Book Review

Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being

Reviewed by James J. Madden

Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being
George Lakoff and Rafael E. Núñez
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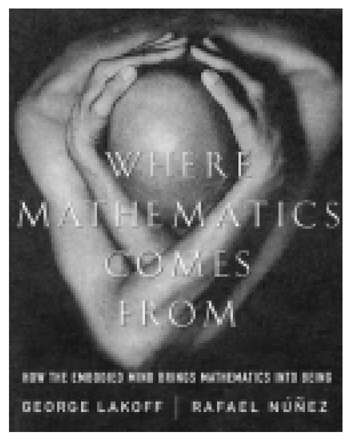
> *A metaphor is an alteration of a woorde from the proper and naturall meanynge, to that which is not proper, and yet agreeth thereunto, by some lykenes that appeareth to be in it.*
>
> —*Thomas Wilson,* The Arte of Rhetorique (1553) [Wi], *page 345*

> *Conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind of thing as if it were another. …It is a grounded, inference-preserving cross-domain mapping—a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say, arithmetic).*
>
> —Where Mathematics Comes From, *page 6*

In his philosophical writings, Poincaré reflected on the origins of mathematical knowledge. His best-known remarks, translated as the essay "Mathematical Creation" in [P1], include speculations about the unconscious processes that precede discovery. This is where we read the famous story of how, while boarding a bus in Coutances, Poincaré suddenly realized the identity of the transformations used to define the Fuchsian functions with those of non-Euclidean geometry.



In his 1905 essay "L'Intuition et la Logique en Mathématiques", which appears in translation in [P1], pages 210–222, Poincaré looks at mathematical production from a different angle. He pictures two different kinds of mathematician. One kind is devoted to explicit logical precision. Ideas must be broken down into definitions and deductions. Even conceptions that seem clear and obvious must be subjected to analysis, cut apart, and examined under the microscope of logic. The other kind of mathematician is guided by geometric intuition, physical analogies, and images derived from experience. This mathematician is like Klein, who proved a theorem in complex function theory by imagining a Riemann surface made of metal and considering how electricity must flow through it.

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Poincaré goes on to argue that logic and intuition play complementary roles in mathematics. Logic provides rigor and certainty by substituting precise notions for vague and ambiguous ones and by moving in sure, syllogistic steps. Logic, however, does not perceive goals and does not grasp that which motivates and organizes our mathematical activity. We may follow the logical trail through an argument yet fail to "see" the idea in it. For this we need intuition, which provides insight, purpose, and direction. But intuition is sometimes ambiguous, and sometimes it even deceives. So ultimately it is only by the combination of logic and intuition that mathematics advances.

How do metaphors function in the mathematical activities of actual people? On this, Lakoff and Núñez are not very clear.

Poincaré set his ideas down at a time when revolutionary advances in mathematical logic were just beginning. He could not have foreseen what the next century would achieve in foundational studies. Today, we cannot claim to have the final word, but we clearly understand much more about logic and the role it plays in mathematics than we did one hundred years ago.

How about intuition? What more do we know about this? A few references come to mind: Hilbert and Cohn-Vossen's book [HC], Hadamard's essay [H], and perhaps a couple of more recent items, particularly S. Dehaene's work on the number sense [Dh] and Devlin's book [De]. I find noteworthy the essay [T], written by a young Oxford mathematics student who went on to become a neuroscientist of high repute at the University of California Los Angeles. Overall, however, it seems that intuition has remained largely unanalyzed and poorly understood.

A good way to approach the book of Lakoff and Núñez is to see it, as the authors suggest in the preface, as an empirical study of the precise nature of clear mathematical intuitions (page xv). Lakoff and Núñez promise in the introduction to give an account of how normal human cognitive mechanisms rooted in the brain are used to formulate mathematical concepts, reason mathematically, and create and understand mathematical ideas. Logic and formal rigor do not figure prominently in this account. Rather, the book is about the intuitive side; the focus is on certain "conceptual metaphors" which, the authors hypothesize, are the basic building blocks of mathematical intuition. Moving beyond cognitive science into philosophy, the authors even suggest that metaphors account for the meaning of mathematical concepts and are the basis of mathematical truth. "Metaphor" is the key word in this book.

