Opinion

In Praise of the Bull Session

Once, I resolved to teach myself German. On page five of a textbook I found the explanation, "German is a declined language...." The next page contained a set of paradigms for declension of German nouns and articles. On the bottom of the page the author had written, "So now there should be no question about the declension of the German noun."

But of course there were questions. Anyone who has struggled to learn a language will recognize that the matter of grammatical form is not settled until it has been integrated on a very low cognitive level, until it has been grasped by both the mind and the heart. Some of this process can happen through introspection, but a powerful tool, for many, is discourse with others.

This column is about learning mathematics, not German. It is prompted by an anonymous quote cited in these pages:[1] "A definition is not something that can be discovered in a bull session." My years in the classroom have convinced me otherwise: that dialogue with other students (what the anonymous writer calls a "bull session") is in fact a powerful tool for learning mathematics, and particularly mathematical definitions, and that the quotation conveys a serious misunderstanding of the processes of teaching and learning.

A definition is logically arbitrary, a matter of convenience. On one level, "learning" it can be accomplished by simple memorization. But the meaning of a definition has not really been learned until it has been integrated on a very low cognitive level, until it has been grasped by both the mind and the heart. Some of this process can happen through introspection, but a more important path, for many, is discourse with others.

Here are some examples of definitions, drawn from a variety of disciplines:

—An *ordered pair* (a, b) is the set $\{a, \{a, b\}\}$.
—A *function* $f : A \to B$ is a set of ordered pairs (a, b), where $a \in A$ and $b \in B$ such that (i) for all $a \in A$ there exists an $(a, b) \in f$, and (ii) if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.
—A *buffer* is a section of memory that loses its contents when read.
—A *plant* is an organism whose cells contain a cell wall.

These definitions can be memorized, but to understand how the definition works, how it separates out a useful class of objects from other objects, how it describes precisely something of which we have, a priori, an intuitive recognition, requires considerable thought.

And definitions are not constant. The middle school student learns that the sine of an angle is a certain ratio in a right triangle. The high school student redefines the sine as the ordinate of a point on the unit circle. The college student may define the sine as the sum of a certain infinite series or as a certain rational function of e^{iz}. Definitions grow with the student's understanding.

[1]*Joan Ferrini-Mundy,* Principles and Standards for School Mathematics: A Guide for Mathematicians, Notices*, September 2000, page 874; Wayne Bishop,* Another View of PSSM, Notices*, January 2001, page 6.*

Historically, many mathematical definitions have been the outcome of significant discussion. The definition of an abstract group or a topology did not solidify into the present form until the beginning of the last (twentieth) century. The definition of a continuous function emerged from more than 150 years of dialogue among mathematicians who are remembered, partly, for just this dialogue.

As a teacher I find that the "bull session" is among my most useful tools. One student can formulate a question that another has been struggling to ask. A student can give an explanation in slightly different words from the teacher, which will reach some students who did not get the idea the first time. Students can ask each other questions that they might be embarrassed to ask the teacher. Most importantly, students are exposed to a variety of ways to express the understanding encapsulated in the definition.

The importance of this process is not limited to definitions. I find that dialogue is of great importance in helping students integrate axioms, theorems, conjectures, and proofs. Again, I am guided partly by the history of the subject. The classic expositions and results did not spring from their creator's forehead full grown. They developed as a result of deep reflection and prolonged discourse.

Why is this view of mathematics so rarely taken by mathematicians? I can think of two reasons. First, the use of discourse in the classroom is sometimes misunderstood to imply that "anything goes". No: the aim of discourse, however wild and wide-ranging, is to come to an understanding of a statement that others have already arrived at. The role of the teacher is to channel the discussion so that it advances this process. A fundamental and hard-won skill of the master teacher is to manage the tension between this role and the need to get students talking about their own, sometimes useless, conceptions.

The second reason that the social nature of mathematical discovery is overlooked is that the process often does not feel like a social one. You struggle with a problem: you take it on the bus, into the shower, even into bed; you try out this or that approach, and, eventually, with luck you get some insight. Where is the social aspect to this process?

Well, where does the insight come from? It is likely that the process of mathematical discovery is a deeply unconscious one. We may be able to explain how our discovery makes sense, but we often cannot explain how we thought of it. And it is often the case, when we reflect on our thought, that the germ of the idea came from outside us. That is, the idea developed through some form of discourse, perhaps through reading or writing, with others.

