From Rotating Needles to Stability of Waves: Emerging Connections between Combinatorics, Analysis, and PDE

Terence Tao

Introduction

In 1917 S. Kakeya posed the *Kakeya needle problem*: What is the smallest area required to rotate a unit line segment (a "needle") by 180 degrees in the plane? Rotating around the midpoint requires $\pi/4$ units of area, whereas a "three-point U-turn" requires $\pi/8$. In 1927 the problem was solved by A. Besicovitch, who gave the surprising answer that one could rotate a needle using arbitrarily small area.

At first glance, Kakeya's problem and Besicovitch's resolution appear to be little more than mathematical curiosities. However, in the last three decades it has gradually been realized that this type of problem is connected to many other, seemingly unrelated, problems in number theory, geometric combinatorics, arithmetic combinatorics, oscillatory integrals, and even the analysis of dispersive and wave equations.

The purpose of this article is to discuss the interconnections between these fields, with an emphasis on the connection with oscillatory integrals and PDE. Two previous surveys ([7] and [1]) have focused on the connections between Kakeya-type problems and other problems in discrete combinatorics and number theory.

Terence Tao is professor of mathematics at UCLA. His e-mail address is `tao@math.ucla.edu`.

These areas are very active, but despite much recent progress our understanding of the problems and their relationships to each other is far from complete. Ideas from other fields may well be needed to make substantial new breakthroughs.

Kakeya-Type Problems

Besicovitch's solution to the Kakeya needle problem relied on two observations. The first observation, which is elementary, is that one can translate a needle to any location using arbitrarily small area; see Figure 1. The second observation is that one can construct open subsets of $\mathbf{R}^2$ of arbitrarily small area which contain a unit line segment in every direction. A typical way to construct such sets (not Besicovitch's original construction) is sketched in Figure 2; for a more detailed construction see [7].

For any $n \geq 2$ define a *Besicovitch set* to be a subset of $\mathbf{R}^n$ which contains a unit line segment in every direction. For any such n the construction of Besicovitch shows that such sets can have arbitrarily small Lebesgue measure and can even be made to have measure zero. Intuitively this means that it is possible to compress a large number of nonparallel unit line segments into an arbitrarily small set.

In applications one wishes to obtain more quantitative understanding of this compression effect by introducing a spatial discretization. For instance, one can replace unit line segments by $1 \times \delta$ tubes for some $0 < \delta \ll 1$ and ask for the optimal compression of these tubes. Equivalently, one can ask for bounds of the volume of the δ-neighbourhood of a Besicovitch set.

Rather surprisingly, these bounds are logarithmic in two dimensions. It is known that the δ-neighbourhood of a Besicovitch set in $\mathbf{R}^2$ must have area at least $C/\log(1/\delta)$;[1] this basically follows from the geometric observation that the area of the intersection of two $1 \times \delta$ rectangles varies inversely with the angle between the long axes of the rectangles. Recently, U. Keich has shown that this bound is sharp.

This observation can be rephrased in terms of the *Minkowski dimension* of the Besicovitch set. Recall that a bounded set E has Minkowski dimension α or less if and only if for every $0 < \delta \ll 1$ and $0 < \varepsilon \ll 1$ one can cover E by at most $C_\varepsilon \delta^{-\alpha+\varepsilon}$ balls of radius δ. From the previous discussion we thus see that Besicovitch sets in the plane must have Minkowski dimension 2.

It is unknown if the analogous property holds in $\mathbf{R}^n$. In one of its principal formulations, the *Kakeya conjecture* states that every Besicovitch set in $\mathbf{R}^n$ has Minkowski dimension n. (There is also a corresponding conjecture for the Hausdorff dimension, but for simplicity we shall not discuss this variant.)

Equivalently, the Kakeya conjecture asserts that the volume of the δ-neighbourhood of a Besicovitch set in $\mathbf{R}^n$ is bounded below by $C_{n,\varepsilon}\delta^{\varepsilon}$ for any $\varepsilon > 0$ and $0 < \delta \ll 1$.

The Kakeya conjecture is remarkably difficult. It remains open in three and higher dimensions, although rapid progress has been made in the last few years. The best-known lower bound for the Minkowski dimension at this time of writing is

$$\max\left(\frac{n+2}{2} + 10^{-10}, \frac{4n+3}{7}\right),$$

although I expect further improvements to follow very soon.

One can discretize the conjecture. Let Ω be a maximal δ-separated subset of the sphere S^{n-1} (so that Ω has cardinality approximately δ^{1-n}), and for each $\omega \in \Omega$ let T_ω be a $\delta \times 1$ tube oriented in the direction ω. The Kakeya conjecture then asserts logarithmic-type lower bounds on the quantity $|\bigcup_{\omega\in\Omega} T_\omega|$.

The above formulation is reminiscent of existing results in combinatorics concerning the number of incidences between lines and points,

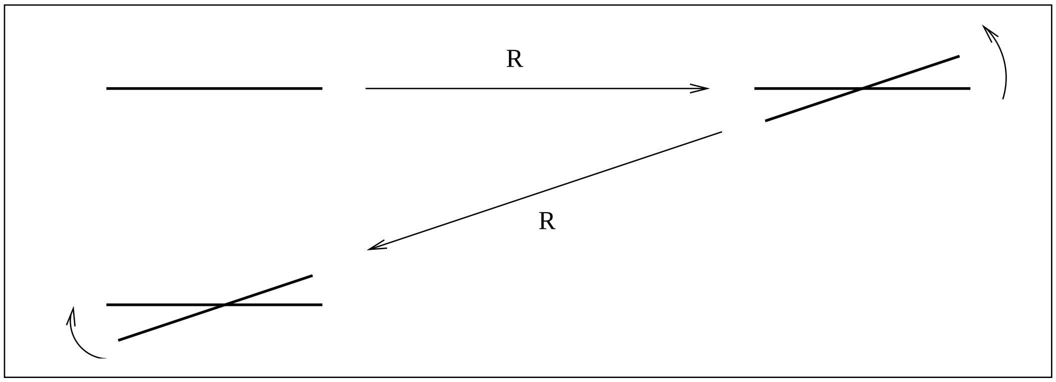


Figure 1. To translate a needle, slide it by R units, rotate by roughly $1/R$, slide it back, and rotate back. This costs $O(1/R)$ units of area, where R is arbitrary.

