Logical Dilemmas: The Life and Work of Kurt Gödel and *Gödel: A Life of Logic*

Reviewed by Martin Davis

Logical Dilemmas: The Life and Work of Kurt Gödel
John W. Dawson Jr.
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Gödel: A Life of Logic
John L. Casti and Werner DePauli
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In its March 29, 1999, issue *TIME* magazine provided its picks for the twenty greatest "scientists and thinkers" of the twentieth century. Kurt Gödel was on the list along with his good friend Albert Einstein. Alan Turing was the other mathematician among the twenty. Since we may agree that Einstein should be regarded primarily as a physicist, it turns out that the mathematicians *TIME* selected as the great thinkers of the past century were a pair of logicians! Whatever one may think about these two, the work of the overwhelming majority of mathematicians has been quite unaffected by what they accomplished.

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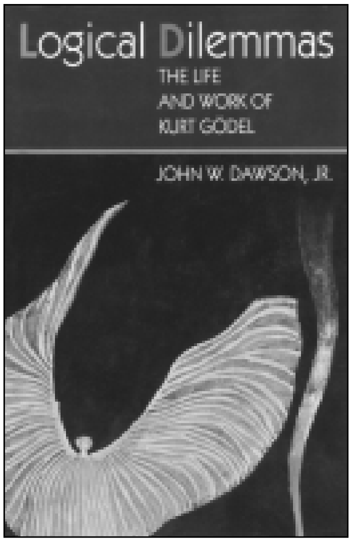
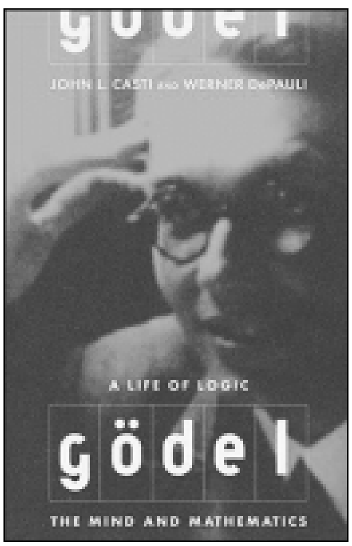
Turing's main influence on most mathematicians, and indeed on the population at large, stems from his role as progenitor of the computer (see [1]), and it was for this role that *TIME* selected him.

In Douglas Hofstadter's *TIME* article on Gödel, John von Neumann is quoted proclaiming that Gödel's "achievement…is singular and monumental…a landmark which will remain visible far in space and time.…" Although Gödel did a number of other very important things, it is the "achievement" to which von Neumann referred, Gödel's incompleteness theorem, that has caught the imagination of the educated public. Indeed, Hofstadter himself played an important part in bringing Gödel's work to the attention of a general audience by writing his Pulitzer Prize-winning book [5], a whimsical, artful work, full of amusing dialogues and connections with music, art, and artificial intelligence.

Kurt Gödel was a very strange man, and his life is as interesting as his scientific work. Both of the books under review bring the two together but are intended for quite different audiences. John Dawson explains in his preface that although he has not "presumed any acquaintance with modern mathematical logic," he has assumed that his readers "possess a modicum of mathematical understanding." In fact, his book would likely be tough going for anyone who had not studied mathematics at the graduate level. John L. Casti and Werner DePauli, on the other hand, have directed their book at a general audience. How well they have succeeded will be discussed later.

John Dawson faced a formidable task when he began writing his definitive biography of Kurt Gödel. He needed to tell the story of a very peculiar and eccentric man in a way that did not minimize his peculiarities but did not sensationalize them either and that placed his story in the context of his time and his remarkable accomplishments. As Dawson himself puts it in his preface, "The problem is to make the ideas underlying his work comprehensible to non-specialists without lapsing into oversimplification or distortion, and to reconcile his personality with his achievements." As the official cataloger of the huge mass of documents of various kinds that Gödel left behind and as coeditor of his *Collected Works*, Dawson was uniquely suited for this responsibility. It also helped that his wife, Cheryl Dawson, had gone to the trouble of learning the obscure "Gabelsberger" shorthand that Gödel used for his personal notes.





