Jacques-Louis Lions (1928–2001)

Peter D. Lax, Enrico Magenes, and Roger Temam

Roger Temam

Jacques-Louis Lions was a scientist of remarkable prescience and immense energy. His vision extended to the development of entire areas of mathematical science. He understood that mathematics can make a great contribution to science, and he worked to see this goal realized.

He authored or coauthored 20 books and nearly 600 articles, and his mathematical legacy is extremely important. Founder of the French school of applied mathematics, he also had a considerable influence worldwide on many mathematicians and many mathematical institutions. Beyond all his personal achievements, his greatest satisfaction came from the success of his son, Pierre-Louis, who was awarded one of the Fields Medals at the International Congress of Mathematicians in Zurich in 1994.

Jacques-Louis Lions was born and grew up in the south of France in the charming town of Grasse, which produces many of the flowers used in the French perfume industry. His father was mayor of Grasse for nearly thirty years, a politician who is said to have been generously devoted to the good of his constituency. Lions's wife and lifelong companion, Andrée, was born and raised in the south of France too; they met in the Résistance during World War II. Lions was very attached to Grasse and the surrounding area; the family would return there during the academic recesses, and he would work there between professional trips.

After high school Jacques-Louis studied for one year at the University of Nice, which was then a small one-year college. His potential was detected by an oral examiner who advised him to prepare for the competitive École Normale Supérieure, which he entered the next year (1947) and attended until 1950. He then joined the Centre National de la Recherche Scientifique (CNRS) and started research under the direction of Laurent Schwartz. Shortly before, Schwartz had completed the theory of distributions, for which he was awarded the Fields Medal in 1950. Schwartz foresaw that the theory of partial differential equations (PDE) should be revisited and completely redeveloped in the context of distribution theory, and he engaged several of his doctoral students in that direction, including Bernard Malgrange, François Treves, and Jacques-Louis Lions. In his thesis Lions developed the basis of the variational theory of linear elliptic and evolution equations, one of the approaches to these problems commonly used nowadays. After completing his thesis (thèse d'état), he developed his own research program, working hard and without interruption to the end of his life, even when he assumed other important and time-consuming responsibilities.

Jacques-Louis Lions passed away on May 17, 2001, at the age of seventy-three, after fighting his illness very discreetly the last few months. He is

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Roger Temam, both a student and a collaborator of Jacques-Louis Lions, thanks the friends, colleagues, and collaborators of Lions who suggested additions to this article; in particular, Alain Bensoussan provided information concerning INRIA and CNES. A version of this article appeared in SIAM News ***34** (2001), no. 6, 2-4.*

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survived by his wife, Andrée; his son, Pierre-Louis, and daughter-in-law, Lila; and his grandson, Dorian.

Scientific Work

The mathematical work of Jacques-Louis Lions is at the same time very diverse and well unified. His work is described appropriately by the title (which he chose) for his chair at the Collège de France: "Analyse Mathématique des Systèmes et de leur Contrôle". The systems he had in mind are those described by linear and nonlinear partial differential equations. Analysis meant here everything from the most abstract existence theorems to approximation and numerical issues and computer implementations; control would come later.

In the early 1950s Lions started to develop the building blocks of what would be his "Analyse des Systèmes". He first addressed, alone or in collaboration, many issues in linear PDE theory and in distribution theory, including a work with J. Deny still commonly used and quoted nowadays. In the late 1950s he started his first two major and lasting works.

On the one hand, Lions started attending the Séminaire of Jean Leray at the Collège de France, and under his influence became interested in nonlinear partial differential equations, in particular the incompressible Navier-Stokes equations. The mathematical analysis of the Navier-Stokes equations, which had been dormant since the pioneering work of Leray in the 1930s, came back to life in 1951 with the article of Eberhard Hopf establishing the long-time existence of weak solutions for bounded domains in dimension three. The contributions of Lions to the subject are twofold. He and Giovanni Prodi (elder brother of the current president of the European Union) independently proved the uniqueness of weak solutions in dimension two, and they published this result together in 1959. Also, as part of his better understanding of evolution equations, Lions was able to considerably shorten the proof of Hopf and make the result more accessible; he thus contributed, with O. A. Ladyzhenskaya, J. Serrin, and others, to the beginning of the modern theory of mathematical fluid dynamics.

