Quantum Spaces and Their Noncommutative Topology

Joachim Cuntz

Noncommutative geometry studies the geometry of "quantum spaces". Put a little more prosaically, this means the "geometric properties" of noncommutative algebras (say, over the field $\mathbb{C}$ of complex numbers). Such algebras include, for instance,

- algebras of pseudodifferential operators, algebras of leafwise differential operators on foliated manifolds, algebras of differential forms, group algebras or convolution algebras for groupoids;
- noncommutative or "quantized" versions of familiar algebras such as algebras of functions on spheres, on tori, on simplicial complexes, or on classifying spaces;
- genuinely new noncommutative algebras, for instance, ones motivated by quantum mechanics.

The underlying philosophy is based on the observation that various categories of spaces can be completely described by the (commutative) algebras of functions on them (a locally compact space by the algebra of continuous functions, a smooth manifold by the algebra of smooth functions, an affine algebraic variety by its coordinate ring). The idea then is that a noncommutative algebra can be viewed as an algebra of functions on a virtual "noncommutative space". This approach is very flexible: for instance, it covers the algebra of functions on a manifold, the algebra of pseudodifferential operators, and the algebra of differential forms all on the same footing.

Now, what is a "geometric" property of a noncommutative algebra? How can one describe characteristic classes or additional structures like a Riemannian metric for a noncommutative algebra? These questions are what noncommutative geometry is all about; see the fascinating book by Connes [5].

The two fundamental "machines" of noncommutative geometry are cyclic homology and (bivariant) topological K-theory. Cyclic theory can be viewed as a far-reaching generalization of the classical de Rham cohomology, while bivariant K-theory includes the topological K-theory of Atiyah-Hirzebruch as a very special case.

Bivariant K-theory was first defined and developed by Kasparov on the category of C^*-algebras, thereby unifying and decisively extending previous work by Atiyah-Hirzebruch, Brown-Douglas-Fillmore, and others. Kasparov also applied his bivariant theory to obtain striking positive results on the Novikov conjecture. Very recently, it was discovered that bivariant topological K-theories can be defined on a wide variety of topological algebras ranging from discrete algebras and very general locally convex algebras to Banach algebras or C^*-algebras.

Cyclic theory is a homology theory developed independently by Connes and by Tsygan, who were motivated by different aspects of K-theoretic constructions. It was immediately realized that cyclic homology has close connections with de Rham theory, Lie algebra homology, group cohomology, and index theorems.

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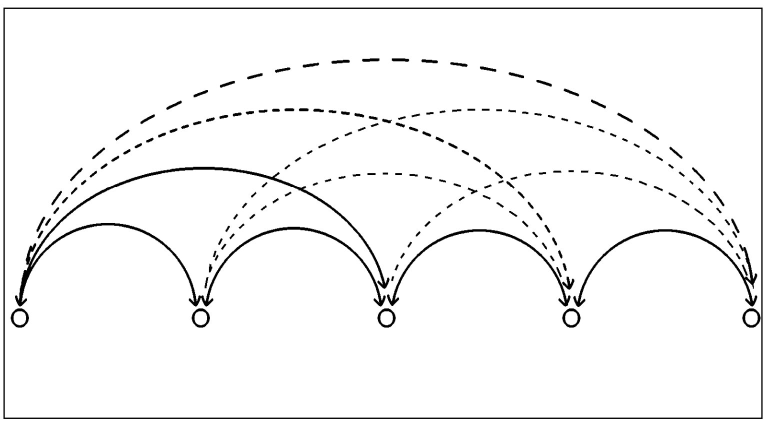
It is important to note that the new theories are by no means simply generalizations of classical constructions. In fact, in the commutative case they provide a new approach and a quite unexpected interpretation of the well-known classical theories. Essential properties of the two theories become visible only in the noncommutative category. For instance, both theories have certain universality properties in this setting.