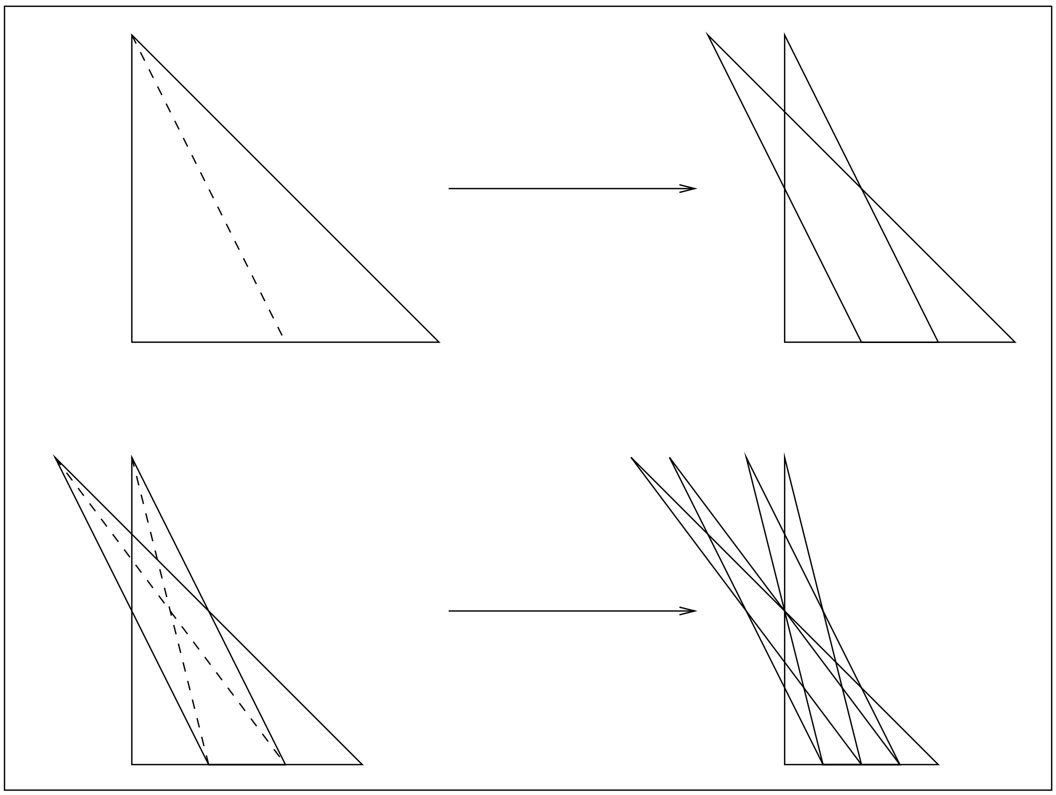


Figure 2. The iterative construction of a Besicovitch set. Each stage consists of the union of triangles. To pass to the next stage, the triangles are bisected and shifted together to decrease their area.

[1] *Throughout this article, the letter C denotes a constant which varies from line to line.*

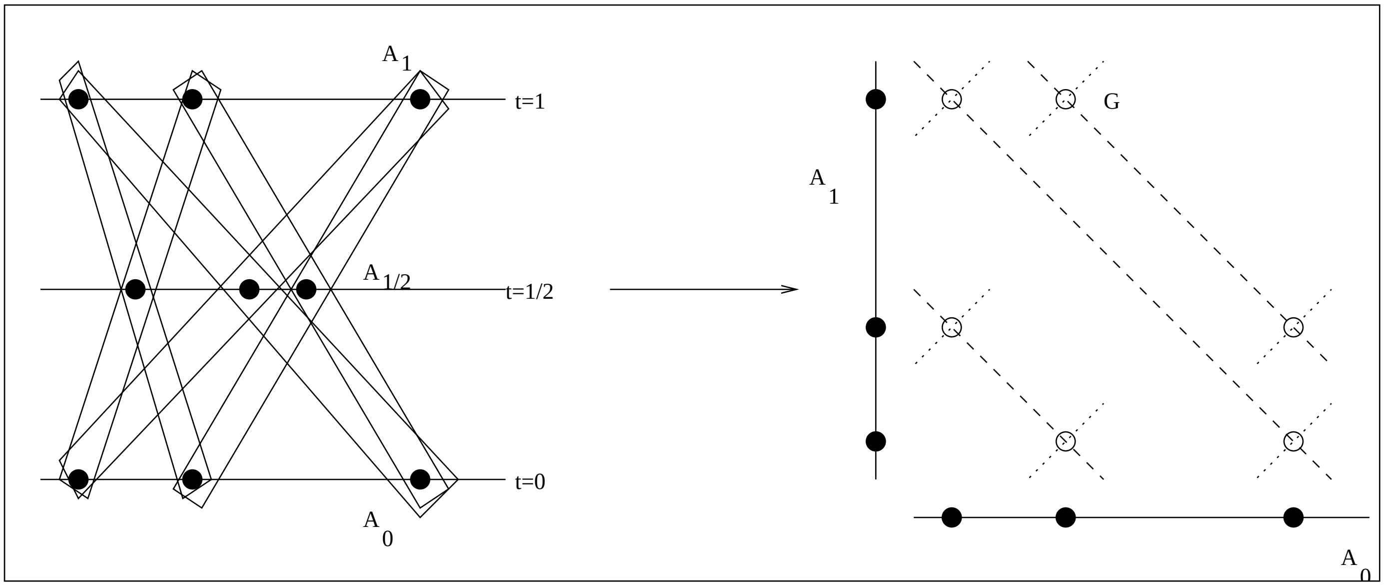


Figure 3. The left picture depicts six tubes T_ω pointing in different directions and the three discretized slices A_0, $A_{1/2}$, A_1, which in this case are three-element sets. The right picture depicts the set G associated to this collection of tubes. Note that the maps $(a,b) \to a$, $(a,b) \to b$, $(a,b) \to a+b$ have small range (mapping to A_0, A_1, and $2A_{1/2}$ respectively), but the map $(a,b) \to a-b$ is one-to-one.

although a formal connection cannot be made because the nature of the intersection of two $\delta \times 1$ tubes depends on the angle between the tubes, whereas the intersection of two lines is a point regardless of what angle the lines make. However, it is plausible that one can use the ideas from combinatorial incidence geometry to obtain progress on this problem. For instance, it is fairly straightforward to show that the Minkowski dimension of Besicovitch sets is at least $(n+1)/2$ purely by using the fact that given any two points that are a distance roughly 1 apart, there is essentially only one $\delta \times 1$ tube which can contain them both.

In the 1990s, work by J. Bourgain, T. Wolff, W. Schlag, A. Vargas, N. Katz, I. Laba, the author, and others pushed these ideas further. For instance, the lower bound of $(n+2)/2$ for the Minkowski dimension was shown in 1995 by Wolff and relies on the δ-discretized version of the geometric statements that every nondegenerate triangle lies in a unique two-dimensional plane and every such plane contains only a one-parameter set of directions. However, there appears to be a limit to what can be achieved purely by applying elementary incidence geometry facts and standard combinatorial tools (such as those from extremal graph theory). More sophisticated geometric analysis seems to reveal that a counterexample to the Kakeya conjecture, if it exists, must have certain rigid structural properties (for instance, the line segments through any given point should all lie in a hyperplane). Such ideas have led to a very small recent improvement in the Minkowski bound to $(n+2)/2 + 10^{-10}$, but they are clearly insufficient to resolve the full conjecture.

