Curvature, Combinatorics, and the Fourier Transform

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This article is dedicated to the memory of Thomas Wolff

In their 1978 paper "Problems in harmonic analysis related to curvature" [SteinWainger78], E. M. Stein and S. Wainger studied a variety of operators defined over hypersurfaces and other lower-dimensional submanifolds of $\mathbb{R}^d$. The recurring theme is that the behavior of these operators is governed, to various degrees, by the Gaussian curvature of the underlying manifold—the determinant of the differential of the Gauss map taking a point on the manifold to the unit normal at that point. In recent years these ideas have undergone further development, not only in the context of harmonic analysis, but also in related problems in combinatorics and analytic number theory. This article is not intended to be an exhaustive survey of the most recent advances in these areas. Rather, the purpose of this article is to describe some of these connections by way of examples accessible to a wide range of mathematicians.

The Erdős Distance Problem

One of the most dramatic examples of the role of Gaussian curvature occurs in a discrete setting. When S is a set of N distinct points in $\mathbb{R}^d$, let $D(S)$ denote the set of distances between pairs of points of S: namely, $D(S) = \{|x-y| : x, y \in S\}$, where $|\cdot|$ is a distance function. The total number of distances between different elements of S is, of course, $\frac{N!}{2!(N-2)!}$. However, some of these distances may be equal. P. Erdős posed the question: What is the smallest number of distinct distances that can actually occur when the cardinality of S is large? Although this problem is open, the answer remarkably depends on whether the "distance" between x and y is the standard Euclidean distance

$$\sqrt{(x_1-y_1)^2+\cdots+(x_d-y_d)^2}$$

or the taxi-cab distance $|x_1-y_1|+\cdots+|x_d-y_d|$. The reason is that the boundary of the unit ball in the Euclidean distance has nonvanishing curvature, while the curvature of the boundary of the unit ball in the taxi-cab distance vanishes at most points (the points where it is defined). To put this bluntly, squares have flat sides, and circles have curved boundaries. This simple-looking theme drives most of the concepts described in this article.

Let $i_d(N,2)$ denote the minimum possible number of distinct Euclidean distances between points of a set of cardinality N in $\mathbb{R}^d$. Similarly, let $i_d(N,1)$ denote the minimum possible number of distinct taxi-cab distances. We will first show that $i_d(N,p) \gtrsim N^{\frac{1}{d}}$ for both $p=1$ and $p=2$. Then we will show that the presence of curvature when $p=2$ allows for a significant improvement in this estimate.[1]

Curvature-Independent Estimate

This was first proved by Erdős in 1946 [Erdős46]. Suppose we are working in the plane, so that $d=2$. If N points are specified, then their convex hull is some polygon. Let P_1 denote an arbitrary vertex of that polygon, and let K denote the number of different distances occurring among the distances P_1P_i for $i=2,\dots,N$. If M is the maximum number of times that the same distance occurs, then $KM \geq N-1$. If r is a distance that occurs M times,

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[1]*Notation: Throughout this article $a \lesssim b$ means that there exists a uniform constant C such that $|a| \leq C|b|$. The symbol $a \gtrsim b$ is defined analogously, and $a \approx b$ means that both $a \lesssim b$ and $a \gtrsim b$.*

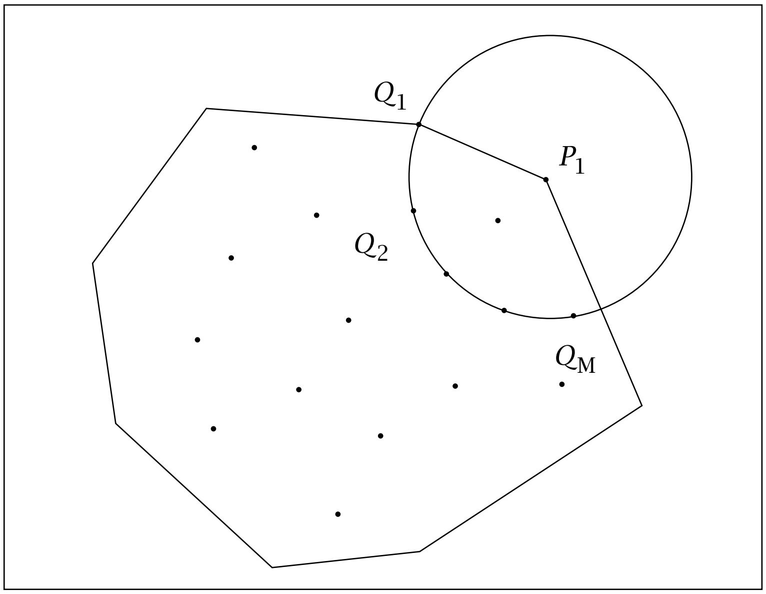

Figure 1.