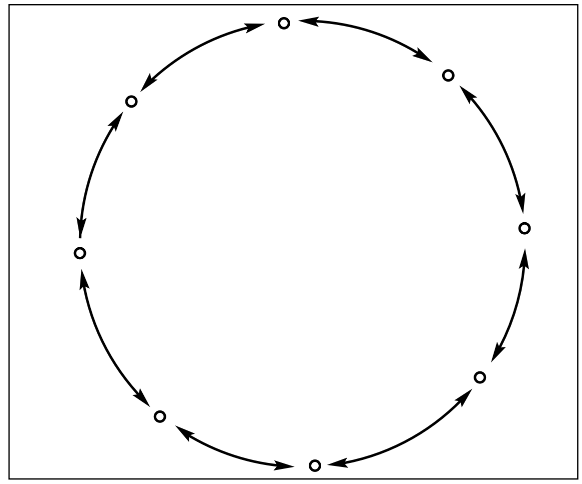
Let us have a look at the kind of geometric information that the two theories give us for a number of simple "quantum spaces". The formal definition of the cyclic and K-theory classes mentioned in these examples will be explained in the subsequent section. The technical definition is not necessary for an intuitive grasp of the situation.

1. **The space with n points and noncommutative connections.** This space has n points and arrows between every two points. As an algebra it is described by the algebra $M_n(\mathbb{C})$ of $n \times n$-matrices (the functions on the n points corresponding to the diagonal matrices). Both

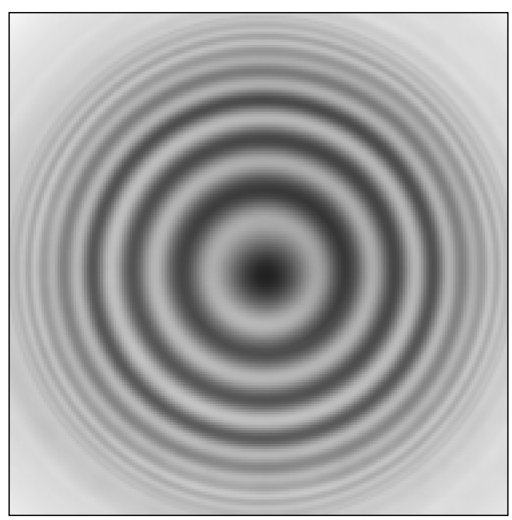


K-theory and cyclic theory see one even cohomology class and no odd classes. In both theories the nontrivial even class is 0-dimensional, and there are no higher cohomology classes. Since there is one class representing the equivalence class of the n points and no higher dimensional classes, $M_n(\mathbb{C})$ looks like a connected 0-dimensional space.

This is the simplest case of a convolution algebra for a groupoid. In general a (topological) groupoid consists of a space of objects and a family of (invertible) arrows which can be considered as above as noncommutative paths between the objects. For the noncommutative homology theories, different points connected by an arrow will be homologous. Higher homology classes can also arise from configurations of arrows (like a loop of arrows), from mixed configurations involving arrows and objects, or even from linear combinations of such things. Consider, for instance, the algebra determined by linear combinations of all possible paths in the following graph. It possesses, besides the 0-dimensional class given by the equivalence class of the points, a one-dimensional class coming from the path around the circle (contrary to the case of $M_n(\mathbb{C})$, we assume here that this path is different from the trivial path).

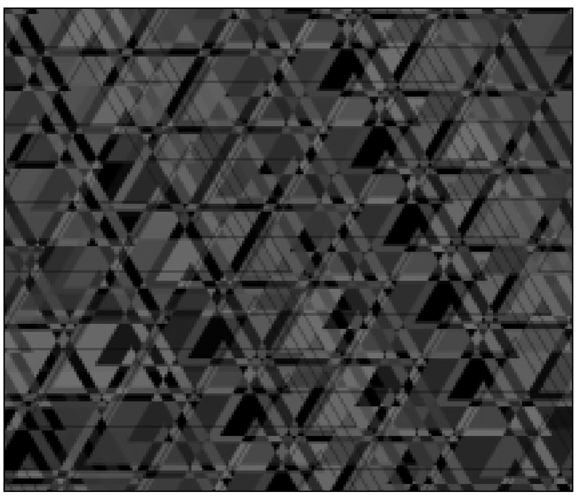


2. **The phase space in quantum mechanics.** This is described by the unital algebra $A(p,q)$ generated by two generators p and q satisfying the Heisenberg relation $pq - qp = 1$ (sometimes this is called the Weyl algebra). There is at present no calculation of the K-theory for this algebra (or its C^∞-completion). The cyclic theory sees one two-dimensional cohomology class and no classes in dimensions different from 2. Thus we have here a noncommutative space that is two-dimensional (say, looks like a 2-plane). However, not only does this "space" have no points, it does not even have any equivalence class of a point.