The Kakeya problem is a representative member of a much larger family of problems of a similar flavour (but with more technical formulations). For instance, one can define a β-set to be a subset of the plane which contains a β-dimensional subset of a unit line segment in every direction. It is then an open problem to determine, for given β, the smallest possible dimension of a β-set. Low-dimensional examples of such sets arise in the work of H. Furstenberg, and it seems that one needs to understand these generalizations of Besicovitch sets in order to fully exploit the connection between Kakeya problems and oscillatory integrals, which we discuss below. Other variants include replacing line segments by circles or light rays, considering finite geometry analogues of these problems, or replacing the quantity $|\bigcup_\omega T_\omega|$ by the variant $\|\sum_\omega \chi_{T_\omega}\|_p$ (the relevant conjecture here is known as the *Kakeya maximal function conjecture*). Another interesting member of this family is the *Falconer distance set conjecture*, which asserts that whenever E is a compact one-dimensional subset of $\mathbf{R}^2$, the *distance set* $\{|x-y| : x, y \in E\}$ is a one-dimensional subset of $\mathbf{R}$. The discrete version of this is the *Erdős distance problem*—what is the least number of distances determined by n points?—and is also unsolved. For a thorough survey of most of these questions, we refer the reader to [7]; see also [3].

The Connection with Arithmetic Combinatorics

The Kakeya problem looks very geometrical, and it is natural to apply elementary incidence geometry to bear on this problem. Although this approach has had some success, it does not seem sufficient to solve the problem.

In 1998 Bourgain introduced a new type of argument, based on *arithmetic* combinatorics (the combinatorics of sums and differences), which gave improved results on this problem, especially in high dimensions. The connection between Kakeya problems and the combinatorics of addition can already be seen by considering the analogy between line segments and arithmetic progressions. (Indeed, the Kakeya conjecture can be reformulated in terms of arithmetic progressions, and this can be used to connect the Kakeya conjecture to several difficult conjectures in number theory, such as the Montgomery conjectures for generic Dirichlet series. We will not discuss this connection here, but refer the reader to [1].)

Bourgain's argument relies on the following "three-slice" idea. Let Ω and T_ω be as in the previous section. We may assume that the tubes T_ω are contained in a fixed ball. Suppose that $|\bigcup_{\omega\in\Omega} T_\omega|$ is comparable to δ^α for some constant α; our objective is to give upper bounds on α and eventually to show that α must be zero.

By choosing an appropriate set of coordinates, one can ensure that each of the three slices

$$X_t := \left\{x \in \mathbf{R}^{n-1} : (x,t) \in \bigcup_{\omega\in\Omega} T_\omega\right\},$$

$t = 0, 1/2, 1$, has measure comparable to δ^α. Because of the δ-discretized nature of the problem, one can also assume that the discrete set

$$A_t := X_t \cap \delta\mathbf{Z}^{n-1}$$

has cardinality comparable to $\delta^{\alpha+1-n}$ for $t = 0, 1/2, 1$.

Morally speaking, every tube T_ω intersects each of the sets A_0, $A_{1/2}$, A_1 in exactly one point. Assuming this, we see that every tube T_ω is associated with an element of $A_0 \times A_1$. Because two points determine a line, these elements are essentially disjoint as ω varies. Let G denote the set of all pairs of $A_0 \times A_1$ obtained this way. Thus G has cardinality about δ^{1-n}.

The sum set

$$\{a + b : (a,b) \in G\}$$

of G is essentially contained inside a dilate of the set $A_{1/2}$; this reflects the fact that the intersection of T_ω with $A_{1/2}$ is essentially the midpoint of the intersection of T_ω with A_0 and A_1. In particular, the sum set is quite small, having cardinality only $\delta^{\alpha+1-n}$. On the other hand, the difference set

$$\{a - b : (a,b) \in G\}$$

of G is quite large, because the tubes T_ω all point in different directions. Indeed, this set has the same cardinality as G, i.e., about δ^{1-n}.

Thus, if α is nonzero, there is a large discrepancy in size between the sum set and difference set of G. In principle this should lead to a bound on α, especially in view of standard inequalities relating the cardinalities of sum sets and difference sets, such as

$$|A - B| \le \frac{|A + B|^3}{|A||B|}.$$

(A summary of such inequalities can be found in [5].) However, these arguments (which are mostly graph-theoretical) do not seem to adapt well to the Kakeya application, because we are working with only a subset G of $A_0 \times A_1$ rather than all of $A_0 \times A_1$.

To overcome this problem, Bourgain adapted a recent argument of W. T. Gowers which allows one to pass from arithmetic information on a subset of a Cartesian product to arithmetic information on a full Cartesian product. A typical result is:

Theorem. *Let A, B be finite subsets of a torsion-free abelian group with cardinality at most N, and suppose that there exists a set $G \subset A \times B$ of cardinality at least αN^2 such that the sum set $\{a + b : (a,b) \in G\}$ has cardinality at most N. Then there exist subsets A', B' of A, B respectively such that $A' - B'$ has cardinality at most $\alpha^{-13}N$ and A', B' have cardinality at least $\alpha^9 N$.*

Roughly speaking, this theorem states that if most of $A + B$ is contained in a small set, then by refining A and B slightly, one can make *all* of $A - B$ be contained in a small set also. Such results are reminiscent of standard combinatorial theorems concerning the size of sum and difference sets, but the innovation in Gowers's arguments is that the control on A' and B' is polynomial in α. (Previous combinatorial techniques gave bounds which were exponential or worse, which is not sufficient for Kakeya applications.)

Recently [2] Katz and the author have obtained the bound $(4n + 3)/7$ by using control on both the sum set $A_0 + A_1$ and the variant $A_0 + 2A_1$, which corresponds to the slice $A_{2/3}$.

These results have remarkably elementary proofs. Apart from some randomization arguments, the proofs rely mainly on standard combinatorial tools such as the pigeonhole principle and Cauchy-Schwarz inequality, as well as on basic arithmetic facts such as

$$a + b = c + d \iff a - d = c - b,$$

$$a - b = (a - b') - (a' - b') + (a' - b),$$

and

$$a_0 + 2b_0 = a_1 + 2b_1, \quad b_0' = b_1' \\ \implies a_1 - b_1' = 2(a_0 + b_0) - 2b_1 - (a_0 + b_0').$$

Further progress has been made by pursuing these methods, though it seems that we are still quite far from a full resolution of the Kakeya problem, and some new ideas are almost certainly needed.