Much of the book is devoted to the examination of prominent topics in standard mathematics curricula. These topics include grade school arithmetic, algebraic structures and their models, logic, set theory, limits, real numbers, and a little bit of nonstandard analysis. Chapters 13 and 14 have a historical orientation, discussing the contributions of Dedekind and Weierstrass to the foundations of analysis. Chapters 15 and 16 treat philosophical issues. The book ends with an extended discussion of the equation $e^{\pi i} + 1 = 0$ intended to illustrate "mathematical idea analysis", a technique that was invented by the authors and that they use to uncover the metaphorical elements in mathematical ideas.

The arguments do not follow a direct path. The book builds on many fronts, elaborates subgoals, and spins off subsidiary projects. While reading the book, I sometimes found it difficult to keep track of what the authors were aiming at. For this review, therefore, I shall simplify things by picking out three major strands and commenting on them separately. I have mentioned them already. They are:

- a hypothesis about the role of metaphors in mathematical cognition,
- a philosophical position about mathematical truth, and
- the technique of mathematical idea analysis.

In many ways the three strands are interdependent—a point to be remembered, even though my discussion is divided into three parts. I find the metaphor hypothesis the most interesting, and accordingly I devote more space to this than to the other two items.

Let me state the metaphor hypothesis as I interpret it. When people think about mathematics—even very deep, advanced mathematics—they somehow activate links to mundane experiences that occur in everyday functioning in the world and links to other mathematical experiences as well. These links are not logical or deductive—often, they are not even conscious. They involve very complex pattern-matching, by means of which people transfer abilities and concepts that are relevant or adaptive in familiar, natural settings to new settings that are less familiar and more abstract. The lifting or "retooling" of cognitive categories and abilities in this fashion is what the authors call *conceptual* *metaphor*. The second quote at the beginning of this review is as close to a definition of this as anything that the authors give.

Lakoff and Núñez present the metaphor hypothesis within the framework of a general theory of human cognition that Lakoff himself played an important role in creating; see [LJ]. In addition to the idea that most abstract concepts are metaphorical, this theory has two other basic tenets. One is that thought is mostly unconscious and mostly "involves automatic, immediate, implicit rather than explicit understanding" (page 28). Conceptual metaphors may be unconscious; they support and influence our thinking without our necessarily being aware of them. The other tenet is that human conceptual structures are deeply influenced by the particulars of our concrete, physical being. We reason with the same equipment we use to observe our immediate surroundings, move about in them, and interact with people and things. Even the most abstract concepts, if analyzed properly, show the marks of their origin in, and dependence on, basic perceptual and motor schemata. In summary, all concepts—mathematical concepts in particular—are metaphorical and rest upon unconscious understandings that originate in bodily experience.

Let us look at some examples. There are various kinds of conceptual metaphors. *Grounding metaphors* transfer conceptual abilities acquired in concrete experiences (like putting things in piles or traveling) to abstract domains like arithmetic. *Linking metaphors* make connections between different abstract domains. What struck Poincaré as he stepped aboard the bus, for example, was a linking metaphor at a very high level that had somehow evolved in his unconscious and then made its way to the surface.

Basic arithmetic has several grounding metaphors, all discussed in Chapter 3. One of these metaphors connects experiences with collections of objects on the one hand and the basic arithmetic operations on the other. Joining two collections, for example, corresponds to addition, while splitting a collection into many subcollections of equal size corresponds to division. The commutative law of addition corresponds to the fact that when two collections are thrown together, it does not matter which goes first. Understanding the commutative law presumably involves some sort of reference back to this property of collections and thus the activation of this metaphor. Other aspects of arithmetic are associated with other grounding metaphors. Adding positive and negative numbers may be understood metaphorically by referring to forward and backward trips along a linear path.