3. **The noncommutative 2-torus.** This is the involutive unital algebra A_θ given by power

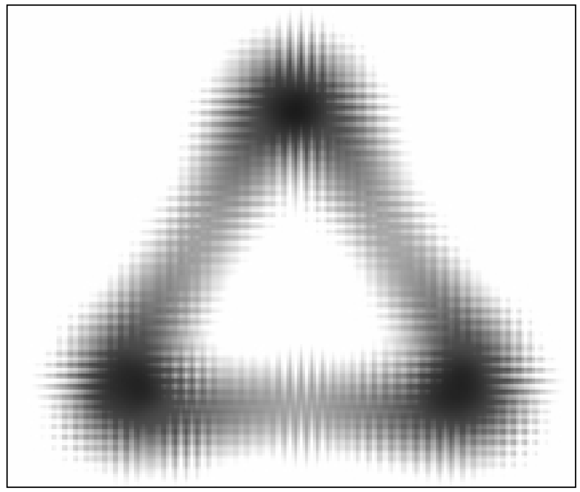
series with rapidly decreasing coefficients in two generators u and v satisfying the relations $uu^* = u^*u = vv^* = v^*v = 1$ and $vu = e^{2\pi i\theta}uv$ for a fixed real number $\theta \in [0, 1]$. Each pair $\{u, u^*\}$ and $\{v, v^*\}$ of generators generates a commutative subalgebra isomorphic to the algebra of C^∞-functions on the circle. If $\theta = 0$, then these subalgebras commute and A_θ will be isomorphic to the algebra of C^∞-functions on the 2-torus $S^1 \times S^1$. The K-theory for A_θ contains two even- and two odd-dimensional classes. The cyclic theory shows more precisely that one of the even classes is 0-dimensional and the other one is 2-dimensional. The two odd classes are both 1-dimensional (and are represented by the 1-forms $u^{-1}du$ and $v^{-1}dv$). Thus, from this point of view the noncommutative torus looks exactly like an ordinary 2-torus.



4. **Noncommutative simplicial complexes.** Let Σ be a finite simplicial complex given by its set of vertices V and by its simplices, represented by finite subsets F of V. We can associate with Σ a noncommutative algebra C_Σ in the following way. Let C_Σ be the unital algebra given by power series with rapidly decreasing coefficients in generators h_s ($s \in V$) satisfying the following relations:

   - $\sum_{s\in V} h_s = 1$;
   - if $\{s_0, s_1, \ldots, s_n\}$ is not in Σ, then the product $h_{s_0}h_{s_1}\ldots h_{s_n}$ is zero.

   (Note that when we introduce the additional relation that the generators commute, we get an algebra isomorphic to an algebra of C^∞ functions on the geometric realization of Σ.) The K-theory and the periodic cyclic homology for C_Σ are isomorphic respectively to the K-theory and the $\mathbb{Z}/2$-graded singular cohomology of the geometric realization of Σ. The dimension of a cyclic cohomology class for C_Σ is, however, much larger than the dimension d of the corresponding commutative homology class (it is of the order of 3^d). There is also a degree filtration on the K-homology for locally convex algebras. The K-homology degree of a K-homology class for C_Σ is the same as the dimension of the corresponding commutative homology class.



In these examples the K-theory and the cyclic classes have been used to describe the "shape" of a noncommutative space. This is by no means the only function of these invariants. They also are the main tool to describe other topological information, such as gluing data in extensions and indices of operators.