One possibility may be that one would have to use combinatorial estimates on *product sets* in addition to sum sets and difference sets, since one has control of $\{a + tb : (a,b) \in G\}$ for all

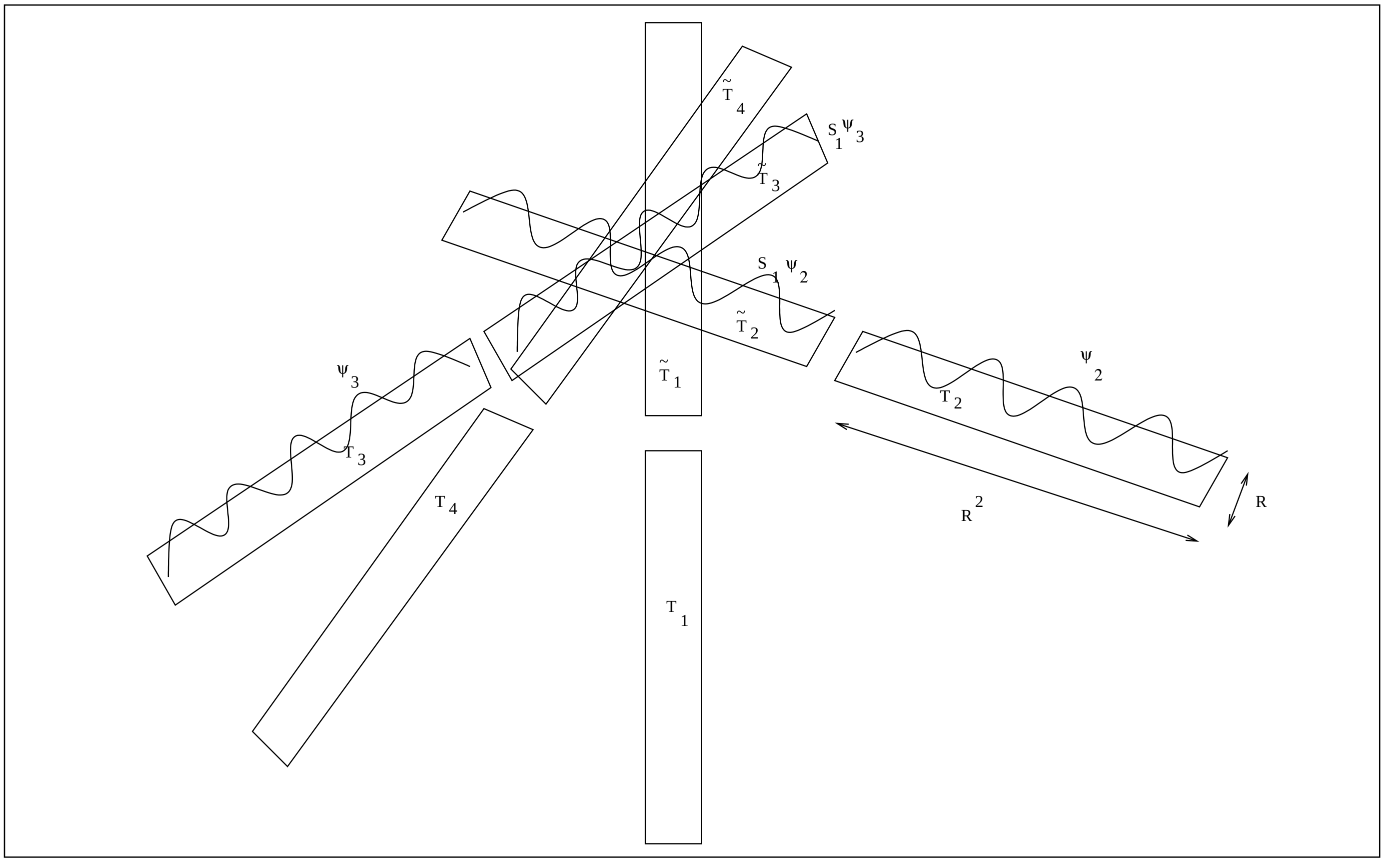


Figure 4. Four tubes T_i, their shifts $\tilde{T}_i$, and the wave packets ψ_i. The interference between the functions $S_1\psi_i$ will cause the L^p norm to be large when $p > 2$.

$t \in [0, 1]$. Discrete versions of such estimates exist; for instance, G. Elekes has recently shown the bound

$$\max(|A \cdot A|, |A + A|) \geq C^{-1}|A|^{5/4}$$

for all finite sets of integers A. However, these bounds do not adapt well to the continuous Kakeya setting because of the difficulty in discretizing both addition and multiplication simultaneously. A good test problem in this setting is the *Erdős ring problem*: Determine whether there exists a (Borel) subring of $\mathbf{R}$ with Hausdorff dimension exactly 1/2. This problem is known to be connected with the β-set problem and the Falconer distance set problem.

Interestingly, the Kakeya problem is also connected to another aspect of arithmetic combinatorics, namely that of locating arithmetic progressions in sparse sets. (A famous instance of this is an old conjecture of Erdős, which is still open, that the primes contain infinitely many arithmetic progressions of arbitrary length.) This difficulty arises in the Hausdorff dimension formulation of the Kakeya problem, and also in some more quantitative variants, because of the difficulty in selecting a "good" set of three slices in arithmetic progression in which to run the above argument. The combinatorial tools developed for that problem by Gowers and others may well have further applications to the Kakeya problem in the future.

Applications to the Fourier Transform

Historically, the first applications of the Kakeya problem to analysis arose in the study of Fourier summation in the 1970s.

If f is a test function on $\mathbf{R}^n$, we can define the Fourier transform $\hat{f}$ by

$$\hat{f}(\xi) := \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} f(x)\, dx.$$

One then has the inversion formula

$$f(x) = \int_{\mathbf{R}^n} e^{2\pi i x \cdot \xi} \hat{f}(\xi)\, d\xi.$$

Now suppose that f is a more general function, such as a function in the Lebesgue space $L^p(\mathbf{R}^n)$. The Fourier inversion formula still holds true in the sense of distributions, but one is interested in more quantitative convergence statements. Specifically, we could ask whether the partial Fourier integrals

$$S_R f(x) := \int_{|\xi| \leq R} e^{2\pi i x \cdot \xi} \hat{f}(\xi)\, d\xi$$

converge to f in, say, L^p norm. (The pointwise convergence question is also interesting, say for L^2 functions f, but this seems extremely difficult to show in two and higher dimensions. In one dimension this was proven in a famous paper by L. Carleson.) By the uniform boundedness principle, this is equivalent to asking whether the linear operators S_R are bounded on $L^p(\mathbf{R}^n)$ uniformly in R. By scale invariance it suffices to show this for S_1:

$$\|S_1 f\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)}.$$

The operator S_1 is known as the *ball multiplier* or (when $n = 2$) the *disk multiplier*, because of the formula

$$\widehat{S_1 f} = \chi_B \hat{f}$$

where B is the unit ball in $\mathbf{R}^n$. In one dimension it is a classical result of Riesz that this operator is bounded on every L^p, $1 < p < \infty$, and so Fourier integrals converge in L^p norm. (Indeed, in one dimension S_1 is essentially the Hilbert transform.) In higher dimensions S_1 is bounded in L^2, thanks to Plancherel's theorem; however, the behaviour in L^p is more subtle. One has an explicit kernel representation which roughly looks like

$$S_1 f(x) \approx \int \frac{e^{\pm i|x-y|}}{(1+|x-y|)^{(n+1)/2}} f(y)\, dy;$$

to be more precise, one must use Bessel functions instead of $e^{\pm i|x-y|}$. The kernel is in L^p only when $p > \frac{2n}{n+1}$, so it might seem natural by duality arguments to conjecture that S_1 is bounded when $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. In 1971, however, C. Fefferman proved the surprising

Theorem. *If $n > 1$, then S_1 is unbounded on L^p for every $p \neq 2$.*

In particular, one does not have L^p convergence for the Fourier inversion formula in higher dimensions unless $p = 2$.