One might react to this with the feeling that it is all pretty trivial. Of course, once the arithmetic metaphors are internalized, using them is as easy as riding a bike. But the skills involved in using arithmetic, or in riding a bike, are cognitively quite complex. This is most obvious in the fact that *learning* them is not at all trivial. I would even suggest that evidence in favor of the metaphor hypothesis can be found in the fact that exposure to different metaphors influences the learning process. In the appendix of [MC], Robert Moses discusses a strategy he developed for teaching arithmetic with signed numbers. Moses does not mention metaphors explicitly, but in the terminology of Lakoff and Núñez, what he did was hypothesize that certain children were failing to progress because they were inappropriately bound by the collection metaphor. So Moses developed a teaching strategy intended to strengthen the trip metaphor, and this strategy succeeded quite well.

Here is another example of a grounding metaphor. In Chapter 8 Lakoff and Núñez introduce what is the single most important metaphor in the book, the "Basic Metaphor of Infinity" (BMI), and illustrate its occurrence in several mathematical contexts. Like all conceptual metaphors, the BMI involves a correspondence between a "source domain", which is familiar and often concrete, and a "target domain", which is less familiar and usually more abstract. In the BMI, the source is the general idea of an iterative process that reaches a completion. Examples would be walking to a destination or picking all the berries off a bush. The target is any iterative process that potentially goes on and on, like counting "One, two, three… ." The BMI simply shifts the idea of a completed process from its natural context into a new context, like counting, in which the idea does not exactly fit but to which it bears a likeness or analogy. Thus, we can reason about the collection of all natural numbers by extending or generalizing the patterns by which we reason about things like the set of all steps taken on a walk somewhere. It is not claimed that the use of this metaphor is conscious, but just that there is a common pattern. We are prepared to think about infinite processes by experiencing finite ones, and the descriptive categories we apply in the finite case have analogues in the infinite. Of course, there are vast differences between the logic of finite and infinite processes, but the authors do not seem very concerned about such differences. I suppose that the authors consider such details to be peripheral to the cognitive science, despite their mathematical importance.

How do metaphors function in the mathematical activities of actual people? On this, Lakoff and Núñez are not very clear. When they do talk about the mathematical activities of real people, they describe them in generic terms: people entertain ideas or "use cognitive mechanisms" of one sort or another to "conceptualize" this or that. Presumably, when an individual is engaged in mathematical work, that person is guided by metaphors that are somehow represented in his or her own brain. The details would depend on the specific task or problem. Unfortunately, Lakoff and Núñez do not provide any illustrations of what they suppose goes on in "real time", so this is about as much as I can say.

This brings me to my first main criticism concerning the metaphor hypothesis: What is the quality of the evidence for it? If, as the authors say on page 1, their goal is to determine "what mechanisms of the human brain and mind allow human beings to formulate mathematical ideas and reason mathematically," then one would expect some data about the actual thoughts and actions of people in the process of doing mathematics. Such data do exist; the work of Robert Moses provides one example. However, essentially the only kind of evidence Lakoff and Núñez provide comes from the examination of the contents of texts and curricula. How much can we infer about the "basic cognitive mechanisms" used in mathematics from what we find in texts and curricula? A study of navigation based on the standard manuals would tell us very little indeed about how the task was actually accomplished on the bridge of a large ship. How exactly do people use metaphors when they are learning new material, solving problems, proving theorems, and communicating with one another? I would like to have seen direct support for the metaphor hypothesis from the observation of mathematical behaviors. After a demonstration that metaphors are indeed as common as the authors believe, I would want a detailed examination of the *ways* metaphors are used in a wide variety of settings. The authors report no such information, and in fact they acknowledge the lack of direct empirical support for their hypotheses in many places. Carefully designed studies might lead to very different ideas about how metaphors function in mathematics or mathematics learning. For example, at the AMS meeting in New Orleans in January 2001, Eric Hsu and Michael Oehrtman, mathematics education research postdocs at the Dana Center at the University of Texas at Austin, spoke about their study of calculus learners. They found students making up their own, often dysfunctional, metaphors, and they raised the fascinating question of how it is that some learners shed the idiosyncratic metaphors they initially build and acquire the ones that are standard.

