Number: From Ahmes to Cantor

Reviewed by Bryan Birch

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Midhat Gazalé
Princeton University Press, 2000
ISBN 0-691-00515-X
Hardcover, $29.95, 272 pages

Midhat Gazalé has always been fascinated by numbers, and in this book he sets out to transmit his fascination to the common reader by giving an account of the development of numbers from the early systems of ancient Mesopotamia to the transfinite ordinals of Cantor. The author's intended audience is a person with some mathematical knowledge but without mathematical sophistication, and he speaks as an engineer rather than as a mathematician. In his very first chapter he stresses Plato's distinction between $\alpha\rho\iota\theta\mu\epsilon\tau\iota\kappa\eta$, nowadays usually translated as mathematics, in which proof is paramount, and $\lambda o\gamma\iota\sigma\tau\iota\kappa\eta$, the art of calculation, which the Greeks were less good at and which is what most of the world (but not the reviewer) usually means by arithmetic nowadays. His book is about $\lambda o\gamma\iota\sigma\tau\iota\kappa\eta$, not about $\alpha\rho\iota\theta\mu\epsilon\tau\iota\kappa\eta$ in Plato's sense!

The author's first chapter is a delight to read; he speaks of the genesis of number systems, from early Mesopotamia and Egypt to the Indo-Arabic decimal system adopted by the modern world. His style is to give an overview of what happened, garnished liberally with literary references and anecdotes. He does not try to give a formal history; the chapter starts with the Sumerians and ends with Simon Stevin, and, in between, time jumps back and forth as he leaps from continent to continent. He picks out various highlights (there are more details in Georges Ifrah's *Universal History of Numbers*) to illuminate his story; and though there are many developments that he omits, his choice is for the most part excellent. The story of how a positional number system was developed by the Babylonians (albeit in the scale of 60) and then deliberately discarded by the Greeks in favour of an inferior alphabetical notation is particularly fascinating, as it involves quite recently discovered history. Tacked on to the end of the chapter is a section of "marginalia"; this seems less carefully prepared and contains various bits and pieces, the biggest of which is a version of the early history of the development of computers.

This first chapter is a very natural preparation for the next two, which form the centrepiece of the book. The author develops the arithmetic of positional number systems in great detail. He starts off by setting out the division algorithm and then dives in to show how to expand a positive integer γ in terms of a given mixed basis $b = (m_0, m_1, m_2, \dots)$, where the m_i are positive integers greater than or equal to 2: there is a unique expansion $\gamma = \sum_{i=0}^{N} \delta_i \pi_i$, where $\pi_0 = 1$ and for each $i \geq 1$, π_i is defined as $\prod_{j=0}^{i-1} m_j$, the digits δ_i are integers in the range $0 \leq \delta_i < m_i$, and N is the least integer with $\pi_N > \gamma$. (So if every m_i is 10, we get the ordinary decimals.) He goes on to show how to convert from one such basis to another, how to add and multiply, and how to express fractions: if $0 < \alpha < 1$ and $(m'_1, m'_2, \dots)$ is a mixed basis, we may express α uniquely as $\sum_{i\geq 1} a_i/\pi'_i$, where this time $\pi'_i = \prod_{j=1}^{i} m'_j$ and $0 \leq a_i < m'_i$, and we may write $\alpha = \cdot a_1 a_2 a_3 \dots$ with respect to this basis. Though utterly elementary, the arithmetic of these mixed systems is decidedly complicated, not to say

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tiresome, though some of the fractions are nice; the "exponential" basis with $m_i' = i+1$, with respect to which the exponential has the expansion $e - 2 = \cdot 11111\ldots$, is really very pretty. (This is very typical of the sort of example that Gazalé's book is full of; indeed, he goes on to show that given any positive irrational α less than 1, we may find a mixed basis with respect to which the expansion of α is $\cdot 11111\ldots$!) Considering the inconvenience of the arithmetic involved, it is surprising how many such systems have been used. The Babylonian system to base 60 was really a mixed-base system, with $b = (10, 6, 10, 6, 10, 6, \ldots)$; the Mayan system was another. More recently, the avoirdupois system of weights and measures is possibly the most complicated of all widely used mixed-base systems; very oddly, the author does not refer to it as an example of a mixed basis, though it is in the marginalia of Chapter 1 as an example of Anglo-Saxon foolishness.