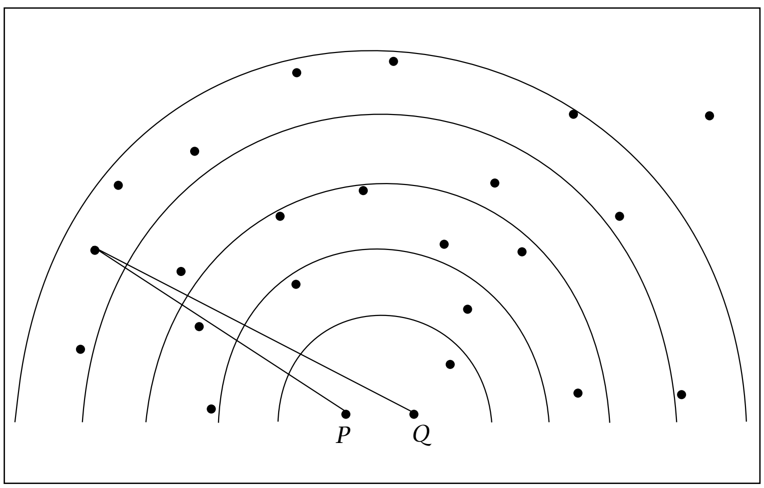

Figure 2.

then there are M of the points on the circle with center P_1 and radius r, all of which lie on the same semicircle, since P_1 is a vertex of a convex polygon containing the P_i. (See Figure 1.) We may label these points by $Q_1, \dots, Q_M$ in such a way that $Q_1Q_2 < Q_1Q_3 < \cdots < Q_1Q_M$. It follows that $i_2(N, p) \geq \max\{M - 1, \frac{N-1}{M}\}$, which is minimal when $M(M - 1) = N - 1$. Thus $i_2(N, p) \gtrsim N^{\frac{1}{2}}$. A similar argument in higher dimensions proves that $i_d(N, p) \gtrsim N^{\frac{1}{d}}$. (When $p = 1$, namely, when we are working with the taxi-cab metric, this argument needs a slight modification because the string of inequalities $Q_1Q_2 < Q_1Q_3 < \cdots < Q_1Q_M$ need not hold. However, these inequalities do hold if the points $Q_1, \dots, Q_M$ lie on a single edge of the "circle" in the taxi-cab metric. We cannot be sure that they all lie on the same edge, but there must be an edge that contains at least a quarter of these points, and the rest of the argument goes through.)

What Makes Us Think That Curvature Will Help?

Consider a subset of the two-dimensional integer lattice $L_N = \{(k_1, k_2) \in \mathbb{Z}^2 : 0 \leq k_i \leq \sqrt{N}\}$. If we consider the $l^1(\mathbb{R}^2)$ (taxi-cab) distance defined above, it is not difficult to see that the number of distinct distances among the points of L_N is, approximately, a constant multiple of $\sqrt{N}$. It follows that $i_2(N, 1) \lesssim N^{\frac{1}{2}}$, which in view of the previous paragraph means that $i_2(N, 1) \approx N^{\frac{1}{2}}$.

On the other hand, when the distance is Euclidean ($p = 2$), a fact from elementary number theory (see, e.g., [Landau69]) says that the number of distances among the points of L_N is, approximately, $N/\sqrt{\log(N)}$. This example leads to a conjecture that $i_2(N, 2) \gtrsim N/\sqrt{\log(N)}$, which is much better than the result for $i_2(N, 1)$.

Curvature to the Rescue

The conjecture we just formulated is still open. However, we can use an argument of L. Moser to obtain a significant improvement of the curvature-independent estimate $i_d(N, 2) \gtrsim N^{\frac{1}{d}}$ by exploiting the nonvanishing Gaussian curvature of the ball in the Euclidean metric.

For simplicity, suppose that the dimension $d = 2$, and let S be a planar set of cardinality N. Via dilation, we may assume that the minimal distance between distinct points of S is 1. Moreover, suppose that S is well distributed in the sense that every Euclidean ball of radius 10 whose center lies in the convex hull of S contains points of S. (This latter assumption does not hold in general, but it is valid in the applications we shall discuss later in this article.) For example, the lattice L_N defined above easily satisfies these conditions. We shall see[2] that $i_2(N, 2) \gtrsim N^{\frac{3}{4}}$. Indeed, choose two points P and Q in S such that $|P - Q| = 1$. Draw enough concentric half-annuli of width 1 centered at the midpoint of P and Q to cover a positive fraction of the points in S. (See Figure 2.) Let us consider the "worst case" scenario where we need $\approx N^{\frac{1}{2}}$ such annuli. At the moment, by measuring distances from the points in these annuli to either P or Q, we already have $\approx N^{\frac{1}{2}}$ distances, which is exactly the result obtained above without any assumptions on the distance function. However, we are using the Euclidean distance function, so it is time to take advantage of it!