Mathematics is constantly absorbing what it learns about itself by gazing at itself.

My second main criticism concerns lack of precision in the concept of metaphor. By rough page count, more than half the text is devoted to displaying mathematical metaphors of one sort or another. After a while, the notion of metaphor seems to become a catchall. In the discussion beginning on page 384, for example, "metaphor" is used to refer to the following: the algebra/geometry dictionary in analytic geometry; the definition of function addition, $(f+g)(x) := f(x) + g(x)$; the "Unit Circle Blend", which involves various things one might attend to in a diagram showing the unit circle at the origin in a Cartesian coordinate system, together with a central angle; the "Trigonometry Metaphor", i.e., thinking of the cosine and sine of θ as the x- and y-coordinates of the point reached after moving counterclockwise from $(1,0)$ along the unit circle through an angle of θ; the "Recurrence Is Circularity Metaphor", which refers to connections between mathematical concepts and the language of recurrence used in nonmathematical settings to describe things like the seasons; and, finally, polar coordinates. Other parts of the book add yet more variety. In abstract algebra the relationship of an abstract structure (e.g., a group) to a model of that structure (a group of rotations) is an example of a metaphor. Cauchy, Dedekind, and Weierstrass contributed to the foundations of analysis by creating the "Arithmetic Cut Metaphor", the "Spaces Are Sets of Points Metaphor", and many others. When mathematicians write axioms, they are using the "Essence Metaphor". Set-theoretic foundations give us an instance of the "Formal Reduction Metaphor". With examples of so many differing kinds serving such diverse functions, the notion of metaphor begins to lose its meaning. If I had been given the original definition and a couple of examples and then had gone looking for conceptual metaphors in mathematics, I would never have come back with many of the things listed in this paragraph. The idea that metaphors play a role in mathematical thinking is quite attractive, but what is needed is a notion specific and precise enough so that people working independently and without consulting one another can discover the same metaphors and agree on the functions they perform. I do not think we have this yet.

Let me turn now to the philosophical parts of the book. The authors devote many pages to portraying a sort of philosophy/ideology that they call the "Romance of Mathematics", which they contrast with their own philosophy. This is a good rhetorical strategy, since the so-called "romance" would probably be disliked by all readers except some superficial and self-congratulatory mathematicians. For the sake of brevity, I will not comment on the “romance” but just go directly to the authors’ philosophical ideas.

According to my reading, Lakoff and Núñez want us to view mathematics as a natural human activity, about which their cognitive theory informs us. They would like us to use words like “meaning”, “existence”, and “truth” to describe aspects of this activity. In particular, they say that mathematical entities are “metaphorical entities” that exist “conceptually” only “in the minds of beings with [appropriate] metaphorical ideas” (pages 368–9). They also say that when we speak of the truth of a mathematical statement, we must speak only relative to a particular person, and we may mean no more than this: that person’s understanding of the statement accords with his or her understanding of the subject matter and the situation at hand; see page 366. Such a view of mathematical truth appears to be at odds with the reality of how mathematicians communicate. If *my* mathematics depends on the metaphors that happen to be in my head, and *your* mathematics depends on the metaphors in yours, then how is it that we can share mathematical ideas? And why is it that we agree on so much?

Lakoff and Núñez make a couple of hypotheses that might address these objections. First, they claim that the metaphors on which mathematics is based are not arbitrary. Many grounding metaphors, in particular, are forced on us by our physical nature. Second, they claim that natural metaphors have a very elaborate and precise structure; see page 375. These, of course, are empirical hypotheses. Some day we might acquire good evidence for or against them. If they are borne out, they might support the kind of naturalistic approach to the philosophy of mathematics that the authors have begun to sketch. In my opinion, though, a naturalistic approach should certainly not dismiss the way mathematicians share definitions with one another, understand and criticize one another’s reasoning, and use a precise, if artificial, logical language to put their ideas in writing so that those ideas can be judged by the world. Surely we can acknowledge a role for intuition without ignoring the ways that logic and conventional rigor support the kind of knowledge that mathematicians build and share.

