Conferences

AMS Short Course

Symbolic Dynamics and Its Applications

San Diego, California, January 4–5, 2002

This program is under the direction of **Susan Williams**, University of South Alabama. Please refer to the website maintained by the organizer at `http://mathstat.usouthal.edu/williams/ams.html/` for the most current information.

Lecture notes will be available to those who register for this course. Advance registration fees: $80 AMS/MAA members; $110 nonmembers; ($35/student/unemployed/emeritus); on-site registration fees: $100 AMS/MAA members; $130 nonmembers; ($50/student/unemployed/emeritus). Registration and housing information can be found in this issue of the *Notices*; see the section "Registering in Advance and Hotel Accommodations" in the announcement for the meetings in San Diego.

A symbolic dynamical system is a dynamical system with a discrete state space. Such systems were first considered as a tool for analyzing general dynamical systems by discretizing space. We partition the state space into finitely many pieces, each identified with a symbol in some finite alphabet, and then associate to each point of the space the sequence of symbols corresponding to the partition elements visited by its trajectory. Ideally, we look for a partition for which the set of symbolic trajectories has a simple description and dynamical properties that mirror those of the original system. This technique was first employed by Hadamard in 1898 to study geodesic flows on surfaces of negative curvature. It has been sucessfully extended to a wide range of dynamical systems, including hyperbolic diffeomorphisms and complex dynamical systems.

The study of symbolic systems in their own right was initiated in a 1938 paper of Marston Morse and Gustav Hedlund. Subsequent developments in information theory brought an added impetus. Strings of symbols are natural objects in mathematical communication and coding theory, which remain vital areas of application of symbolic dynamics.

The *full (two-sided) shift* on an alphabet A is the sequence space $A^{\mathbb{Z}}$, given the product topology, together with the *shift map* which shifts each coordinate of the sequence to the left. Any closed, shift-invariant subset is a symbolic dynamical system. An important class are the *shifts of finite type*, which are defined by a finite set of local constraints, that is, by forbidding the occurrence of symbol strings in a certain finite list. The definition of a symbolic dynamical system may be expanded in several ways. The finite alphabet may be replaced by a countable one. The dynamics of one-sided sequences is also studied and applied to the analysis of general noninvertible dynamical systems. The past decade has seen substantial development in the theory of multidimensional symbolic systems, with applications to tilings, modeling of materials, and coding of images.

The course will begin with an introductory survey of symbolic dynamics: its history and the fundamental definitions, results, and examples. This will be followed by a series of lectures on selected applications and new directions in the field.

Combining Modulation Codes and Error-Correction Codes

Brian Marcus, IBM Almaden Research Center

Synopsis

In the abstract, a channel is a "black box" with inputs and outputs. The inputs represent messages that are transmitted through the channel. The outputs are supposed to faithfully represent the inputs. However, distortions in the channel can adversely affect the output. For this reason, coding is applied to protect the messages.

One usually thinks of a channel as a communications system in which information is sent from one point in space to another. Examples of communications systems include telephones, cellphones, digital subscriber lines, and deep space communications. But recording systems, such as magnetic/optical disk/tape drive systems, can also be viewed as channels. Current recording applications require storage devices to have very high immunity against errors. On the other hand, the ever-growing demand for storage forces the designers of such devices to write more data per unit area, thereby making the system less reliable. In magnetic recording systems this is manifested in the effects of interference between successive transitions in magnetization (called intersymbol interference), inaccurate clocking, and random noise.

A modulation encoder, also known as a constrained encoder or line encoder, transforms arbitrary user data sequences into sequences, also called codewords, that satisfy a given constraint. In the most general terms, the purpose of a modulation code is to improve the performance of the system by matching the characteristics of the recorded signals to those of the channel; the recorded signals are thereby constrained in such a way as to reduce the likelihood of error. For instance, run length constraints, which bound the runs of zeros in an encoded data stream, help to mitigate the problems of intersymbol interference and inaccurate clocking.

In addition to modulation coding, an error-correction code (ECC) may be used to protect the data against random noise sources. A good ECC has the property that any two distinct codewords differ enough so as to be distinguishable even after being subjected to a certain amount of channel noise. While both error-correction coding and constrained coding have been active for fifty years, the former enjoys much greater notoriety.

What is the difference between an error-correction code and a modulation code? One difference is that the "goodness" of an error-correction code is measured by how the different codewords relate to one another (e.g., in how many bit locations must any two distinct codewords differ?), whereas the "goodness" of a modulation code is measured by properties of the individual codewords (e.g., how well does each codeword pass through the channel?).