The other major thrust Lions began at that time was his work with Enrico Magenes on nonhomogeneous boundary value problems, which led eventually to the publication of a three-volume book in 1968. As part of this work, they made a systematic study of Sobolev spaces (introduced by Sobolev in the late 1930s), developing a wealth of results on spaces with fractional exponents, the many different characterizations of Sobolev spaces, and the theory of linear elliptic and parabolic equations in such spaces. A parallel theory needed for these purposes is the theory of interpolation between Hilbert or between Banach spaces, that is, the construction of spaces intermediate—in the sense of topology and set inclusion—between two given spaces. Lions made substantial contributions to interpolation; he had initiated interpolation between Hilbert spaces during a one-year postdoctoral visit at the University of Kansas in 1953 with Nachman Aronszajn. All this work is very "linear", but its importance and subsequent use in nonlinear problems is considerable.

Then, in the early 1960s, a new adventure started. Lions met (in spirit) another of his intellectual mentors, John von Neumann, who had previously designed the first computers and used numerical methods for the solution of partial differential equations from fluid mechanics and meteorology. At a time when the French mathematical school was almost exclusively engaged in the development of the Bourbaki program, Lions dreamed, almost alone in France, that there was an important future for mathematics in another direction. While continuing to produce high-level theoretical work in partial differential equations, he threw himself into the subject of numerical methods.

Although he did not publish yet in the area, he started the French school of numerical analysis. In a loft belonging to the CNRS (Institut Blaise Pascal)—away from the official center of Paris mathematical life, the (old) Institut Henri Poincaré—he started to teach a graduate course in numerical analysis, and he engaged graduate students in this program—in particular, and in chronological order, Jean Céa, Pierre-Arnaud Raviart, and Jean-Pierre Aubin, followed by many others. The mimeographed notes of the course he taught at the Institut Blaise Pascal were used to start numerical analysis programs in many places worldwide at a time when few books devoted to the numerical analysis of partial differential equations were available. In his course Lions used from the beginning the variational theory of boundary value problems, which he developed himself in his thesis and subsequently. This point of view was further developed in the first theses that he directed in numerical analysis, thus producing an appropriate framework for the mathematical development of finite element methods and for many other important subsequent and contemporary developments in numerical analysis. In this way Lions played a significant, albeit indirect, role in research on the numerical analysis of PDEs.

Lions began his professional career as professor at the University of Nancy (1954–1962) before going to the University of Paris (1962–1973) and then to the famous Collège de France from 1973 until his retirement in 1998. He was also part-time professor at the École Polytechnique (1966–1986), the alma mater of many of his graduate students.

Besides being involved in numerical analysis, he produced, as mentioned above, high-level theoretical work. Following an idea of George Minty and Felix Browder, he was involved in the development of the theory of strongly nonlinear equations that are monotone in their highest arguments. In his one joint paper with Jean Leray, he published in 1965 one of the most general results in that direction, extending and considerably simplifying an earlier result of Mark Vishik. Together with Guido Stampacchia, he published in 1965 and 1967 two articles developing the theory of variational inequalities. Subsequently Lions continued to develop this theory individually, with Haim Brezis, and later with Georges Duvaut in a book devoted to applications of variational inequalities to many concrete and specific problems in continuum mechanics and physics (1972). His 1969 book on nonlinear PDEs, including results by himself, colleagues, and students, was very exhaustive at the time. This valuable book, written in French, deserves to be better known; unfortunately the rights to the English translation were acquired by a company that went bankrupt. Nonetheless, this book played a role similar to the Institut Blaise Pascal mimeographed notes, being used worldwide to start courses or research in nonlinear partial differential equations.

The late 1960s were again the beginning of a new orientation for Lions in his theoretical and applied work. He was then scientific director at the newly created Institut de Recherche en Informatique et Automatique (IRIA, later INRIA). Alain Bensoussan and Roland Glowinski became his students and then his collaborators at IRIA. His work there included the numerical analysis of variational inequalities, leading to a two-volume book coauthored by Glowinski and R. Tremolières and published in 1976.

The French word "automatique" means automatic control or, more generally, control. Control is not usually paired with computer science; its presence in the name of IRIA can be attributed, in part, to the influence of Pierre Faurre, a younger scientist and industrialist highly respected by Lions (and who prematurely passed away a few months before him). At IRIA Lions discovered "systems theory", which became a new interest for him and a new component of his activity. He worked then on control theory, the area that lent its name to the second part of the title of his chair at the Collège de France.