As the reader might guess, my German is still primitive. My rote memorization of verb forms has not been given meaning through their use in dialogue. And my mind's hold on what I have managed to memorize is weak, unsupported by the network of associations that such dialogue would create. In contrast, the "bull session" is alive and well in my classroom and plays a significant role in helping my students reach a level of understanding of mathematics that cannot be achieved by rote.

—Mark Saul
Associate Editor

Letters to the Editor

Smale and Politics

Robion Kirby, in his review of Batterson's biography of Steve Smale (December *Notices*, pages 1408–1411), displays magnanimous generosity toward the faults of the American Right. After all, forty-odd years have passed since the Red-hunt expelled us and others from American universities, and threatened Smale and many more with the same expulsion. Kirby has by now achieved the detachment to forgive the expellers: their motives were so pure, and those they attacked so vile, that "means even as extreme as McCarthyism" become understandable.

Indeed. What does your avowed libertarianism mean, Professor Kirby? That a little tyranny is a good thing, provided it is applied against us?

Moving to the 1960s, Kirby is charitable toward the American war in Vietnam, saying it was possibly "correct (perhaps necessary, perhaps even honorable)"—though he concedes it was harmful, and he might have noted that it was certainly illegal, as the Constitution allows the executive to make war only under a declaration of war by Congress. Kirby goes on to snipe at Smale as the wrong kind of anti-war protester.

Well, we may be awfully slow forgetters, but we still think the Vietnam war was evil, and we still honor Steve Smale for his labors to stop it.

—Chandler Davis
University of Toronto
—Lee Lorch
York University
(Received January 16, 2001)

Response to Davis-Lorch Letter

The Davis-Lorch letter, particularly in its first paragraph, totally misrepresents what I wrote, and I urge the readers to consult my book review and decide for themselves.

—Rob Kirby
University of California, Berkeley
(Received February 19, 2001)

Listing Foreign Ph.D.'s

I noted with some interest the listing in the February *Notices* of Ph.D.'s awarded by U.S. institutions in the 1999–2000 academic year. I was interested to see who had graduated from certain schools I have been associated with and in what areas their theses were. I noted a few graduate students I was personally acquainted with who had graduated, and looked for a couple of names of former undergraduates who might have completed degrees. The list is also a good reference, in addition to institutional Web pages, for information regarding areas of graduate study for students contemplating graduate school. Finally, I can see the list playing a role for both individuals and institutions in the employment market.

I wonder if it might not be a good idea to extend the list somewhat. I have similar interests in relation to Canadian universities. I know someone pursuing a degree in Canada and would be interested to note when he finishes. Sometimes students want to apply to graduate school there, or we get applications for faculty positions from people who studied in Canada. Also, it would be good to get notice of doctoral dissertations of other U.S. citizens (who would typically be AMS members) who recently completed dissertations in other countries.

In truth, it is partly the omission of my name many years ago when I completed my doctorate at the University of London that prompts these suggestions.

—Walter S. Sizer
Minnesota State University, Moorhead
(Received January 24, 2001)

Massachusetts Signing Bonus

I was quite disturbed by your "Mathematical Opportunities" announcement of the "Massachusetts Signing Bonus Program" in the February 2001 *Notices*. There are many reasons that this controversial program should not have been included as such an "announcement", among them:

1. It is a political ploy to legitimize this hotly debated program to draw "professionals" into high school teaching via $20,000 signing bonuses,

2. Its purpose is the recruitment of high school teachers—straying quite far from the mission of the *Notices* as far as I can tell, and,

3. It's an advertisement!! Our college (a Massachusetts public college) had to pay hundreds of dollars to advertise our open, tenure-track position for Ph.D. mathematicians on EIMS (Employment Information in the Mathematical Sciences) and would have had to pay several hundred dollars more to have an ad in the *Notices*.

In the future I hope you would filter political propaganda a bit more judiciously.

—Julian F. Fleron
Westfield State College
(Received January 25, 2001)

The *Notices* invites readers to submit letters and opinion pieces on topics related to mathematics. Electronic submissions are preferred; see the masthead for addresses. Opinion pieces are usually one printed page in length (about 800 words). Letters are normally less than one page long, and shorter letters are preferred.