On the other hand, this distinction is not hard and fast. Clearly, if an error-correction code is to have any value at all, then its codewords cannot be completely arbitrary and therefore must be constrained. Conversely, in recent years there has been a great deal of interest in constrained codes that also have error-correction properties. Such developments have contributed to a blurring of the lines between these two types of coding. Nevertheless, each subject has its own emphases and fundamental problems that are shaped by the distinction posed in the preceding paragraph.

Ideally the messages recorded on a channel should be determined by a single code that has both "pairwise" error-correction properties as well as "individual" modulation properties. In this talk we will survey some methods for combining modulation codes with error correction codes. Part of this will involve connections with symbolic dynamics: an algorithm for constructing modulation codes based on state splitting of shifts of finite type and a method of constructing combined modulation/error-correction codes based on the follower set description of sofic shifts. The talk will assume only a minimal familiarity with symbolic dynamics and coding theory. The references below provide much more than is needed.

Reading List

[1] B. Marcus, R. Roth, P. Siegel, Constrained systems and coding for recording channels, Chapter 20 of *Handbook of Coding Theory* (V. Pless, C. Huffman, and R. Brualdi, eds.), Elsevier, 1998.

[2] S. B. Wicker, *Error Control Coding in Digital Communication and Storage*, Prentice-Hall, 1995 (Chapters 1, 4, 5, and 8).

Complex Dynamics and Symbolic Dynamics

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Synopsis

As so often happens in mathematics, there is a surprising connection between two quite distinct subfields of dynamical systems theory, namely, the structure of the automorphism group of the one-sided shift map on d-symbols and the topology of the analogue of the Mandelbrot set for degree d polynomials of one complex variable. In this lecture we will give an elementary overview of both of these topics, highlighting the tools that relate them.

Let Σ_d denote the space of (one-sided) sequences of integers $0, 1, \dots d-1$, and let σ denote the usual shift map $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots)$. An important question in symbolic dynamics concerns the group of automorphisms of the shift, i.e., maps $\eta : \Sigma_d \to \Sigma_d$ that commute with the shift map. For one-sided shift maps, this group is well understood thanks to work of Hedlund [6], Boyle, Franks, and Kitchens [3], and Ashley [1]. The automorphism group for the 2-shift is simple: There is only one nontrivial element, namely, the automorphism that interchanges the two symbols 0 and 1. For the d-shift the group is infinitely generated with a rich algebraic structure.

Turning now to complex dynamics, consider first the dynamics of quadratic polynomials of the form $Q_c(z) = z^2 + c$. As is well known, the interesting dynamics of this map takes place on the Julia set [7], J_c. This set assumes one of two topological types: either J_c is connected or J_c is a Cantor set. In the latter case, the action of Q_c on the Julia set is equivalent to the one-sided 2-shift. The well-known Mandelbrot set $\mathcal{M}$ is a picture of this dichotomy: If c lies in $\mathcal{M}$, the Julia set is connected; outside $\mathcal{M}$, J_c is a Cantor set. If we follow a closed loop in the complement of $\mathcal{M}$, the return to the original position induces an automorphism of the shift. If this loop winds once around $\mathcal{M}$, then the induced automorphism is the nontrivial element of the group; on the other hand, if the loop does not contain $\mathcal{M}$, the trivial automorphism is induced.

The main goal of this lecture is to describe a similar phenomenon that occurs for polynomials of higher degree. Here the analogue of $\mathcal{M}$ lies in complex $d-1$, dimensional space and has a rich topology. Following loops around various portions of this space again induces an automorphism of the d-shift. We will describe how one can generate every automorphism of the shift in this manner, thus yielding a surjection from the fundamental group of this space onto the group of automorphisms [2].

For background on complex dynamics, we suggest the proceedings from several previous AMS Short Courses [4], [5]. For background on the relevant symbolic dynamics, see [8].

References

[1] J. Ashley, Marker automorphisms of the one-sided d-shift, *Ergodic Theory Dynam. Systems* **10** (1990), 247–262.

[2] P. Blanchard, R. Devaney, and L. Keen, The dynamics of complex polynomials and automorphisms of the shift, *Inventiones Math.* **104** (1991), 545-580.

[3] M. Boyle, J. Franks, and B. Kitchens, Automorphisms of the one-sided shift and subshifts of finite type, *Ergodic Theory Dynam. Systems* **10** (1990), 421–449.

[4] R. Devaney, *Complex Dynamical Systems: The Mathematics behind the Mandelbrot and Julia Sets*, Amer. Math. Soc., 1994.

[5] R. Devaney and L. Keen, *Chaos and Fractals: The Mathematics behind the Computer Graphics*, Amer. Math. Soc., 1989.

[6] G. Hedlund, Endomorphisms and Automorphisms of the Shift Dynamical System, *Math. Systems Theory* **3** (1969), 320-375.

[7] L. Keen, Julia sets, *Chaos and Fractals: The Mathematics behind the Computer Graphics*, Amer. Math. Soc. 1989, pp. 57–74.