Dawson begins with Kurt as an inquisitive child, forever asking "Why?"—in his family he was "Herr Warum" ("Mr. Why"). In Brno (then part of the Austro-Hungarian Empire, today in the Czech Republic) Kurt maintained perfect grades in the German-language schools he attended. After his bout with rheumatic fever, which left him convinced that his heart had been impaired, he became a lifelong hypochondriac. For his university education he was drawn to Vienna, 68 miles south of Brno, where he soon decided on mathematics as his field of study.

Interrupting the biography with a chapter providing a capsule history of mathematical logic to 1928, Dawson continues with a discussion of Gödel's dissertation. The problem Gödel chose had to do with the basic rules of logical deduction used in mathematical proofs, rules that had first been worked out by Gottlob Frege in his *Begriffsschrift* of 1879. Frege's key discovery was that in addition to the propositional connectives—usually written $\neg \vee \wedge$ and $\supset$—whose rules had been found by Boole, it was necessary to use the existential and universal quantifiers—written nowadays $\exists$ and $\forall$—to uncover the logical structure of mathematical propositions. The basic steps in a mathematical proof then can be seen as amounting to applying appropriate rules for manipulating these symbols. Of course, for very good reasons proofs are not presented at that level of detail. But in principle they could be. These rules can be given in various equivalent ways, and we needn't worry about the details. Roughly speaking, the quantifiers get in the way of taking the steps called for by the propositional connectives. So one way of thinking of the rules (logicians call this *natural deduction*) is that the rules specify how to remove quantifiers safely and how to restore them. Mathematicians do this all the time, proceeding intuitively, and don't need to know the rules. But it becomes crucial to be precise about them when one is proving theorems about what can or cannot be proved.

The problem that Gödel took as his dissertation topic, proposed by Hilbert in 1928, was to show that Frege's rules are complete, that by their means any valid inference could be justified. In the dissertation the proof of completeness used familiar methods but in a daring way. However, Gödel's epochal paper on undecidability, which also dealt with a completeness problem proposed by Hilbert, used entirely novel methods. The second problem in Hilbert's famous 1900 list asked for a consistency proof for arithmetic. Working with Hilbert's ideas, such researchers as Ackermann, von Neumann, and Herbrand had been attempting to find such a proof for a system based on Frege's rules of logic, with a language appropriate for the arithmetic of positive integers and with formal versions of Peano's postulates as axioms. In an address in Bologna in 1928, Hilbert asked for a proof that this system is complete in the sense that any proposition expressible in its language would be provable or refutable from the axioms. What Gödel proved was that not only are these systems incomplete but there is no hope for achieving arithmetic completeness by means of more powerful systems. Finally, he dealt a mortal blow to the efforts to prove the consistency of arithmetic by proving that formal systems were generally incapable of proving their own consistency.

One result of the sensation created by Gödel's incompleteness theorem was an invitation to visit the newly established Institute for Advanced Study in Princeton. Gödel arrived in the fall of 1933, three years after he had announced that theorem. In December of that year he delivered an address entitled "The present situation in the foundations of mathematics" at a joint meeting of the Mathematical Association of America and the AMS in Cambridge, Massachusetts (Dawson, p. 100; [4], vol. III, pp. 45–53). In this talk he maintained that the problem of giving an adequate foundation for the whole of mathematics that avoided the familiar paradoxes (like Bertrand Russell's class of all those classes not members of themselves) had found a unique solution. This was to think of the sets needed for mathematics as occurring in *levels* or *types*. Beginning with a set of "individuals" (for example, the natural numbers), one can form the set consisting of these individuals together with all of the sets of these same individuals. At each new level, one can adjoin to the elements of the previous level all sets whose members are in that previous level. In this manner one obtains types $V_0, V_1, \ldots$. But, as Gödel emphasized, there is no reason to stop there. One can go on to $V_\omega = \bigcup_{n=0}^{\infty} V_n$ and continue the process. Although Gödel did not pause to mention the fact, it turns out that there is no loss in generality in beginning the process with the empty set, and that has become the standard practice. Moreover, if one proceeds in that manner, one can define each subsequent type V' to be simply the *power set* of the previous type, V, that is, the set of all subsets of V. Writing "$\mathcal{P}$" for the *power set* operation, i.e.,

$$\mathcal{P}(x) = \{y \mid y \subseteq x\},$$

one has:

$$V_0 = \emptyset;\ V_{n+1} = \mathcal{P}(V_n);\ V_\omega = \bigcup_{n=0}^{\infty} V_n.$$