By the pigeon-hole principle a positive proportion of our annuli actually have $\approx N^{\frac{1}{2}}$ points, since the cardinality of S is N. Consider one of these annuli more closely. Suppose that by measuring distances to P and Q we find that the points of this annulus contribute exactly k distances d_1,

[2] *This result, and in fact a stronger estimate $i_2(N, 2) \gtrsim N^{\frac{6}{7}}$ due to J. Solymosi and C. D. Tóth, still holds without the assumption that S is well distributed. An earlier estimate with the exponent $\frac{4}{5}$ is due to F. Chung, E. Szemerédi, and W. T. Trotter.*

$d_2, \dots, d_k$. Let A_i ($i = 1, 2, \dots, k$) denote the set of points in the annulus whose distance to P is d_i, and let $B_1, B_2, \dots, B_k$ be defined similarly with respect to Q. By the pigeon-hole principle, perhaps after a relabeling of the A_i's and B_i's, at least one A_i contains $\gtrsim \frac{N^{\frac{1}{2}}}{k}$ elements of S. On the other hand,

$$A_i = \bigcup_{j=1}^{k} A_i \cap B_j, \tag{1.1}$$

and each of the intersections can support at most one point, since the intersection of two semicircles with different centers is at most one point. This is where the curvature is being used, since a similar assertion about two circles with respect to the taxi-cab distance would be false! It follows from (1.1) that

$$\frac{N^{\frac{1}{2}}}{k} \lesssim k, \tag{1.2}$$

which implies that $k \gtrsim N^{\frac{1}{4}}$. We conclude that $i_2(N, 2) \gtrsim N^{\frac{3}{4}}$, as claimed. It is straightforward to check that when $p > 1$, the argument for the Euclidean distance works just as well for the $l^p(\mathbb{R}^d)$ distance defined in dimension d by

$$(|x_1 - y_1|^p + \cdots + |x_d - y_d|^p)^{\frac{1}{p}}.$$

Thus, $i_d(N, p) \gtrsim N^{\frac{1}{d}}$ when $p \geq 1$, and $i_d(N, p) \gtrsim N^{\frac{3}{2d}}$ when $1 < p < \infty$. In other words, even a little bit of curvature suffices to improve the estimate.

Applications to the Fuglede Conjecture

Which domains can be used to tile Euclidean space? For example, we can tile the plane either with squares or with hexagons. B. Fuglede conjectured that domains whose translates tile Euclidean space can be characterized through Fourier analysis. A domain D in $\mathbb{R}^d$ is called *spectral* if there exists a discrete subset A of $\mathbb{R}^d$ such that the set of exponential functions $\{e^{2\pi i x \cdot a}\}_{a \in A}$ forms an orthogonal basis for the space $L^2(D)$ of square-integrable functions on D.

In the celebrated example where D is the cube $[0, 1]^d$, one can take the set A to be the integer lattice. If D is the regular hexagon, one can take A to be the "honeycomb" lattice. In both cases, D tiles $\mathbb{R}^2$ by translation (that is, one can cover $\mathbb{R}^2$ by translates of D without overlaps, except possibly at the boundary). On the other hand, Fuglede proved by a direct computation that the disc is not spectral. Since the disc does not tile $\mathbb{R}^2$ by translation, this led Fuglede to conjecture that D is spectral if and only if D tiles by translation. Fuglede proved this conjecture under an additional hypothesis that either a spectrum or a tiling set for D is a lattice, in which case the problem essentially reduces to the Poisson Summation Formula, which relates the sum of a function restricted to the integer lattice to the sum of the Fourier transform of this function restricted to the appropriately defined dual lattice.