[8] D. LIND and B. MARCUS, *Symbolic Dynamics and Coding*, Cambridge Univ. Press, 1995.

Multi-Dimensional Symbolic Dynamics

Douglas Lind, University of Washington

Synopsis

Dynamics has traditionally studied the iterates of a single transformation, modeling the time evolution of a physical system. However, many physical and mathematical systems have other symmetries as well. This leads directly to the study of the joint action of several commuting transformations. This lecture will introduce simple examples of such actions, describe some of their properties and significant theorems, and discuss a few of the many open problems. The main message is that the study of joint actions is much more than a routine generalization of a single transformation and that genuinely new and deep phenomena occur which are only now being understood.

We will begin by describing the higher-dimensional analogue of a shift of finite type, which consists of all d-dimensional arrays of symbols from a finite alphabet subject to a finite number of local rules or conditions. Such arrays can be shifted in each of the d coordinate directions, given d commuting invertible transformations, or what amounts to the same thing, an action of the lattice $\mathbb{Z}^d$ of d-tuples of integers. Already a deep distinction arises between $d = 1$, where it is quite easy to describe the space of such arrays, and $d \geq 2$, where there is no general algorithm which will decide, given the set of local rules, whether or not the space is empty!

A good example to keep in mind is that of Wang tiles. Imagine a finite set of unit squares, with each square having its four edges colored (two different edges are allowed to have the same color). The corresponding two-dimensional shift of finite type is the set of all tilings of the plane by copies of these squares, subject to the local rule that overlapping edges must have the same color. Different collections of Wang tiles lead to quite different shifts of finite type, some of which are well understood and some still mysterious.

Although multidimensional shifts of finite type are in general still quite difficult to understand, there is one class of commuting transformations for which a systematic and powerful theory has been recently developed. These are *algebraic* actions, which are commuting automorphisms of compact abelian groups. Here is a simple but very instructive example: consider all two-dimensional arrays of 0's and 1's subject to the condition that, at each site, the sum of the digits at the site itself, the one to the right, and the one above is even. It is not hard to see that this set is a compact abelian group under coordinate-wise operations and that the horizontal and vertical shifts are commuting group automorphisms. We will see how very concrete examples like this can be thoroughly investigated using tools from harmonic analysis and commutative algebra.

Finally, we will conclude with a taste of open problems, such as Furstenburg's conjecture about measures simultaneously invariant under several commuting transformations, Lehmer's conjecture on Mahler measure and its connections with entropy, and the computation of information capacity that arises in holographic data storage.

Reading List

[1] DOUGLAS LIND and BRIAN MARCUS, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, 1995.

[2] KLAUS SCHMIDT, *Dynamical Systems of Algebraic Origin*, Birkhäuser, 1995.

Symbolic Dynamics and Tilings of $\mathbb{R}^d$

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Synopsis

Let X_p be the set of all of tilings of $\mathbb{R}^d$, $d \geq 1$, by translations of a fixed finite set of basic tile shapes (called *prototiles*). We assume that tilings $x \in X_p$ satisfy the *local finiteness* condition: For any $R > 0$ there are, up to a small translation, only finitely many pictures in an R-ball sampled from any $x \in X_p$. It then follows that X_p is compact metrizable in the tiling topology: two tilings are ϵ-close if they agree after an ϵ translation in a $1/\epsilon$ ball around $\mathbf{0}$. We allow $\mathbb{R}^d$ to act on X by translation and denote this action by $T^{\mathbf{t}}x$. This action is clearly continuous.

Tiling dynamics studies the pair (X_p, T) using dynamical systems theory. More generally, one studies (X, T), where X is a closed T-invariant subspace of X_p.

The interpretation of (X_p, T) or (X, T) is that they are multidimensional continuous-time symbolic dynamical systems. In particular, we think of the prototiles as the symbols, i.e., p is the alphabet. Carrying this analogy a little further and using the language of symbolic dynamics, X_p is the full p shift and X is a subshift. In some cases X is determined by a set of "local matching rules", which restrict the types of allowed tile adjacencies. This occurs, for example, in the famous *Penrose tilings*. We view such an (X, T) as a tiling "shift of finite type". In other cases when X is defined via a "tiling inflation", (X, T) generalizes the idea of a substitution dynamical system.

The tiling topology on X is reminiscent of the product topology on a one-dimensional discrete shift space $\{1, 2, \ldots, n\}^{\mathbb{Z}}$. There are, however, interesting new complications that arise for tiling systems. First, the acting group is continuous (i.e., $\mathbb{R}^d$ instead of $\mathbb{Z}^d$ or $\mathbb{Z}$), making geometry play a significant role in the theory. For example, allowing rotations as well as translations to act on X leads to applications to the study of the symmetries of quasicrystals.