(It is not difficult to see by induction that for $n = 0, 1, \ldots, V_n \subseteq V_{n+1}$.) But there is still no need to stop. One can go on to $V_{\omega+1} = \mathcal{P}(V_\omega)$, etc. Contemporary set theorists work with the *cumulative hierarchy*

$$V_0 = \emptyset; \qquad V_{\alpha+1} = \mathcal{P}(V_\alpha);$$
$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha \quad (\lambda \text{ a limit ordinal}).$$

Gödel explained that a suitable foundation for mathematics consists of axioms for this hierarchy of types, with Frege's rules of logical inference being used to obtain theorems from the axioms. (It may be mentioned in passing that this set-theoretic foundation for mathematics has been widely accepted, as can be seen in the introductory sections on sets and mappings in typical beginning graduate-level textbooks.) A system of axioms for set theory, he further explained, can be understood as consisting of closure properties (i.e., properties that enable one to proceed from the existence of given sets to the existence of other sets formed from them). By forming the least set closed under those operations, one obtains a set belonging to the hierarchy; however, the very existence of that set cannot be proved from those axioms. (This is because one could use the existence of this set to prove the consistency of the given axioms from those axioms, which Gödel had shown to be impossible.) This set "can be considered as a new domain of individuals and used as a starting point for creating still higher types." He continued, explaining the relationship between this situation and his incompleteness theorem:

> …we are confronted by a strange situation. We set out to find a formal system [of axioms] for mathematics and instead of that found an infinity of systems, and whichever system you choose …, there is one…whose axioms are stronger.
>
> But…this character of our systems…is in perfect accord with certain facts which can be established quite independently.…For any formal system you can construct a proposition—in fact a proposition of the arithmetic of integers—which is certainly true if the given system is free from contradictions but cannot be proved in the given system. Now if the system under consideration (call it S) is based on the theory of types, it turns out that…this proposition becomes a provable theorem if you add to S the next higher type and the axioms concerning it ([4], vol. III, pp. 47–48).

In the years immediately following Gödel's breakthrough, his life inevitably became entangled in the sinister events developing in Germany and Austria. In 1940, when it was almost too late, he finally decided to emigrate to America. By this time he had made several extensive visits to the U.S. Suffering from depression, he had felt forced to abort one of these visits and had spent some time in mental institutions. He managed to keep secret a romance he had developed with Adele Porkert, an attractive dancer who had previously been married, so that when he and Adele married, her very existence came as a surprise to his friends and colleagues. Gödel only resolved to leave Vienna when, to his surprise, he was found fit for "garrison duty" in the German army. Leaving was not easy because World War II had already broken out. In addition to bureaucratic difficulties on both sides of the Atlantic, that ocean was no longer safe. Gödel with his bride traveled across Siberia (the pact between Germany and the Soviet Union still being in force), across the Pacific to

Kurt Gödel

California, and finally by train to Princeton, where he was to remain for the rest of his life.

In the meantime Gödel had made another breakthrough regarding a fundamental problem: Cantor's continuum hypothesis (CH). This is the assertion that infinite sets of real numbers come in only two sizes, i.e., that each such set is either countable or has the same cardinality as the set of all real numbers. Famously, Cantor had tried very hard to prove this with no success. The status of CH was the first problem in Hilbert's famous list in his 1900 address. What Gödel was able to prove was that if the usual systems of axioms for set theory (including, in particular, the so-called Zermelo-Fraenkel axioms) are consistent, then they remain consistent if the axiom of choice and CH are added as additional axioms (Dawson, pp. 115–122). (As usual we write ZF for the Zermelo-Fraenkel axioms, and ZFC if the axiom of choice is included.)