Roughly speaking, the idea is as follows. By duality it suffices to consider the case $p > 2$. Let R be a large number, and let T be a cylindrical tube in $\mathbf{R}^n$ with length R and radius $\sqrt{R}$ and oriented in some direction ω_T. Let ψ_T be a bump function adapted to the tube T, and let $\tilde{T}$ be a shift of T by $2R$ units in the ω_T direction. Then a computation shows that

$$|S_1(e^{2\pi i \omega_T \cdot x}\psi_T(x))| \approx 1$$

for all $x \in \tilde{T}$.

To exploit this computation, one uses the Besicovitch construction to find a collection $\{T\}$ of tubes as above which are disjoint but whose shifts $\tilde{T}$ have significant overlap. More precisely, we assume that

$$\left|\bigcup_T \tilde{T}\right| \leq \frac{1}{K}\sum_T |T|$$

for some K which grows in R (the standard construction in Figure 2 gives $K \sim \log(R)/\log\log(R)$). Then we consider the function

$$f(x) = \sum_T \epsilon_T e^{2\pi i \omega_T \cdot x}\psi_T(x),$$

where $\epsilon_T = \pm 1$ are randomized signs. Using Khinchin's inequality (which roughly states that one has the formula $|\sum_T \epsilon_T f_T| \sim (\sum_T |f_T|^2)^{1/2}$ with very high probability), one can eventually compute that

$$\|S_1 f\|_p \geq C^{-1} K^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}\|f\|_p.$$

Since K is unbounded, we thus see that S_1 is unbounded.

Fefferman's theorem is an example of how a geometric construction can be used to show the unboundedness of various oscillatory integral operators. The point is that while the action of these operators on general functions is rather complicated, their action on "wave packets" such as $e^{2\pi i \omega_T \cdot x}\psi_T(x)$ is fairly easy to analyze. One can then generate a large class of functions to test the operator on by superimposing several wave packets together and possibly randomizing the coefficients to simplify the computation.

The counterexample provided by the Besicovitch construction is very weak (only growing logarithmically in the scale R) and can be eliminated if one mollifies the disk multiplier slightly. For instance, the counterexample does not prohibit the slightly smoother *Bochner-Riesz operator* S_1^ε, defined by

$$\widehat{S_1^\varepsilon(\xi) f} = (1-|\xi|)^\varepsilon \chi_B(\xi)\hat{f}(\xi),$$

from being bounded for $\varepsilon > 0$, because the analogous computation gives

$$|S_1^\varepsilon(e^{2\pi i \omega_T \cdot x}\psi_T(x))| \approx R^{-\varepsilon}$$

for all $x \in \tilde{T}$. Indeed, *the Bochner-Riesz conjecture* asserts that S_1^ε is indeed bounded on L^p for all $\varepsilon > 0$ and $2n/(n+1) < p < 2n/(n-1)$. (For other values of p one needs $\varepsilon > n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$.) This conjecture was proven by L. Carleson and P. Sjölin in 1972 in two dimensions, but the higher-dimensional problem is quite challenging, and only partial progress has been made so far. This conjecture would imply that the partial Fourier integrals will converge in L^p if one uses a Cesàro summation method (such as the Fejér summation method, which corresponds to $\varepsilon = 1$).

The Bochner-Riesz conjecture would be disproved if one could find a collection of disjoint tubes T for which the compression factor K had some power dependence on R as opposed to logarithmic, i.e., if $K \geq C^{-1}R^\varepsilon$ for some $\varepsilon > 0$. A more precise statement is known, namely, that the failure of the Kakeya conjecture would imply the failure of the Bochner-Riesz conjecture. (More succinctly, Bochner-Riesz implies Kakeya.)

In 1991 Bourgain introduced a method in which these types of implications could be reversed, so that progress on the Kakeya problem would (for instance) imply progress on the Bochner-Riesz conjecture. The key observation is that every function can be decomposed into a linear combination of wave packets by applying standard cutoffs both in physical space (by pointwise multiplication) and in frequency space (using the Fourier transform). After applying the Bochner-Riesz operator to the wave packets individually, one has to reassemble the wave packets and obtain estimates for the sum. Kakeya estimates play an important role in this, since the wave packets are essentially supported on tubes; however, this is not the full story, since these packets also carry some oscillation, and one must develop tools to deal with the possible cancellation between wave packets. The known techniques to deal with this cancellation, mostly based on L^2 methods, are imperfect, so that even if one had a complete solution to the Kakeya conjecture, one could not then completely solve the Bochner-Riesz conjecture. Nevertheless, the best-known results on Bochner-Riesz (e.g., in three dimensions the conjecture is known [6] for $p > 26/7$ and for $p < 26/19$) have been obtained by utilizing the best-known quantitative estimates of Kakeya type.

These techniques apply to a wide range of oscillatory integrals. A typical question, the *(adjoint) restriction problem*, concerns Fourier transforms of measures. Let $d\sigma$ be surface measure on, say, the unit sphere S^{n-1}. The Fourier transform $\widehat{d\sigma}$ of this measure can be computed explicitly using Bessel functions and decays like $|x|^{-(n-1)/2}$ at infinity. In particular, it is in the class $L^p(\mathbf{R}^n)$ for all $p > 2n/(n-1)$. The *restriction conjecture* asserts that the same statement holds if $\widehat{d\sigma}$ is replaced by $\widehat{f\,d\sigma}$ for any bounded function f on the sphere. This question originally arose from studying the restriction phenomenon (that a Fourier transform of a rough function can be meaningfully restricted to a curved surface such as a sphere, but not to a flat surface like a hyperplane); it is also related to the question of obtaining L^p estimates on eigenfunctions of the Laplacian on the torus (although the eigenfunction problem is far more difficult due to number theoretic issues), as well as L^p estimates on solutions to dispersive PDE, as we shall see below.

The restriction conjecture is logically implied by the Bochner-Riesz conjecture and is slightly easier to deal with technically. It has essentially the same amount of progress as Bochner-Riesz; for instance, it is completely solved in two dimensions and is known to be true [6] for $p > 26/7$ in three dimensions. One uses the same techniques, namely wave packet decomposition of the initial function f, Kakeya information, and L^2 estimates to handle the cancellation, in order to obtain these results.

There is an endless set of permutations on these types of oscillatory integral problems: more general phases and weights, square function and maximal estimates, more exotic function spaces, bilinear and multilinear variants, etc. There are some additional rescaling arguments available in the bilinear case, as well as some L^2-based estimates, but apart from this there are few effective tools known outside of Bourgain's wave packet analysis to attack these types of problems.