The Fuglede conjecture is far from being proved in any dimension, though there is considerable recent progress.[3] N. Katz, S. Pedersen, and the author proved that the unit ball B_d is not spectral in any dimension $d \geq 2$. Let us see how curvature enters the proof. Suppose that B_d were spectral, and let A denote a putative spectrum. We want to estimate both the number of points of A in a cube of sidelength R and the number of distances between these points. Our method is to study the Fourier transform of the characteristic function χ_{B_d} of B_d (the function that is equal to 1 when $x \in B_d$ and 0 otherwise). Recall that the Fourier transform is defined by the formula

$$\widehat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x)\, dx. \tag{1.3}$$

From the assumption that the set of exponentials $\{e^{2\pi i x \cdot a}\}_{a \in A}$ is complete, it is not difficult to show that $\sum_{a \in A} |\widehat{\chi}_{B_d}(\xi - a)|^2 \equiv 1$. Next, we claim that the cardinality of the intersection of A with a cube of sidelength R is $\approx R^d$. To see this, consider a cube $Q_r(y)$ of sidelength r centered at any point y in this intersection. We have

$$1 \equiv \sum_{a \in A} |\widehat{\chi}_{B_d}(\xi - a)|^2 = \sum_{a \in Q_r(y) \cap A} |\widehat{\chi}_{B_d}(\xi - a)|^2 + \sum_{a \notin Q_r(y)} |\widehat{\chi}_{B_d}(\xi - a)|^2.$$

One can check that if A is a spectrum, then the second sum is smaller than $\frac{1}{2}$ when r exceeds some number r_0 depending only on the geometric properties of B_d. It follows that $Q_r(y) \cap A$ is not empty for $r > r_0$, and our claim is established if R is sufficiently large.

The explicit form of the Fourier transform $\widehat{\chi}_{B_d}(\xi)$ of the characteristic function of the unit ball is

$$\int_{B_d} e^{-2\pi i x \cdot \xi}\, dx. \tag{1.4}$$

The assumption of orthogonality implies that if a and a' are distinct points of A, then the integral in (1.4) equals 0 when $\xi = a - a'$. It follows that the distance between any two points of A is bounded below by some positive number. It is a classical result that the Fourier transform of a radial function (a function that depends only on the Euclidean distance of its argument from a point) is also radial. This tells us that the integral in (1.4) depends only on $|\xi|$. Moreover, the zeroes of this function are very close to the zeroes of $\cos\left(|\xi| - \frac{\pi d}{4}\right)$. See, for example, [SteinWeiss71] for this fact and related background. It follows that in the intersection of

[3] *A number of interesting breakthroughs on the Fuglede conjecture have recently been made by P. Jorgensen, M. Kolountzakis, I. Laba, J. Lagarias, J. Reeds, T. Tao, Y. Wang, and others.*

A and the cube of radius R there are $\approx R^d$ points connected by only $\approx R$ distinct distances. We saw in the previous section that the curvature of the Euclidean distance function makes this impossible.

Distribution of Lattice Points in Convex Domains

Gauss observed in the middle of the nineteenth century that if D is a convex domain in $\mathbb{R}^d$, then the number $N(t)$ of lattice points inside the scaled domain tD equals $t^d|D|$, up to an error $R(t) \lesssim t^{d-1}$. This is because the error cannot exceed the number of lattice points that live a distance at most $\frac{\sqrt{d}}{2}$ away from the boundary. In general, this result cannot be improved. We will show that curvature leads to a better bound on the error term $R(t)$.

If we take D to be the square $[-1,1]^d$, then if t is an integer, the number of lattice points on the boundary of tD is $\approx t^{d-1}$. It follows that the estimate $R(t) \lesssim t^{d-1}$ cannot be improved for the square. Let us now try to take advantage of curvature. The estimate we just used in the case of the square does not work for the unit ball, for example, because, roughly speaking, the boundary of the ball is curved, which allows it to avoid collecting many lattice points. It is a classical result due to E. Landau that if the boundary of D has everywhere nonvanishing Gaussian curvature, then $R(t) \lesssim t^{d-2+\frac{2}{d+1}}$.

V. Jarník proved in dimension 2 that the power $d-2+\frac{2}{d+1}$ is "natural" in the sense that one can construct a convex domain of diameter $\approx t$ such that the curvature is bounded below by $\approx \frac{1}{t^2}$ and the boundary contains $\approx t^{\frac{2}{3}}$ lattice points. Jarník's construction was recently refined by E. Sawyer, A. Seeger, and the author. They constructed a *fixed* convex planar set D, whose boundary has everywhere nonvanishing Gaussian curvature, such that the boundary of tD contains $\approx t^{\frac{2}{3}}$ lattice points for a sequence of t's going to infinity. However, for planar domains with some smoothness, better estimates on $R(t)$ are possible. A recent result of M. Huxley says that if the boundary has everywhere nonvanishing curvature and is three times continuously differentiable, then $R(t) \lesssim t^{\frac{46}{73}}$. In higher dimensions, there has also been a series of improvements by E. Krätzel and W. G. Nowak and, most recently, by W. Müller. The case of the ball is completely solved in dimensions ≥ 4, partly because the number-theoretic problem of determining the number of ways of writing an integer as a sum of four or more squares is sufficiently well understood. See [Huxley96] and the references contained therein.