The main tool used in the proof is a modification of the cumulative hierarchy that was discussed above. The language of set theory can be used not only to express propositions but also to define sets. For example, the formula

$$\neg(\exists y)(y \in x) \\ \vee [(\exists y)(y \in x) \wedge (\forall z)(z \in x \supset \neg(\exists y)(y \in z))]$$

can be satisfied only if x is either the empty set or contains a single element, namely the empty set. One says that this formula *defines* the set $\{\emptyset, \{\emptyset\}\}$. In general, given any set S, one can consider subsets of S definable by formulas of the language of set theory. The formulas used in these definitions are allowed to contain "parameters" standing for particular elements of S. Thus if $S = \{\emptyset, \{\emptyset\}\}$, the formula $x = \{\emptyset\}$ containing the parameter $\{\emptyset\}$ defines the subset $\{\{\emptyset\}\}$ of S. Let us write $\mathcal{D}(S)$ for the collection of all subsets of a given set S that can be defined in this manner. Evidently for any set S, $\mathcal{D}(S) \subseteq \mathcal{P}(S)$. Note that if S is countably infinite, this inclusion is proper: $\mathcal{P}(S)$ is uncountable (in fact having the cardinality of the continuum), whereas, because there are only countably many formulas, $\mathcal{D}(S)$ is countable. Now just as the cumulative hierarchy is defined by indefinitely iterating the power set operation $\mathcal{P}$, Gödel defined what he called the *constructible sets* as those obtained beginning with $\emptyset$ and indefinitely iterating the $\mathcal{D}$ operation. The precise definition is again by transfinite recursion:

$$L_0 = \emptyset; \qquad L_{\alpha+1} = \mathcal{D}(L_\alpha);$$
$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \quad (\lambda \text{ a limit ordinal}).$$

The constructible sets are then those belonging to any one of the L_α. Gödel introduced the proposition

A: *Every set is constructible*

and proved the following:

- A is consistent with ZF.
- A implies the axiom of choice.
- A implies CH.
- In fact, A implies that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ (the so-called "generalized continuum hypothesis").

In his 1938 announcement in the *Proceedings of the National Academy of Sciences* ([4], vol. II, pp. 26–27), Gödel said:

> The proposition A added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague concept of an arbitrary infinite set in a definite way.

A decade later in an expository article about CH ([4], vol. II, pp. 176–187), his position was quite different. Espousing a "realist" notion of *set* based on the cumulative hierarchy, it was clear that he no longer regarded the notion as "vague". Regarding CH, he predicted that it would be found to be not only consistent with ZFC, as he had shown, but actually independent, a prediction that was fulfilled in 1962 by Paul Cohen. However, Gödel warned against accepting this independence as ending the matter. Rather, he concluded with the daring prediction:

> …the continuum problem…will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

In particular Gödel speculated that the kind of axiom that could do the trick might be a so-called *large-cardinal axiom*, that is, an axiom not provable from the Zermelo-Fraenkel axioms that implies the existence of levels of the cumulative hierarchy of enormous size. However, because of the wide applicability of Paul Cohen's methods, it became clear that large-cardinal axioms by themselves could not settle CH, and Gödel's prediction seemed increasingly far-fetched. However, very recently Hugh Woodin [10] has developed new methods that do suggest that Gödel was right after all. The two-part article "The Continuum Hypothesis," by Hugh Woodin, appeared in the June/July 2001 and August 2001 issues of the *Notices*.

Gödel returned to the theme of his 1933 Cambridge lecture in 1951 when he delivered the twenty-fifth annual Josiah Willard Gibbs Lecture at an AMS meeting in Providence.

> The phenomenon of the inexhaustibility of mathematics…always is present…all of mathematics is reducible to abstract set theory.…Now if one attacks [the] problem [of axiomatizing set theory], the result is quite different from what one would have expected.…one is

> faced with an infinite series of axioms, which can be extended further and further, without any end being visible…there can never be an end…because the very formulation of the axioms up to a certain stage gives rise to the next axiom. It is true that in the mathematics of today the higher levels of this hierarchy are practically never used.…it is not altogether unlikely that this character of present-day mathematics may have something to do with…its inability to prove certain fundamental theorems, such as, for example, Riemann's hypothesis.…For…the axioms for sets of high levels…have consequences even for…the theory of integers. To be more exact, each of these set-theoretical axioms entails the solution of certain diophantine problems which had been undecidable from the previous axioms (Dawson, pp. 197–200; [4], vol. III, pp. 305–308).

Making use of the work leading to the unsolvability of Hilbert's tenth problem (on Diophantine equations) [2], one can be very specific about this last comment:

Theorem. *There is a polynomial $p(a, x_1, \ldots, x_n)$ with integer coefficients having the following property: Corresponding to any consistent list of axioms in the language of set theory, there is an integer z_0 such that while the equation*

$$p(z_0, x_1, \ldots, x_n) = 0$$

has no solutions in natural numbers, that fact cannot be proved from those axioms.