In this article we sketch a uniform approach to cyclic theory and bivariant K-theory, which in fact can be made to work for many different categories of algebras. This approach emphasizes the analogy of cyclic theory with de Rham theory and the connection between K-theory and extensions. It leads in a natural way to the fundamental properties of both theories. We will also explain the construction of the bivariant Chern-Connes character taking bivariant K-theory to bivariant cyclic theory. The existence of this multiplicative transformation has been obtained in full generality only very recently (important special cases had been considered by Connes and others, e.g., [4], [12]) as a result of progress both on the cyclic homology side and on the K-theory side [9], [6]. It is a vast generalization of the classical Chern character in differential geometry and allows one to associate "characteristic classes" with K-theoretic objects.

I am indebted to my sons, Nicolas and Michael, for the illustrations to the examples above. Since these pictures have no technical meaning, they are only meant to provide a kind of suggestive visualization of the corresponding quantum spaces.

The Noncommutative de Rham Complex—or How to Extract Commutative Information from a Quantum Space

Any noncommutative algebra A can be abelianized by dividing out the ideal generated by all commutators. This procedure, however, destroys the relevant information in nearly all interesting cases. In fact, the abelian quotient typically is zero (this is the case in examples 1, 2, and 3 above).

A more promising approach consists in dividing only by the linear space of commutators or, dually, in considering traces on A. A trace is by definition a linear functional f on A such that

$$f(xy) = f(yx)$$

for all x and y in A. To describe topological information, the strategy then is to consider homotopy classes of traces. What is homotopy for traces, and how can this be formulated algebraically?

An answer is provided by the simple X-complex introduced by Quillen [14] in connection with differential graded algebras and then used systematically in [7], [8], [9].

Let A be an algebra. The space $\Omega^1 A$ of abstract 1-forms over A is defined as the bimodule consisting of linear combinations of expressions of the form $xd(y)$ with $x \in \tilde{A}$ and $y \in A$, where $\tilde{A}$ is the unitization of A. The bimodule structure is given by the rules

$$a(xdy) = axdy \quad (xdy)a = xd(ya) - xyda, \; a \in A$$

(that is, one introduces the relation $d(xy) = xdy + d(x)y$). A trace on $\Omega^1 A$ is a linear functional f such that $f(a\omega) = f(\omega a)$ for $a \in A$ and $\omega \in \Omega^1 A$.

The (dual) X-complex $X'(A)$ is the $\mathbb{Z}/2$-graded complex

$$\{\text{functionals on } A\} \underset{\delta}{\overset{\beta}{\rightleftarrows}} \{\text{traces on } \Omega^1 A\}.$$

The boundary operators are defined by $(\delta f)(x) = f(dx)$ and $(\beta f)(xdy) = f([x, y])$. It is straightforward to check that $\beta\delta = \delta\beta = 0$.

This complex has only two homology groups, namely, the even and the odd homology. The even homology HX^{ev} is the space of traces—linear functionals on A that vanish on commutators—divided by the space of "derivatives" of traces—linear functionals of the form $f \circ d$, where f is a trace on $\Omega^1 A$. This quotient can be considered rightfully as the space of homotopy classes of traces on A. Arguing similarly for the odd homology, we obtain

$$HX^{ev}(A) = \{\text{homotopy classes of traces on } A\}$$
$$HX^{od}(A) = \{\text{classes of closed traces on } \Omega^1 A\}$$

where a trace f is closed if $f \circ d = 0$.

There is a natural complex $X(A)$ for which $X'(A)$ is the dual: namely,

$$A \underset{\beta}{\overset{\delta}{\rightleftarrows}} \Omega^1 A_\natural$$

where $\Omega^1 A_\natural$ is the quotient $\Omega^1 A/[A, \Omega^1 A]$ of $\Omega^1 A$ by commutators, $\natural : \Omega^1 A \to \Omega^1 A_\natural$ is the quotient map, and the boundary operators are defined by $\delta(x) = \natural(dx)$ and $\beta(\natural(xdy)) = [x, y]$.

It is certainly somewhat surprising that the extremely simple complex $X(A)$ should play the role of the de Rham complex in noncommutative geometry. In the commutative case it obviously does not reduce to the de Rham complex. Its analogy with the de Rham complex will now be explained. This also leads to a new interpretation for the classical de Rham theory.