Let us briefly get technical to see how to exploit curvature to prove Landau's estimate for the remainder term $R(t)$. Suppose that the Gaussian curvature of the boundary of D is bounded below by 1. Let ρ be a smooth compactly supported function whose integral is 1, and let $\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$. The convolution of two functions f and g is defined by

$$(f * g)(x) = \int f(x-y)g(y)\,dy. \tag{2.1}$$

Convolving with the approximate identity ρ_ϵ smooths a function out without changing its values much. Consequently,

$$N_\epsilon(t) = \sum_{k\in\mathbb{Z}^d} (\chi_{tD} * \rho_\epsilon)(k) \tag{2.2}$$

approximately counts the number of lattice points in tD.

It is a classical result that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the two functions. In view of the Poisson Summation Formula, (2.2) becomes

$$N_\epsilon(t) = t^d|D| + t^d \sum_{k\neq(0,\dots,0)} \widehat{\chi}_D(tk)\widehat{\rho}(\epsilon k)$$
$$= t^d|D| + R_\epsilon(t). \tag{2.3}$$

Using the presence of curvature, we will see that $|R_\epsilon(t)| \lesssim t^{\frac{d-1}{2}}\epsilon^{-\frac{d-1}{2}}$. On the other hand, since $N_\epsilon(t-\epsilon) \leq N(t) \leq N_\epsilon(t+\epsilon)$, we can subtract $t^d|D|$ from each term, compare our estimate with $t^{d-1}\epsilon$, and conclude by setting $\epsilon = t^{-\frac{d-1}{d+1}}$ that

$$R(t) \lesssim t^{d-2+\frac{2}{d+1}}. \tag{2.4}$$

The estimate on $R_\epsilon(t)$ follows once we have good information about the rate of decay of $\widehat{\chi}_D$ at infinity. One can show by a fairly direct calculation in the case of the ball, and by a slightly more complicated argument in general, that if the boundary of D has everywhere nonvanishing Gaussian curvature, then

$$|\widehat{\chi}_D(\xi)| \lesssim |\xi|^{-1}|\xi|^{-\frac{d-1}{2}} \tag{2.5}$$

when $|\xi|$ is large. See, for example, [Stein93, Chapter VIII]. The factor $|\xi|^{-1}$ comes from reducing the integral to the boundary of D, while the remaining factor is the result of curvature. So how does (2.4) follow? By the discussion above, it is enough to show that $|R_\epsilon(t)| \lesssim t^{\frac{d-1}{2}}\epsilon^{-\frac{d-1}{2}}$. With (2.5) in hand, we see that

$$R_\epsilon(t) \lesssim t^d \textstyle\sum_{k\neq(0,\dots,0)} |tk|^{-1}|tk|^{-\frac{d-1}{2}}(1+|\epsilon k|)^{-N}$$

for any $N > 0$. Replacing sums by integrals, we see that this expression is $\lesssim t^{\frac{d-1}{2}}\epsilon^{-\frac{d-1}{2}}$ as desired.

How the curvature enters the picture in establishing the estimate (2.5) is easiest to see in two dimensions. Let D be an arbitrary convex set in the plane. Let e_θ denote a unit vector determined by the angle θ, and set

(2.6) $$S_\theta = \sup_{x \in D}(x \cdot e_\theta).$$

For small positive ϵ, define a region $A_D(\epsilon, \theta)$ via

(2.7) $$A_D(\epsilon, \theta) = \{x \in D : S_\theta - \epsilon < (x \cdot e_\theta) < S_\theta\}$$

(see Figure 3).

It is not hard to establish via an integration by parts argument[4] that

(2.8) $$|\hat{\chi}_D(re^{i\theta})| \lesssim |A_D(r^{-1}, \theta)| + |A_D(r^{-1}, \theta + \pi)|,$$

where the absolute value signs on the right-hand side denote area.

If the boundary of D has everywhere nonvanishing Gaussian curvature, then $A_D(r^{-1}, \theta)$ is a "lune" of width $\approx r^{-1}$ and length $\approx r^{-\frac{1}{2}}$, where the constants buried in the $\approx$ notation depend on the actual value of the curvature. It follows that $|A_D(r^{-1}, \theta)| \approx r^{-\frac{3}{2}}$, which, of course, implies (2.5) in two dimensions. In higher dimensions (2.5) does not hold in such full generality. However, it does essentially hold under the assumption of convexity and everywhere nonvanishing Gaussian curvature.