Here, the word "list" is intended to imply that if there are infinitely many axioms, then there is an algorithm that can systematically generate them. Note that the polynomial p is fixed, so that while the unprovable fact will become provable if one appropriately strengthens the given axioms (for example, as Gödel proposed, by proceeding to the next level of the cumulative hierarchy), there will be a new unprovable fact differing from the previous one only by a change in the value z_0 of the parameter a. It should be noted that all of this is entirely constructive: the polynomial p can be given explicitly, and the number z_0 can be computed explicitly from the axioms. However, it must be admitted that the polynomial p is by no means beautiful to behold [6]. Moreover, the constant z_0, being in effect a numerical encoding of the axioms, will be enormous.

In his Gibbs lecture, Gödel was quite willing to stick his neck out and to predict emphatically that "Some kind of set-theoretical number theory, still to be discovered, would certainly reach much further" than what can be accomplished with "analytic number theory." Alas, forty years later this "set-theoretical number theory" still remains "to be discovered," and the working lives of number theorists remain almost entirely unaffected by Gödel's discoveries.

Two areas where one might claim that Gödel's vision of the effectiveness of axioms going beyond ZFC has been vindicated are descriptive set theory and finite combinatorics. The hierarchy of projective sets in Euclidean space can be defined as follows: one begins with the Borel sets in n dimensions (for arbitrary n) and iterates the operations of *projection* to a lower-dimensional space and *complementation* (i.e., for $E \subseteq R_n$, forming $R_n - E$). Investigations of the projective hierarchy between the world wars (mostly in Eastern Europe) came to a dead end with problems that seemed utterly out of reach. Work in the 1960s made it clear that these problems were indeed beyond the scope of the Zermelo-Fraenkel axioms. However, it turned out that a very plausible axiom ("projective determinacy") resolved all of these problems in a very satisfactory manner ([8], [7], Chap. 6). Finally it has been shown that this axiom is itself a consequence of *large-cardinal axioms*, axioms that assert the existence of levels of the cumulative hierarchy of enormous size [9].

In finite combinatorics it was in the context of Ramsey theory that examples of genuine mathematical interest were found that required "the higher levels" of the cumulative hierarchy for their proof. Harvey Friedman has found examples that even go beyond ZFC (see his paper [3], which also has references to previous work). Most recently, he has used large-cardinal axioms to obtain some particularly striking results.

As usual, let Z be the set of integers. Friedman calls a subset of Z *bi-infinite* if it has infinitely many positive elements as well as infinitely many negative elements. Given $A, B, C \subseteq Z$, one says that A, B *disjointly cover* C if

$$A \cup B \supseteq C \quad \text{and} \quad A \cap B = \emptyset.$$

For $x \in Z^n$, say $x = (x_1, \ldots, x_n)$, write

$$|x| = \max_{1 \le i \le n} |x_i|.$$

Friedman considers *multivariate functions* on the integers, i.e., functions that map Z^n into Z for some n. For such a function f and for $A \subseteq Z$, he writes

$$fA = \{f(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in A\}$$

and says that f is of *expansive linear growth* if for some $p, q > 1$, the inequalities $p|x| \le f(x) \le q|x|$ hold for all sufficiently large $|x|$. Friedman has shown that although the following proposition is not provable from ZFC, it can be proved using a large-cardinal axiom:

Bi-infinite Disjoint Cover Theorem. *Let f, g be multivariate functions on Z of expansive linear growth. Then there exist bi-infinite $A, B, C \subseteq Z$ such that C, gB disjointly cover fA and C, gC disjointly cover fB.*

Among the various possible variants of this theorem, it might be worth mentioning that the proof of the proposition obtained by simply replacing "C, gB" by "B, gB" not only requires no large-cardinal axiom but in fact can be carried out using axioms far weaker than ZF. Friedman sees propositions like these in the context of a general development he calls *Boolean Relation Theory*, which he predicts will have wide ramifications, intersecting many disciplines, with the use of large-cardinal axioms frequently being necessary.

Pages could be filled listing and correcting the errors in [Casti and DePauli's] book. Probably the worst is the misstatement of Gödel's incompleteness theorem.