Let me move now to the third and last of the three major strands on which I promised to comment. The authors argue that mathematical ideas occur within elaborate networks of interconnected metaphors. “Mathematical idea analysis” is the technique of teasing apart the network to reveal the metaphorical parts. The reason for doing this is to “characterize in precise cognitive terms the mathematical ideas in the cognitive unconscious that go unformalized and undescribed when a formalization of conscious mathematical ideas is done” (page 375).

The appendix contains an extended discussion of the equation $e^{\pi i} + 1 = 0$ that is intended to illustrate the method. The authors say that they want to characterize the meaning of the equation and provide an understanding of it (page 384). What a mathematician will find here is a detailed description of the geometric interpretation of complex exponentiation. Building from basic high-school-level ideas up to college-level complex analysis, the treatment is, at different times, insightful, entertaining, opaque, interesting, breezy, ponderous, and muddled.[1] Mathematicians will find the images and metaphors discussed here to be very familiar, scarcely unconscious. Many come directly from the pages of calculus books and are things most of us describe over and over to our calculus classes. To me, these mental images are certainly very helpful in understanding. The mystery has always been how difficult it is for students to absorb them and use them productively.

Lakoff and Núñez seem to want to open up that whatever-it-is that makes mathematics into a coherent, meaningful whole and expose it for all to see and appreciate. If only there were a way! Poincaré himself speculated about how best to teach mathematics, concluding finally that understanding means different things to different people at different times and for different purposes; see

[1] *See the website* `http://www.unifr.ch/perso/nunezr/errata.html` *for a list of errata. One error in the first printing (pages 416–8) was fairly serious. The “correction” that I found at the website and that appears in the recently distributed softcover edition of the book is a kind of mathematical red herring—a passage that appears to answer a question but in fact misleads. The authors preface this passage by criticizing other texts for failing to answer “the most basic of questions: Why* should *this particular limit,* $\lim_{n\to\infty}(1+\frac{1}{n})^n$*, be the base of the exponential function that is its own derivative?” (pages 415–6). In the revised explanation (errata 416–8), the unacknowledged assumption that* $\lim_{n\to\infty}(1+\frac{1}{n})^n$ exists *is used in an essential way. (There are situations that are analogous in all explicit details but in which the pattern of explanation that the authors use would “explain” a falsehood.) When the authors complete their proffered explanation, they make a point of saying that, once the meanings and metaphors are combined with some “simple algebraic manipulations,” there is “nothing mysterious” (errata, page 419) about the result. I would retort that what is really the most basic question is: How do we know that there is any number at all to which the values of* $(1+\frac{1}{n})^n$ *tend as* n *increases? The authors create the appearance of simplicity only by suppressing this issue, and in fact they create a path that can lead to error. The book* [Do]*, on the other hand, contains a nice self-contained treatment of this limit, beginning, appropriately, with a non-mysterious demonstration of its existence; see page 45. The central claims of the book under review, which are in cognitive science and philosophy, may not be threatened by such mathematical infelicities, but perhaps they serve as useful reminders of the importance of logical discipline in mathematics.*

[P1], page 432. Surely there will always be good reasons to experiment with new formats for mathematical exposition, and Lakoff and Núñez are not alone in exploring. Zalman Usiskin, for example, has proposed something he calls "concept analysis", intended for people who are learning to teach mathematics; see [U]. Concept analysis examines various methods of representing and defining mathematical ideas, as well as how those methods evolved over time, how they are used, the problems people have understanding them, and strategies that are useful in explaining them.