Indeed, after reducing to the boundary, we need to estimate the Fourier transform of the Lebesgue measure σ on a small piece of a smooth hypersurface S. Suppose that this hypersurface is convex and has finite order of contact with every tangent line. Let $T_x(S)$ denote the tangent plane to S at x, and define the "ball" $B(x, \delta)$ via

(2.9) $$B(x, \delta) = \{y \in S : dist(y, T_x(S)) \le \delta\}.$$

J. Bruna, A. Nagel, and S. Wainger proved in this setting that

(2.10) $$|\hat{\sigma}(\xi)| \lesssim \sup_{x \in S} \left| B\left(x, \frac{1}{|\xi|}\right) \right|.$$

In the case when S has everywhere nonvanishing curvature, $B(x, \delta)$ is approximately a box of dimensions $\approx (\delta^{\frac{1}{2}}, \dots, \delta^{\frac{1}{2}}, \delta)$, which implies (2.5) in view of (2.9) and the fact, mentioned above, that in computing $\hat{\chi}_D(\xi)$, we pick up another factor of $\frac{1}{|\xi|}$ in the process of integrating by parts to reduce the problem to the boundary.

"Arithmetic" Curvature

Let us think about the effect of curvature in this problem a bit more carefully. Choose a point on the boundary of the disc, and consider an outward normal at that point. As we expand the disc the curvature of the circle guarantees that almost all such normals miss the lattice points by a "significant" margin in a sense which can be made

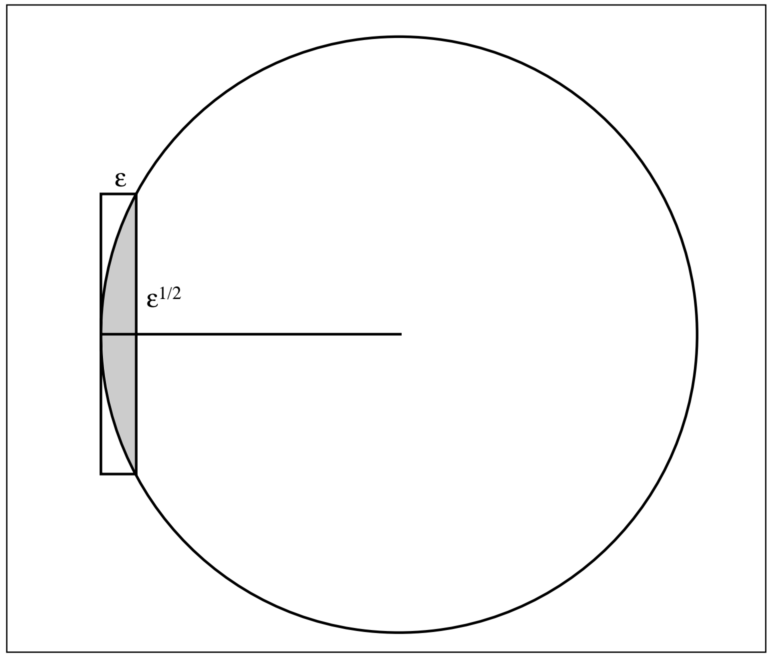

Figure 3.

precise using Diophantine approximations. To put it simply,

$$\text{Curvature} \Rightarrow \text{Many different normals}$$
$$\Rightarrow \text{Few Lattice Points Are Encountered.}$$

Why do we explore this point of view? Let us consider, once again, the case of the unit square with sides parallel to the coordinate axes. As we noted above, the estimate $R(t) \lesssim t$ cannot be improved in this case because lack of curvature prevents the boundary of the cube from escaping from the integer lattice. However, we can actually take advantage of this complete lack of curvature by rotating the domain by a sufficiently irrational angle. The point here is that while the boundary of the unrotated square picks up lattice points by the bushel, the boundary of a properly rotated square picks up hardly any lattice points at all! In summary,

$$\text{No Curvature} \Rightarrow \text{Few Normals}$$
$$\overset{\text{irrational rotation}}{\Longrightarrow} \text{Few Lattice Points.}$$

Using a modification of the Poisson Summation argument given above, one can show that for almost every rotation of the square, or, more generally, any polygon, $R(t) \lesssim \log^\delta(t)$, provided that $\delta > 3$. Using a more sophisticated argument, Skriganov showed that the same result holds for $\delta > 1$. Results of a similar nature were obtained by B. Randol in the 1960s, Y. Colin de Verdière and M. Tarnopolska-Weiss in the 1970s, and others.