In Chapter 3 he goes more deeply into the properties of expansions of rational numbers with respect to periodic bases. He starts off with various fundamental theorems, like the fundamental theorem of arithmetic, Euler's theorem (and a mild extension thereof), and the primitive root theorem. He shows that the expansion of a rational number with respect to a periodic basis is periodic and shows how to find the period. He warns that this is a more difficult chapter, because it contains proofs. I fear that it is more difficult than it need have been, since sometimes he includes proofs, sometimes he omits them, and sometimes he indicates why something is true without really proving it, and it may not be clear to the reader which he is doing.

Chapter 4 is entitled "Real Numbers" and starts off with a nice essay on Kronecker's theme that "God made the integers, all the rest is the work of man"; Gazalé shows some sympathy (appropriate in the context) with Poincaré's dictum that "later mathematicians will regard set theory as a disease from which one has recovered." He gives the standard construction of the rationals from the integers and reasonable indications, with references to Dedekind and Eudoxus, as to how one might go on to define the real numbers, but he does not complete a definition. A really nice part of the chapter is about Pythagorean triangles. He gives a discussion, more detailed than the one in Georges Ifrah's book, of the Plimpton tablet, which contains several Pythagorean triples known to the Babylonians. (So Pythagoras's theorem was believed before the time of Pythagoras, but was there a proof?) The chapter also contains a couple of neat proofs that $\sqrt{2}$ and e are both irrational.

Chapter 5 is about continued fractions, a beautiful subject; sadly, he does not allow himself enough space to do them justice. In particular, it is surprising that he did not say more about their approximation properties at this stage, since they have fundamental relevance for his *cleavages* in the next chapter (so that some of these properties of continued fractions have to be developed piecemeal in Chapter 6). He gives some pretty examples—Euler's continued fraction for e and the periodic continued fractions of quadratic surds—but he proves very little; he very rightly refers the reader who wants to know more to H. Davenport's *Higher Arithmetic*.

In Chapter 6 he introduces the idea of *cleavages*: if μ is a positive number, then the line $y = \mu x$ cleaves the positive integer points of the positive quadrant into two sets, an upper set U consisting of integer points (x, y) for which $y \geq \mu x$ and a lower set L consisting of those (x, y) for which $y < \mu x$. So the cleavage is very much like a Dedekind cut, with $\left(\frac{y}{x}\right) < \mu$ if (x, y) is in the lower set of the cleavage. It is arguable that cleavages will give some amateurs a better idea of what is involved in Dedekind's cuts than does Dedekind's original definition: a picture separating the integers in the first quadrant into two sets is easier to look at than a picture of the rationals.

In his final chapter Gazalé treats infinity. Despite certain felicities (for instance, a clear proof that a set may not be put into 1-1 correspondence with its set of subsets; a reasonable picture of the Cantor set; and Woody Allen's remark that eternity is very boring, particularly near the end) it is inadequate; there are real difficulties in the subject matter which the author has not faced up to. There are also real errors: it is wrong to assert that one cannot order the rationals; there are models of set theory in which the Continuum Hypothesis is false; and at the very end of the book, his explanation of why the sky is black is inadequate, since stars are not points.

To sum up: many ordinary people will thoroughly enjoy this book, even if they give up before the end. The author says clearly that he is not writing for professional mathematicians; if, nonetheless, mathematicians try it, they will begin like ordinary people by thoroughly enjoying it and will become more and more dissatisfied.