The starting point is that even though taking traces is a much milder procedure than abelianizing, it still leads to trivial results for many noncommutative algebras. Indeed, there are natural examples of algebras A for which no nontrivial traces exist. One is led to consider also traces on algebras related to A via an extension. An extension of A is an algebra E that admits A as a quotient (by an ideal I) or, in short, an exact sequence

$$0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0,$$

where the arrows are algebra homomorphisms. Extensions play a fundamental role in noncommutative geometry.

An algebra A is called quasi-free if in any extension of A where the ideal I is nilpotent (that is, $I^k = 0$ for some k), there is a homomorphism $A \to E$ that is a left inverse for the quotient map $E \to A$. This condition is formally nearly identical to the condition of smoothness introduced for commutative algebras and algebraic varieties by Grothendieck. Any free algebra is quasi-free.

The periodic cyclic cohomology $HP^*(A)$ of an algebra A, where $* = ev/od$, is obtained by representing A as a quotient of a quasi-free algebra T by an ideal I and then writing

$$HP^*(A) = \varinjlim_n HX^*(T/I^n).$$

It is an important fact that this definition does not depend on the choice of the quasi-free resolution T. The homology $HX^*(T/I^n)$ is essentially the (non-periodic) ordinary cyclic cohomology $HC^{2n+*}(A)$.

It is not very difficult to see that one can alternatively obtain HP^* by the formula

$$HP^*(A) = \varinjlim_n HX^{*+1}(I^n) \tag{1}$$

where $ev + 1 = od$ and vice versa. In this picture a cyclic cocycle (of dimension $2n - 1$) is described by a trace on the n-th power of the ideal in an extension of A.

This definition of cyclic homology is manifestly quite different from the original definition by Connes or Tsygan. The proof that both definitions give the same result is nontrivial. Note that any algebra A has a canonical quasi-free (even free) resolution given by the tensor algebra TA over A.

There is a striking analogy between periodic cyclic homology and Grothendieck's notion of infinitesimal homology which describes the topology of nonsmooth varieties in algebraic geometry. In this analogy quasi-free algebras play the role, in the noncommutative category, of smooth varieties, and the X-complex corresponds to the de Rham complex.

In fact, in algebraic geometry the infinitesimal cohomology is defined by writing the coordinate ring A of the variety as a quotient S/I of the coordinate ring of a smooth variety S and by taking the de Rham cohomology of the completion $\hat{S} = \varprojlim_n S/I^n$. It can be shown that this does not depend on the choice of the embedding into a smooth variety. This procedure is exactly analogous to the definition of periodic cyclic cohomology above, where we write A as a quotient of a quasi-free algebra T and then take the homology $HX^*(\hat{T})$.

Given two algebras A_1 and A_2, we can also define the bivariant periodic cyclic theory $HP_*(A_1, A_2)$, where $* = 0, 1$, as the homology of the Hom-complex between the X-complexes associated with quasi-free extensions for A_1 and A_2. There is a natural composition product

$$HP_i(A_1, A_2) \times HP_j(A_2, A_3) \longrightarrow HP_{i+j}(A_1, A_3).$$

Any algebra homomorphism $\alpha : A \to B$ determines an element $HP(\alpha)$ in $HP_0(A, B)$.

The periodic cyclic theory $HP_i(A_1, A_2)$ has very good properties. It can be shown that it is invariant under differentiable homotopies and under Morita invariance in both variables. Moreover, by [9] it satisfies "excision" in the following sense.

Let D be any algebra. Every extension $0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$ induces exact sequences in $HP_*(D, \cdot\,)$ and $HP_*(\,\cdot\,, D)$ as follows:

$$\begin{array}{ccccc} HP_0(D,I) & \xrightarrow{\cdot HP(i)} & HP_0(D,A) & \xrightarrow{\cdot HP(q)} & HP_0(D,B) \\ \uparrow & & & & \downarrow \\ HP_1(D,B) & \xleftarrow{\cdot HP(q)} & HP_1(D,A) & \xleftarrow{\cdot HP(i)} & HP_1(D,I) \end{array}$$

and

$$\begin{array}{ccccc} HP_0(I,D) & \xleftarrow{HP(q)\cdot} & HP_0(A,D) & \xleftarrow{HP(i)\cdot} & HP_0(B,D) \\ \downarrow & & & & \uparrow \\ HP_1(B,D) & \xrightarrow{HP(i)\cdot} & HP_1(A,D) & \xrightarrow{HP(q)\cdot} & HP_1(I,D) \end{array}$$

Special cases of these exact sequences had been obtained before in [10] and [16]. They are one of the main tools in the computation of the cyclic homology invariants.