Curvature Is Not Always Your Friend

In the previous examples the presence of curvature was always helpful in the sense that it improved the results in some well-defined way. In this

[4] *A beautiful and thorough description of this result and related issues appears in a paper by L. Brandolini, M. Rigoli, and G. Travaglini in* Revista Matemática Iberoamericana.

section, we shall see that curvature is not always helpful. Our first example will involve the standard Gaussian curvature in a way, and the second example will make use of the notion of "arithmetic" curvature from the preceding section. Roughly speaking, the idea is the following. Presence of curvature often leads to "cancellation" on the Fourier transform side, which leads to favorable results. However, if curvature is introduced in a suitable way on the Fourier transform side, the situation reverses, and curvature, which used to be our friend, becomes an enemy. This is the essence of the following examples.

Nikodým Set and a Fundamental Theorem of Calculus

The Lebesgue differentiation theorem says that if f is an integrable function on $\mathbb{R}^d$, then the average of this function over a cube Q centered at x converges to $f(x)$ for almost every x as Q shrinks to x. The problem becomes much more difficult, however, if we attempt to average f over more eccentric sets. Consider the following question. Let $\mathcal{R}$ be the collection of rectangles centered at the origin with arbitrary orientation. Is it true that

$$\lim_{\{diameter(R)\to 0;\ R\in\mathcal{R}\}} \frac{1}{|R|}\int_R f(x-y)\,dy = f(x) \tag{3.1}$$

for almost every x?

A. Zygmund observed that this question has a negative answer because O. Nikodým proved that there exists a subset N of the unit cube of full measure such that for every $x \in N$ there is a line $l(x)$ that intersects N only at the point x.

Indeed, choose a closed subset $F \subset N$ such that $|F| \geq 1-\epsilon$, where $\epsilon > 0$. It follows that

$$\liminf_{\{diameter(R)\to 0;\ R\in\mathcal{R}\}} \frac{1}{|R|}\int_R \chi_F(x-y)\,dy = 0, \tag{3.2}$$

which contradicts (3.1).

There is curvature in this problem in the sense that the rectangles in $\mathcal{R}$ have arbitrary orientations, so the underlying geometric object is a circle. It is not difficult to see that the above argument breaks down if we restrict the orientation of our rectangles to a finite number of directions, where the underlying geometric shape is a polygon. In fact, a positive result is known to hold in such a case. See [Stein93, Chapter X] and the references contained therein.

Projections and Trigonometric Polynomials

Let $T(x,y)$ be a trigonometric polynomial in two variables (the higher dimensional case is more difficult):

$$T(x,y) = \sum_{(n,m)\in\mathbb{Z}^2} c_{n,m} e^{i(nx+my)}, \tag{3.3}$$

where only finitely many coefficients $c_{n,m}$ are different from 0. Given a set Ω, define a projection P_Ω via

$$P_\Omega T(x,y) = \sum_{(n,m)\in\Omega} c_{n,m} e^{i(nx+my)}. \tag{3.4}$$

It is natural to measure to what extent P_Ω "distorts" polynomials $T(x,y)$. To that end, we say that P_Ω is bounded if there exists a constant C_Ω such that

$$\max_{x,y} |P_\Omega T(x,y)| \leq C_\Omega \max_{x,y} |T(x,y)| \tag{3.5}$$

for every polynomial T.

A beautiful result due to E. Belinsky says that if Ω is a strip of width Δ and slope λ, i.e.,

$$\Omega = \{(n,m) \in \mathbb{Z}^2 : |m - \lambda n| \leq \Delta\},$$
$$\lambda \in \mathbb{R}, \quad \Delta > 0, \tag{3.6}$$

then P_Ω is bounded if and only if λ is rational. The argument we present can be found in [GLMP92], and the author is grateful to A. Podkorytov for pointing it out.

Recall that in the section on the distribution of lattice points inside convex domains, we argued in the context of rotated squares that the "arithmetic" curvature provided by the ubiquitous irrationality of rotations leads to very small error terms in that problem. We are about to see that things are completely reversed here. The case of rational λ is less interesting, so we treat only the irrational case. Using the pigeon-hole principle, or the Dirichlet principle as it is often called in this case, we can find an irreducible fraction $\frac{p}{q}$ such that

$$\left|\lambda - \frac{p}{q}\right| \leq \frac{1}{q^2}. \tag{3.7}$$

Let L be a straight line that passes through the points $(0,0)$ and (q,p). By the irreducibility of $\frac{p}{q}$, all integer points on L are of the form (kq, kp), $k \in \mathbb{Z}$. It follows that for some $N > 0$,

$$\Omega \cap L = \{(kq,kp) : k \in \mathbb{Z},\ |k| \leq N\}. \tag{3.8}$$

We shall now demonstrate that

$$N \geq q\Delta. \tag{3.9}$$

Write $\lambda = \frac{p}{q} + \frac{\delta}{q^2}$, where $|\delta| < 1$. For $|k| \leq q\Delta$, $(kq,kp) \in \Omega \cap L$ when k satisfies the relation

$$|kp - \lambda kq| = \frac{|k\delta|}{q} \leq \frac{|k|}{q} \leq \Delta, \tag{3.10}$$

and (3.9) follows.