One variant of the above problems comes from replacing Euclidean space by a curved manifold. There are some interesting three-dimensional examples of C. Sogge and W. Minicozzi showing that the Kakeya conjecture can fail on such manifolds, which then implies the corresponding failure of oscillatory integral conjectures such as the natural analogue of Bochner-Riesz. This may shed some light on the robustness of Kakeya estimates and their applications in variable coefficient situations. Certainly the arithmetic and geometric techniques used currently to attack Kakeya problems do not adapt well to curved space.

Applications to the Wave Equation

In the previous section we saw how Kakeya-type problems are related to oscillatory integrals. There is also a similar, and in some sense more natural, connection between Kakeya problems and linear evolution equations such as the free Schrödinger, wave, and Airy equations.

For brevity of exposition we shall restrict our attention to the solutions of the free wave equation

$$u_{tt}(t,x) = \Delta u(t,x); \quad u(0,x) = f(x), u_t(0,x) = 0$$

with initial position f and initial velocity zero. However, much of our discussion has analogues for other linear evolution equations such as the Schrödinger equation.

One can solve for u explicitly using the formula

$$u(t) = \cos(t\sqrt{-\Delta})f,$$

but this does not reveal much information about the size and distribution of u. On the other hand, this formula does show that the wave equation is connected to the oscillatory integral problems mentioned earlier. For instance, the disk multiplier S_1 can be rewritten as

$$S_1 f = \chi_{[-2\pi,2\pi]}(\sqrt{-\Delta})f,$$

since S_1 preserves those Fourier modes $e^{2\pi i x\cdot\xi}$ which are (generalized) eigenfunctions of $\sqrt{-\Delta}$ with eigenvalue in $[-2\pi, 2\pi]$ and eliminates all others. In particular, we have the Fourier representation

$$\begin{aligned} S_1 f &= \int_{-\infty}^{\infty} \frac{\sin(2\pi t)}{\pi t} \cos(t\sqrt{-\Delta})f\, dt \\ &= \int_{-\infty}^{\infty} \frac{\sin(2\pi t)}{\pi t} u(t)\, dt \end{aligned}$$

of the disk multiplier in terms of wave evolution operators.

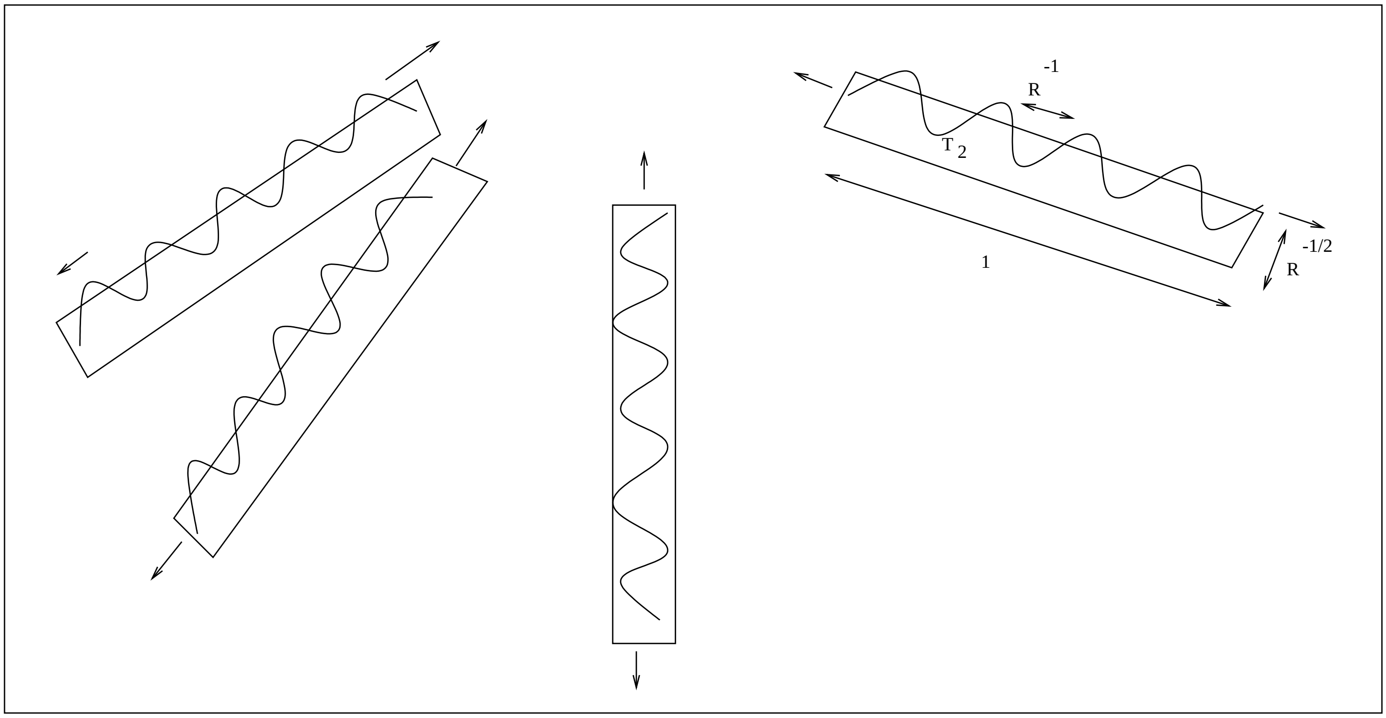


Figure 5. A schematic depiction of the Wolff example at time zero. Because of the zero initial velocity, the wave trains will move in two opposite directions.

A general class of problems is the following: given size and regularity conditions on the initial data f, what type of size and regularity control does one obtain on the solution?

From integration by parts (or from the above explicit formula), one has energy conservation

$$\int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}|u_t(t,x)|^2\, dx = \int \frac{1}{2}|\nabla f(x)|^2$$

for all time t. This conservation law and its generalizations show that u has as much regularity as f when measured in L^2-based spaces. However, L^2 control by itself does not reveal whether u focuses or disperses. To obtain better quantitative control on u, one needs other estimates, such as L^p estimates.

The energy estimate is a *fixed time* estimate: it controls the solution at a specified time t. In the L^p setting, fixed time estimates exist but require a lot of regularity for the initial data and therefore have limited usefulness. A typical estimate is the *decay estimate*

$$\|u(t)\|_{L^\infty(\mathbf{R}^n)} \leq C(1+|t|)^{-(n-1)/2} \sum_{0\leq k\leq s} \|\nabla^k f\|_{L^1(\mathbf{R}^n)}$$

whenever $s > (n+1)/2$ is an integer. The necessity of this many derivatives is demonstrated by the *focussing example*, in which the initial data is spread out near a sphere of radius 1 and the solution u focuses (with an extremely high L^∞ norm) at the origin at time $t = 1$.