This concludes my comments on the three strands I identified earlier. If I think about the portrayal of mathematics in the book as a whole, I find myself disappointed by the pale picture the authors have drawn. In the book, people formulate ideas and reason mathematically, realize things, extend ideas, infer, understand, symbolize, calculate, and, most frequently of all, *conceptualize*. These plain vanilla words scarcely exhaust the kinds of things that go on when people do mathematics. They explore, search for patterns, organize data, keep track of information, make and refine conjectures, monitor their own thinking, develop and execute strategies (or modify or abandon them), check their reasoning, write and rewrite proofs, look for and recognize errors, seek alternate descriptions, look for analogies, consult one another, share ideas, encourage one another, change points of view, learn new theories, translate problems from one language into another, become obsessed, bang their heads against walls, despair, and find light. Any one of these activities is itself enormously complex cognitively—and in social, cultural, and historical dimensions as well. In all this, what role do metaphors play?

Moving to a different perspective, I want to note that there are areas not even hinted at in the book where cognitive science is prepared to contribute to our understanding of mathematical thought. Consider this: Metaphorical ideas are frequently misleading, sometimes just plain wrong. Zariski spent most of his career creating a precise language and theory capable of holding the truths that the Italian geometers had glimpsed intuitively while avoiding the errors into which they fell. What cognitive mechanisms enable people to recognize that a metaphor is not doing the job it is supposed to do and to respond by fashioning better conceptual tools?

From early childhood people comment on their own thinking or on the things they create in order to represent their thinking, and they use this commentary to adjust and correct themselves. In a fascinating article about her own first-grade classroom, Kristine Reed Woleck describes how children talk to themselves and to one another while drawing and revising pictures to depict mathematical ideas, in the process coming to "question, debate, defend, clarify, and refine their mathematical understandings" ([W], page 224). Isn't this a miniature version of what a group of research mathematicians might be found doing in front of a tentative proof sketched on a blackboard? Or what you do alone when you draft a proof and then read, correct, and redraft it?

If mathematical thinking is like other kinds of thinking in its use of metaphors, what distinguishes mathematical thinking may be the exquisite, conscious control that mathematicians exercise over how intuitive structures are used and interpreted. We can step back from our own thinking and critically examine our attempts at meaning-making. This, I would venture, is as fundamental a cognitive mechanism as any mentioned by Lakoff and Núñez. Mathematics is constantly absorbing what it learns about itself by gazing at itself. In Al Cuoco's memorable phrase, "mathematics is its own mirror on the very thinking that creates it" ([CC], page x). In a similar vein, Schoenfeld's classic study [S] showed how important self-observation is in solving mathematical problems. Today we can associate the ability to observe and control our own thought processes with certain clearly demarcated regions of the brain, and we understand much about how these regions function in normal brains and how they fail in certain diseased brains; see [M].

Where does mathematics come from? Poincaré viewed mathematical intuition as that which invents and logic as that which proves. Perhaps in this sense mathematics starts with intuition. But Poincaré also said that without proof there is no understanding, no communication, no meaning ([P2], page 95–6). Most mathematicians would probably agree that mathematics is both invention *and* proof and that it comes from the cooperation of intuition *and* logic.

What about the question I asked earlier: what do we know today about mathematical intuition that we did not know one hundred years ago? Lakoff and Núñez have suggested that intuition is not as formless and elusive as perhaps we had thought. To the contrary, or so they claim, it works on the basic mechanism of metaphor, and it has a profound structure that is dictated by human nature. These are interesting and appealing ideas, or perhaps *intuitions*, about the nature of mathematical intuition.

Among the sciences, mathematics is not alone in building on intuition. Nor is it alone in requiring more. Every science also needs concepts that are precise enough to frame testable hypotheses, and every scientific *theory* needs proof—not in the mathematical sense, but at least in the sense that the theory has been subjected to the most rigorous trials we can devise and has survived. Lakoff and Núñez have shared with us their intuitions about the way the mathematical mind operates. More work remains to be done. We do not know what scientists may some day build upon these intuitions.

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