Now consider those polynomials whose coefficients $c_{n,m}$ are 0 for $(n,m) \notin L$. By the above, such polynomials T have projections of the form $P_\Omega T(x,y) = \sum_{\{k\in\mathbb{Z},\ |k|\leq N\}} a_k e^{ik(qx+py)}$.

A simple one-dimensional calculation that involves writing a trigonometric polynomial as a convolution with the Fejér kernel shows that (3.5) can hold only if $\log(N) \lesssim C_\Omega$. This gives

$$\log(q\Delta) \lesssim \log(N) \lesssim C_\Omega \tag{3.11}$$

by (3.9). This means that (3.5) cannot hold with a bounded constant, since q can be arbitrarily large.

Conclusion

The examples we have discussed follow the following simple pattern of ideas:

$$\text{Curvature} \overset{\text{Fourier transform}}{\Longrightarrow} \text{Cancellation} \Rightarrow \text{Unexpected Result.} \tag{4.1}$$

The very first example of this article, the Erdős Distance Problem, follows the pattern

$$\text{Curvature} \overset{\text{Combinatorics}}{\Longrightarrow} \text{Unexpected Result.} \tag{4.2}$$

However, an application of this result to the Fuglede conjecture puts it firmly into the framework of (4.1). Indeed, the main point in the proof that the ball is not spectral is to use the Erdős Distance Problem to show that the curvature of the boundary of the ball B_d implies that the property

$$\sum_{a \in A} |\hat{\chi}_{B_d}(\xi - a)|^2 \equiv |B_d|^2 \tag{4.3}$$

for all $\xi \in \mathbb{R}^d$ cannot hold for any "reasonable" discrete set A. This means, in effect, that the curvature of the boundary of the ball is preventing cancellation from taking place! The word "cancellation" should be interpreted liberally, since we are summing positive terms. What we mean is that the terms need to be smaller than "expected" in order for (4.3) to hold, since each term is potentially close to 1.

In the case of the lattice points inside a disc, (4.1) takes the form

$$\text{Curvature} \underset{\text{Poisson Summation Formula}}{\overset{\text{Fourier transform}}{\Longrightarrow}} \text{Cancellation} \Rightarrow \text{Good Error Term,} \tag{4.4}$$

and in the case of the square we have

$$\text{No Curvature + Rotation} \underset{\text{Poisson Summation Formula}}{\overset{\text{Fourier transform}}{\Longrightarrow}} \text{Cancellation} \Rightarrow \text{Good Error Term.} \tag{4.5}$$

The notion of cancellation is quite classical here. The curvature causes the trigonometric integral of the form $\int_{\partial D} e^{-2\pi i x \cdot \xi}\, d\sigma(x)$ to be very small when ξ is very large, even though the integrand has constant modulus 1. This integral in turn is the key to obtaining a nontrivial, and often best possible, result.

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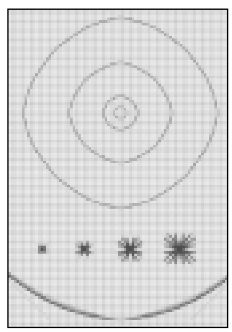
About the Cover



As the article by Alex Iosevich explains, curved paths have fewer lattice points on them than flat ones. But Vojtěch Jarník showed (*Math. Zeitschrift* **24** (1925), 500–518) that curvature alone will not guarantee the best estimates. He constructed an infinite sequence of curves C_N of length roughly N^3, with continuous radius of curvature bounded above by CN^3, containing roughly N^2 points of the integral lattice. The N-th curve interpolates the polygon constructed by starting at an arbitrary point of the lattice and following displacements, in order of their direction, parametrized by all primitive integral vectors whose coordinates have absolute value at most N. The cover illustration shows the first four of these polygons, as well as the corresponding sets of displacement vectors. (More information about Jarník's construction can be found in Chapter 2 of the book by Martin Huxley listed among Iosevich's references.)

The bottom of the illustration shows parts of several more of these polygons, scaled to equal diameters. Darker curves correspond to larger polygons. The picture suggests that the scaled polygons converge to a somewhat ragged limit curve.

—*Bill Casselman* (covers@ams.org)