However, one can obtain much better estimates, requiring far fewer derivatives, if one is willing to average locally in time. The intuitive explanation for this is that it is difficult for a wave to maintain a focus point (which would generate a large L^p norm for $p > 2$) for any length of time. This phenomenon is known as *local smoothing*. A very useful class of local smoothing estimates is known collectively as *Strichartz estimates*. A typical example of a Strichartz estimate is

$$\|u\|_{L^4_{x,t}(\mathbf{R}^{3+1})} \leq C\|(\sqrt{-\Delta})^{1/2} f\|_{L^2_x(\mathbf{R}^3)}$$

in three spatial dimensions. Without the averaging in time, one would require 3/4 of a derivative on the right-hand side rather than 1/2; this can be seen from the Sobolev embedding theorem. These Strichartz estimates are usually proven by combining the energy and decay estimates with some orthogonality arguments.

However, even Strichartz estimates lose some regularity. One may ask if there are L^p estimates other than the energy estimate which do not lose any derivatives at all. Unfortunately, even if one localizes in time and assumes L^∞ control on the initial data, one still cannot do any better than L^2 control, as the following result of Wolff shows:

Theorem. *If $n > 1$ and $p > 2$, then the estimate*

$$\|u\|_{L^p([1,2]\times\mathbf{R}^n)} \leq C\|f\|_{L^\infty(B(0,1))}$$

cannot hold for all bounded f on the unit ball.

The argument proceeds similarly to Fefferman's disk multiplier argument. Let $\{T\}$ be a collection of disjoint tubes arranged using the Besicovitch set construction as in Fefferman's argument, except that we rescale the tubes to have dimensions $1 \times R^{-1/2}$ rather than $R \times \sqrt{R}$. On each of these

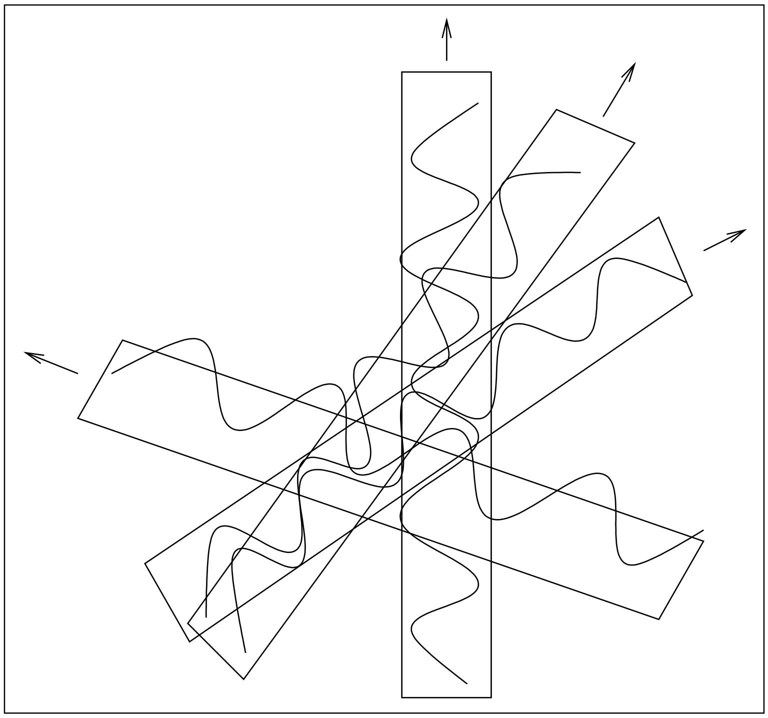


Figure 6. The Wolff example at a time $1 \le t \le 2$; only the incoming wave trains are shown.

tubes T we place a "wave train", which is basically $e^{iRx\cdot\omega_T}$ times a bump function adapted to T. Let f be the sum of all these wave trains (although we may randomize the signs of these trains to simplify computations).

At time zero, the function f has low L^∞ norm. However, as time evolves, each wave train T splits as the superposition of two pulses, one moving in the direction ω_T and the other in the direction $-\omega_T$. For times $1 \le t \le 2$ a large portion of the wave train at T now lives in the shifted tube $\tilde{T}$. Because of the large overlap of these tubes, the L^p norm of u is large for all $1 \le t \le 2$; as with Fefferman's argument, it is about $K^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}$. By letting $R \to \infty$, one can make K unbounded, and this gives the theorem.

Because the Besicovitch construction has a logarithmic compression rate, one could get around this obstruction by requiring an epsilon of regularity on the initial data. The *local smoothing conjecture* of C. Sogge asserts that no further loss of regularity occurs or, more precisely, that

$$\|u\|_{L^p([1,2]\times\mathbf{R}^n)} \le C_{p,\varepsilon}\|(1+\sqrt{-\Delta})^\varepsilon f\|_{L^p(\mathbf{R}^n)}$$

for all $\varepsilon > 0, n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}$, and $2 \le p \le \infty$. This conjecture is easy when $p = 2$ or $p = \infty$; the most interesting case is when $p = 2n/(n-1)$.

The local smoothing conjecture is extremely strong and would imply many of the known estimates on the wave equation. It implies the Kakeya conjecture, for a counterexample to the Kakeya conjecture could be used to strengthen Wolff's argument to disprove the local smoothing conjecture. This conjecture also implies the Bochner-Riesz conjecture; the idea is to write the Bochner-Riesz multiplier S_1^ε in terms of wave operators $\cos(t\sqrt{-\Delta})$ in the manner briefly discussed earlier. However, the conjecture is far from settled; even in two dimensions the conjecture is completely proven for only $p > 74$ (due to T. Wolff), and at the critical exponent $p = 4$ the conjecture is known only for $\varepsilon > 1/8 - 1/88$ [6], [8]. These estimates are all proven via Kakeya methods. Briefly, the idea is to decompose the initial data f (and hence the solution u by linearity) into pieces which are localized in both space and frequency. This decomposes the solution u into wave packets, which are oscillatory functions which travel along light rays. One then uses Kakeya-type arguments to control how many times these light rays intersect each other, together with orthogonality arguments to control to what extent the oscillations of each wave packet reinforce each other. A recently developed, but apparently quite powerful, technique here is *induction on scales*, in which one assumes that the desired estimate is already proven at smaller scales and uses this hypothesis together with the Kakeya strategy just discussed to obtain the same estimate at higher scales.

There are several other wave equation estimates which are related to those discussed here. An active area of research is to obtain good *bilinear* or even *multilinear* estimates on solutions to the wave equation, as opposed to the linear estimates described here; these estimates have direct application to nonlinear wave equations, since one can often use techniques such as the method of Taylor series to write the solution of nonlinear wave equations as a series of multilinear expressions of solutions to the linear wave equation. There are some tantalizing hints that these Kakeya techniques could also be used to handle nonlinear wave equations directly (or wave equations with rough metrics, potential terms, etc.), but these ideas are still in their infancy.