Embedding Quantum Spaces into Smooth Spaces—Extensions and K-Theory

Over the years (see, e.g., [3], [11], [4], [6]) it has become evident that the single most important notion in noncommutative topology is the one of an extension. As already mentioned above, an extension of an algebra A is an exact sequence

$$0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0$$

where the arrows are algebra morphisms. Dually, and intuitively, such an extension corresponds to embedding the quantum space corresponding to A into the quantum space corresponding to E.

To illustrate the kind of topological information contained in an extension, we mention that the content of the Atiyah-Singer index theorem [1] may be viewed as the determination of the class of the extension defined by the pseudodifferential operators on a compact manifold. Other index theorems are concerned with more complicated extensions.

We are now going to sketch a general construction, based on extensions, of (bivariant) topological K-theory that works for many categories of noncommutative algebras and fits nicely with the definition of cyclic theory outlined above. As a result there will be a natural transformation, the bivariant Chern-Connes character, from topological K-theory to cyclic theory.

To be more specific, we will assume from now on that all our algebras are topological algebras with a complete locally convex structure given by a family of seminorms (p_α) satisfying $p_\alpha(xy) \le p_\alpha(x)p_\alpha(y)$ for all x and y in the algebra. For any such locally convex algebra A, the algebraic tensor algebra $A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \oplus \dots$ has a natural completion TA which is a locally convex algebra. There is a canonical algebra homomorphism $TA \to A$ mapping $x_1 \otimes \dots \otimes x_n$ to the product $x_1 x_2 \dots x_n$. We denote the kernel by JA, so that we get a free resolution for A of the form

$$0 \longrightarrow JA \longrightarrow TA \longrightarrow A \longrightarrow 0,$$

which can be viewed as an embedding of the quantum space corresponding to A into the smooth quantum space corresponding to TA. The ideal corresponds to the complement of the image of the quantum space for A. Since JA is again a locally convex algebra, we can iterate the construction and form $J^2A = J(JA)$ and, inductively, $J^nA = J(J^{n-1}A)$.

With a locally convex algebra B, we can also associate the algebra $M_\infty(B)$ of infinite matrices $(b_{ij})_{i,j\in\mathbb{N}}$ with rapidly decreasing matrix elements in B. The corresponding quantum space looks like the one in example (1) of the introduction. It has infinitely many points, which are labeled by $\mathbb{N}$, and arrows between the points, which are labeled by all possible elements of B. A fundamental extension, using pseudodifferential operators on the circle, shows that for any A there exists a canonical homomorphism $J^2A \to M_\infty(B)$. This map can be used to form the inductive limit in the following definition.

Let A and B be locally convex algebras, and $* = 0$ or 1. We define

$$kk_*(A,B) = \varinjlim_k [J^{2k+*}A, M_\infty(B)],$$

where $[J^{2k+*}A, M_\infty(B)]$ denotes the set of (differentiable) homotopy classes of homomorphisms $J^{2k+*}A \to M_\infty(B)$. Provided with the ordinary direct sum addition of maps into matrices, $kk_*(A,B)$ is an abelian group.

Since taking the homology of the X-complex is an algebraic version of taking homotopy classes, this definition is formally remarkably similar to the definition of periodic cyclic theory in equation (1) in the preceding section.

This bifunctor kk_* has the same abstract properties as HP^* (see the preceding section). In particular:

- Every homomorphism $\alpha : A \to B$ induces an element $kk(\alpha)$ in $kk_0(A,B)$.
- There is an associative product $kk_i(A,B) \times kk_j(B,C) \to kk_{i+j}(A,C)$ (where $i,j \in \mathbb{Z}/2$, and A, B, and C are locally convex algebras) that is additive in both variables and that satisfies $kk(\alpha)kk(\beta) = kk(\alpha\beta)$ for two homomorphisms α and β.
- kk_* is homotopy invariant and satisfies excision in both variables (as described in the preceding section). The canonical map $B \to M_\infty(B)$ induces an isomorphism in both variables of kk_*.