The second complication is that the acting group is multi-dimensional. Because of this, the theory acquires all the complexity of $\mathbb{Z}^d$ symbolic dynamics, including the phenomenon of undecidability: For p in general, it is undecidable whether $X_p \neq \emptyset$.

In this lecture we will discuss the foundations of tiling dynamical systems and survey some of the main results. We will discuss various dynamical concepts and properties and describe their counterparts in the theory of tilings. We will also briefly discuss quasicrystals.

Reading List

[1] BRANKO GRÜNBAUM and G. C. SHEPHARD, *Tilings and Patterns*, W. H. Freeman, New York, 1987.

[2] Charles Radin, *Miles of Tiles*, Amer. Math. Soc., Providence, RI, 1999.
[3] E. Arthur Robinson Jr., The dynamical properties of Penrose tilings, *Trans. Amer. Math. Soc.* **348** (1996), 4447–4464.
[4] ____ , The dynamical theory of tilings and quasicrystallography, *Ergodic Theory of* $\mathbf{Z}^d$ *Actions* (Warwick, 1993–1994), Cambridge Univ. Press, Cambridge, 1996, pp. 451–473.
[5] Marjorie Senechal, *Quasicrystals and Geometry*, Cambridge University Press, Cambridge, 1995.
[6] Boris Solomyak, Dynamics of self-similar tilings, *Ergodic Theory Dynam. Systems* **17** (1997), 695–738.

Strong Shift Equivalence and Positive Algebraic K-Theory

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Synopsis

Strong shift equivalence theory emerged from work on the long-standing Shift Equivalence Conjecture in symbolic dynamics. But it turns out to be closely related to areas of mathematics outside dynamics, such as algebraic K-theory, cyclic homology, and topological quantum field theory. We will survey some of these developments.

One way of describing subshifts of finite type is by zero-one transition matrices A arising from Markov partitions for discrete time dynamical systems. Another comes from variable length coding and represents a subshift of finite type by a matrix of the form $I - P$ where P has entries that are polynomials in a variable t with nonnegative integral coefficients. Passing from the first approach to the second is achieved by setting $P = tA$. A subshift of finite type generally has infinitely many Markov partitions and therefore infinitely many presentations in either of the two frameworks. The classification program seeks to determine when two apparently different subshifts of finite type actually have the same dynamical behavior. R. F. Williams formulated an algebraic approach to this problem in [7] by introducing strong shift equivalence and shift equivalence over the nonnegative integers, the latter being much more accessible algebraically. These concepts can be expressed in an elementary fashion by graphs and matrices.

The Shift Equivalence Conjecture/Problem dates from 1974 and asks whether shift equivalence implies strong shift equivalence. Kim and Roush [2] produced the first counterexamples in 1997 for primitive matrices using the Boyle-Krieger sign-gyration-compatibility condition. Subsequently, in [5] another way of detecting primitive counterexamples was found using the algebraic K-theory group K_2. The method comes from an analogy with one-parameter Morse theory, and it uses the polynomial matrix viewpoint. It is part of what might be called positive algebraic K-theory, because multiplication by an elementary matrix transforming $I - P$ to $I - Q$ generates a topological conjugacy in the presence of certain natural positivity conditions.

Polynomial matrix techniques have other applications in symbolic dynamics, such as finding new inert automorphisms of infinite order [4] which are conjecturally detected by the cyclic cohomology Chern character on K_3, characterization of nonzero spectra of nonnegative integral matrices [3], and efficient representation of subshifts of finite type [1]. See the expository account [6] for an overview and references to background literature.

References

[1] M. Boyle and D. Lind, *Small polynomial matrix representations of nonnegative matrices*, in preparation.
[2] K. H. Kim and F. W. Roush, The Williams conjecture is false for irreducible subshifts, *Ann. of Math.* **149** (1999), 545–558.
[3] K. H. Kim, N. Ormes, and F. W. Roush, The spectra of nonnegative integer matrices via formal power series, *J. Amer. Math. Soc.* **13** (2000), 773–806.
[4] K. H. Kim, F. W. Roush, and J. B. Wagoner, Characterization of inert actions on periodic points, Part I and Part II, *Forum Math.* **12** (2000), 565–602 and 671–712.
[5] J. B. Wagoner, Strong shift equivalence and K_2 of the dual numbers, *J. Reine Angew. Math.* **521** (2000), 110–160.
[6] ____ , Strong shift equivalence and the shift equivalence problem, *Bull. Amer. Math. Soc. (N.S.)* **36** (1999), 271–296.
[7] R. F. Williams, Classification of subshifts of finite type, *Ann. of Math.* (2) 98 (1973), 120–153; Errata ibid. **99** (1974), 380–381.