Since u can be written in terms of circular averages of f, there is also a close relationship between wave equation estimates and estimates for circular means. (Such circular means estimates can then be used, for instance, to make progress on the Falconer distance problem mentioned earlier.) There is also an extremely strong square function estimate conjectured for the wave equation which, if true, would imply the local smoothing, Bochner-Riesz, restriction, and Kakeya conjectures. It would also give estimates for other seemingly unrelated objects such as the helix convolution operator $f \mapsto f * d\sigma$, where $d\sigma$ is arclength measure on the helix $\{(\cos t, \sin t, t) : 0 \le t \le 2\pi\}$ in $\mathbf{R}^3$. (The connection arises because the Fourier transform of $d\sigma$ is concentrated near the light cone.) These estimates are quite difficult, and the partial progress which has been made on them

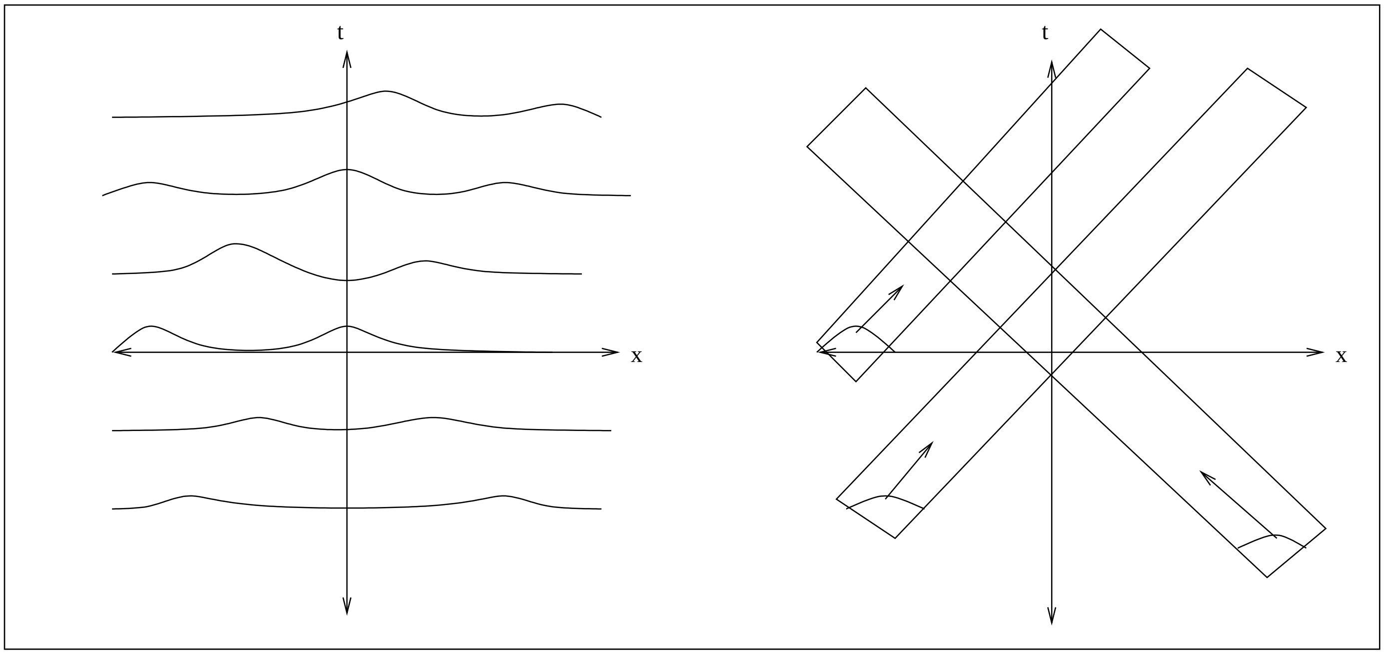


Figure 7. A schematic depiction of how a wave (such as the one drawn on the left) can be written as a superposition of wave packets or "photons". These objects are localized in space and also have a localized direction, and different wave packets are essentially orthogonal. There is no canonical way to perform this decomposition, but one usually uses a combination of spatial cutoffs and cutoffs in Fourier space.

has proceeded via Kakeya estimates. Although these deep wave equation estimates have not yet found significant applications, I am confident that they will do so in the near future.

Acknowledgments

I thank Nets Katz for some helpful references and the referees for very helpful comments. The author is a Clay long-term Prize Fellow and is supported by grants from the Sloan and Packard Foundations. This article is based on a talk given at the Clay millennium event.

References

[1] J. Bourgain, Harmonic analysis and combinatorics: How much may they contribute to each other?, *Mathematics: Frontiers and Perspectives*, IMU/Amer. Math. Soc., 2000, pp. 13–32.

[2] N. Katz and T. Tao, Bounds on arithmetic projections, and applications to the Kakeya conjecture, *Math. Res. Lett.* **6** (1999), 625–630.

[3] ____ , Some connections between the Falconer and Furstenberg conjectures, in preparation.

[4] S. Klainerman and S. Selberg, Bilinear estimates and applications to non-linear wave equations, preprint.

[5] I. Ruzsa, Sums of finite sets, *Number Theory: New York Seminar* (D. V. Chudnovsky, G. V. Chudnovsky, and M. B. Nathanson, eds.), Springer-Verlag, 1996.

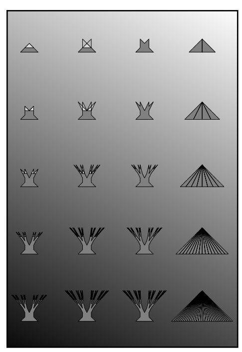
[6] T. Tao and A. Vargas, A bilinear approach to cone multipliers I, II, *GAFA* **10** (2000), 185–258.

[7] T. Wolff, Recent work connected with the Kakeya problem, *Prospects in Mathematics* (Princeton, NJ, 1996), Amer. Math. Soc., Providence, RI, 1999, pp. 129–162.

[8] T. H. Wolff, A sharp bilinear cone restriction estimate, *Ann. of Math.*, to appear.

About the Cover

Around 1926 A. S. Besicovitch showed that there exist planar regions of arbitrarily small area in which one could rotate a segment of fixed length, thus solving what is now known as the Kakeya needle problem. As Terry Tao's article explains, even nowadays Besicovitch's Theorem and variations on it are a fruitful source of analysis.



One of the principal steps in Besicovitch's proof was the construction of certain complicated regions of arbitrarily small area in the plane containing needles of a fixed length and a range of directions. The original construction of such regions was quite complicated. Already in the next year's volume of the *Mathematische Zeitscrift,* O. Perron exhibited a simpler one, involving what are now called "Perron trees", and in the early 1960s I. J. Schoenberg simplified Perron's construction in turn, showing how these trees could be constructed recursively by what he called "sprouting". The cover illustrates Schoenberg's construction and demonstrates visually that the ratio of the areas of the regions in the third and fourth columns has limit 0.

The Mathematical Association of America received a grant from the NSF around 1960 to make a film about this topic. What was essentially a transcript of the film appeared in an often cited article by Besicovitch ("The Kakeya problem", volume 70 of the *American Mathematical Monthly*, 1963, pp. 697-706). The film itself may have been the first professionally produced mathematical animation. Are there any viewable copies of the film left? Is there anybody still around who took part in that project?

—*Bill Casselman* (`covers@ams.org`)