operator, i.e., the orthogonal projection of $L^2(\Omega)$ onto the closed subspace $L^2(\Omega) \cap \{\text{holomorphic functions in } \Omega\}$. The latter is not to be confused with the Szegö operator, i.e., the orthogonal projection of $L^2(\partial\Omega)$ onto the closed subspace $L^2(\partial\Omega) \cap \{\text{CR functions in } \partial\Omega\}$. Not surprisingly, the Neumann, Bergman, and Szegö operators can be expressed by fairly simple integral formulas when Ω is an open ball or the convex side of the hyperquadric $\mathcal{Q}$.

A very effective approach to the study of the $\bar\partial$-Neumann problem, as well as to that of the Bergman and Szegö kernels, is through microlocal analysis and the use of pseudodifferential operators. *Classical* pseudodifferential operators made one of their first appearances in the celebrated article [11]—as a tool to prove the so-called subelliptic estimates for the $\bar\partial$-Neumann problem, among other "noncoercive" boundary value problems.

Acknowledgment

The author wishes to thank P. D. Cordaro for pointing out some misstatements in the original manuscript and A. Knapp for much stylistic improvement. Lack of space has prevented the discussion of many topics and the citations of many works related to the contents of this article.

The photograph of Hans Lewy was provided courtesy of Mrs. Helen Lewy.

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