Gödel's friends and colleagues in Princeton could see that he suffered from eccentricity bordering on paranoia. On more than one occasion he seriously endangered his life by stubbornly refusing to accept medical advice. Finally, when Adele was ill and unable to prepare food he considered safe, he stopped eating and literally starved himself to death. He died on January 14, 1978, at the age of seventy-two. Although Dawson gives a full account of these matters, he always preserves an appropriate tact. In addition to the aspects of Gödel's work discussed in this review, Dawson's masterly biography doesn't omit Gödel's other important scientific contributions: in particular, his functional-based semantics for intuitionistic logic and his novel solution of the equations of general relativity.

Because Gödel's work on undecidability is of such general interest, treatments of his life and work intended for a general audience are very desirable. The book by Casti and DePauli is an effort in this direction. Unfortunately it is deeply disappointing, being marred by serious errors sure to confuse the novice.

In order to explain the idea of proof in mathematics, the authors tell the charming tale of how Gauss as a schoolboy is said to have summed the numbers from 1 to 100 by writing the numbers

1	2	…	50
100	99	…	51

and noting that each column adds up to 101. They then show how the same method can be used to sum the numbers from 1 to n yielding the formula $n(n+1)/2$ (with the caveat that for n odd, 0 must be included). Astonishingly, readers are then told that this proof "is not a proof that the formula holds for *every* positive integer n; it's just a proof for any fixed number.…" This nonsense is followed by a very brief explanation of mathematical induction as the "usual" way the formula is proved. Next comes a piece of utterly gratuitous misinformation:

> There are some philosophers of mathematics who argue that such nonconstructive and/or infinitary principles of inference as mathematical induction should not be admitted into mathematics as a tool of proof.

Of course constructivists have no quarrel with mathematical induction. Perhaps the authors were confusing the finitary rule of inference of mathematical induction with the infinitary nonconstructive ω-rule. The former obtains the conclusion $(\forall n)\mathcal{A}(n)$ from the two premises

$$\mathcal{A}(0) \quad \text{and} \quad (\forall n)[\mathcal{A}(n) \Rightarrow \mathcal{A}(n+1)],$$

while the latter obtains that same conclusion from the infinite set of premises:

$$\mathcal{A}(0), \mathcal{A}(1), \mathcal{A}(2), \ldots.$$

The ω-rule is an interesting thing for logicians to study, but as stated it is not a practical rule of proof—a mathematician ordinarily has that infinite set of premises available only when the desired conclusion has already been obtained some other way.

Unfortunately, the authors' confusion is not limited to this one example. Pages could be filled listing and correcting the errors in this book. Probably the worst is the misstatement of Gödel's incompleteness theorem. As Gödel was at great pains to emphasize and has been explained above, it is a question of *relative* incompleteness: the statement found to be undecidable in a given system is seen to be true in a more comprehensive system obtained using a natural construction. But over and over again the authors give the impression that it is a matter of *absolute* undecidability. Thus, here is their version of what can be inferred from the work on Hilbert's tenth problem:

> There exists a Diophantine equation having no solution—but no theory of mathematics can prove this.

No such Diophantine equation is known. A correct statement (involving a system-dependent parameter) was stated above.

Although Casti and DePauli include Dawson's excellent biography in their list of references, there are many inaccuracies concerning Gödel's life and thought. The example that annoyed me most was the assertion that Gödel "first described himself as a mathematical realist in 1925." What is true is that in 1975, in replies to a questionnaire, he asserted that he had been a mathematical realist (that is, one who accepts mathematical entities such as sets as "real") since 1925. To serious Gödel scholars this is a crucial difference, since there are good reasons to doubt that his assertion was true. For example, Gödel's suggestion in 1938 that his statement "A" (to the effect that all sets are constructible) was a reasonable completion of the "vague" notion of set is not what a "realist" would say. (See also the introductory notes in [4], vol. III, by Solomon Feferman, pp. 36–44, and by me, pp. 156–163.)

To end on a positive note: Casti and DePauli spend some time discussing Gödel's interesting unsuspected solution of the equations of general relativity. In the universe specified by this solution, time travel to the past is possible in principle. I believe their discussion is accurate and interesting. The book concludes with an illuminating exposition of Gregory Chaitin's information-theoretic form of Gödel's theorem.

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