In fact, one can show that kk_0 is the universal functor from the category of locally convex algebras as above into an additive category satisfying the third property.

Even though this construction of K-theory looks really different from the usual approach using projections or projective modules, it turns out that when we specialize the first variable to a point, i.e., to the algebra $\mathbb{C}$ of complex numbers, $kk_*(\mathbb{C},B)$ is nothing but the usual K-group in the case where B is a Banach algebra [2] or a Fréchet algebra [13] (the cases in which the usual K-theory is defined).

Another important property is that any extension, or more generally any "n-step" extension of the form

$$0 \to B \to E_1 \to \cdots \to E_n \to A \to 0,$$

gives an element in $kk_n(A,B)$ where n is counted modulo 2. In homological algebra one uses a well-known product on such extensions, known as the Yoneda product, which consists simply in splicing together two such extensions. This product is compatible with the product

$$kk_n(A,B) \times kk_m(B,C) \to kk_{n+m}(A,C).$$

The pseudodifferential operators on a smooth compact manifold give rise to an extension

$$0 \longrightarrow \Psi_{-1} \longrightarrow \Psi_0 \longrightarrow C^\infty(S^*M) \longrightarrow 0,$$

where Ψ_{-1} and Ψ_0 denote the algebras of pseudodifferential operators of order -1 and 0, respectively, and $C^\infty(S^*M)$ denotes the algebra of smooth functions on the cosphere-bundle of M. The problem solved by the Atiyah-Singer index theorem is exactly the determination of the class in $kk_1(C^\infty(S^*M), \Psi_{-1})$ defined by this extension.

The Bivariant Chern-Connes Character

The most important ingredient in the construction of a bivariant multiplicative transformation from kk_* to the bivariant theory HP_* on the category of locally convex algebras is the universality property of kk_0 mentioned at the end of the preceding section.

Since HP_0 satisfies the properties for which kk_0 is universal, we immediately obtain a transformation

$$ch : kk_0(A,B) \longrightarrow HP_0(A,B)$$

which is compatible with the product.

In trying to extend this to a multiplicative transformation from the $\mathbb{Z}/2$-graded theory kk_* to HP_*, one faces the problem that the product of two odd classes is defined differently in kk_* and in HP_*. It turns out that one has to introduce (somewhat arbitrarily) a factor of $\sqrt{2\pi i}$. With this proviso one does then obtain a transformation

$$ch : kk_*(A,B) \longrightarrow HP_*(A,B)$$

which is multiplicative in full generality.

Both the cyclic cohomology $HP^*(A)$ and the K-homology $kk_*(A,\mathbb{C})$ have a natural (dimension) filtration due to their definitions as inductive limits. This dimension filtration was alluded to in the examples in the introduction.

The behaviour of these filtrations under the Chern-Connes character is well understood due to a very delicate analysis of the boundary map in the cyclic homology long exact sequence by M. Puschnigg and by R. Meyer. Given an element α in $kk_*(A,\mathbb{C})$, the dimension of $ch(\alpha)$ is bounded by 3^d, where d is the K-theoretic dimension of α. This estimate is optimal.

To close this article, we want to emphasize that despite their seemingly abstract definition, the cyclic theory and K-theory invariants can be computed very explicitly for a large variety of noncommutative algebras. Some typical examples were described in the introduction.

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About the Cover

The image on this month's cover arose from Joachim Cuntz's effort to render into visible art his own internal vision of a noncommutative torus, an object otherwise quite abstract. His original idea was then implemented by his son Michael in a program written in Pascal. More explicitly, he says that the construction started out with a triangle in a square, then translated the triangle by integers times a unit along a line with irrational slope; plotted the images thus obtained in a periodic manner; and stopped just before the figure started to seem cluttered.

Many mathematicians carry around inside their heads mental images of the abstractions they work with, and manipulate these objects somehow in conformity with their mental imagery. They probably also make aesthetic judgements of the value of their work according to the visual qualities of the images. These presumably common phenomena remain a rarely explored domain in either art or psychology.

—*Bill Casselman* (covers@ams.org)