Instead of publishing articles, he published directly a research monograph on the optimal control of systems governed by partial differential equations. This unique book, published in 1968, became the reference book on the subject; like others of his books, it was translated into English, Russian, Japanese, and Chinese. He then considerably developed the subject, writing nine books partly or totally devoted to control theory, which were published between 1968 and 1992. Two of them, in 1978 and 1982, were written with Alain Bensoussan: one was devoted to the applications of quasi-variational inequalities to stochastic control, and the other one to impulse control and quasi-variational inequalities. In the 1980s Lions was interested in controllability, and he introduced the Hilbert Uniqueness Method, which he developed in a book published in 1988. This was also the topic of his John von Neumann lecture at the meeting of the Society for Industrial and Applied Mathematics held in Boston in 1986. The question of controllability is whether one can drive a given system to a given state. In his very own style Lions reduced this problem to a problem of analysis that can be studied systematically, and he derived a wealth of results. His research also stimulated much work in both the control and the analysis communities, and it led to advances in the theory of certain partial differential equations, more specifically hyperbolic equations.

Another direction of Lions's research in the late 1970s and during the 1980s was homogenization, whose purpose is the macroscopic description of materials with a complex microscopic structure; PDEs and asymptotic and stochastic analysis are the tools needed here. His first major work appeared in 1978 in a book with Bensoussan and G. Papanicolaou. His former student Luc Tartar continued to develop this subject in a different direction. Lions also followed very closely the related work on G- and Γ-convergence of the Italian school around Ennio De Giorgi.[1]

In the 1990s, while he was president of the Centre National d'Études Spatiales (CNES) and president of the Scientific Council of the National Meteorological Office in France, Lions developed an interest in the mathematical problems of the ocean, the atmosphere, and the environment. The environment was the only subject for which he agreed to get directly involved in politics; in particular, he participated in the Venice Environment Initiative Network (VEIN), an international project which, however, did not come to maturity. On the scientific side, in one of his books on control he introduced and studied the concept of Sentinels for the control and detection of pollution, a concept that produced very good results in a thesis devoted to concrete applications. With two collaborators he wrote a series of eleven articles and a monograph on the mathematical problems raised by the primitive equations governing the motion of the atmosphere, the ocean, and the coupled atmosphere and ocean, and by related asymptotic and numerical issues. Lately he had been working on domain decomposition methods, a subject on which he wrote as early as 1972. His recent work

[1] *Concerning De Giorgi, see the obituary by Lions and François Murat,* Notices ***44*** *(1997), 1095–1096.*

with Olivier Pironneau on this subject led to nine articles or announcements of results, and a book was in progress.

Another massive work of Lions is the nine-volume series that he published and edited with Robert Dautray in 1988, *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*, seen by some as a modern version of the book by Courant and Hilbert. This series was translated into English as a six-volume series. During this work and on many other occasions, Dautray noted Lions's deep insight into the physics of the problems; Lions sometimes raised physically relevant questions not contemplated by the physicists with whom he was interacting. Lions also started to publish and edit with Philippe G. Ciarlet the *Handbook for Numerical Analysis* series, which Ciarlet intends to bring to completion; together they also edited two series of books in applied mathematics.

This brief description of the scientific work of Jacques-Louis Lions does not give a proper idea of its very considerable impact nor of the tremendous activity behind it, which included the original courses and lectures he gave, plenary lectures at major international congresses, seminars in small departments (often in developing countries), frequent travels to distant destinations, and hundreds of pages of faxes that he exchanged weekly with his collaborators.

We have also alluded to his students: Lions attracted many young people around him, both from France and from foreign countries. The complete list of his graduate students should appear in the near future on the Mathematics Genealogy website.[2] He had at least fifty students for Ph.D. theses or thèses d'état or habilitations corresponding to the postdoctoral level. All his students were delighted and amazed at how quickly he would read their drafts and how available he would be to each of them individually. Another reason for his success as thesis and postdoctoral advisor is that he could determine very quickly what research would suit a new student, and he would tailor new problems adapted to the student's abilities. Many of his students themselves became well-established mathematicians, and by the end of Lions's life he had scientific descendants of the sixth generation. In France most of those working in the numerical analysis of PDEs are his scientific descendants, and so are a significant number of those working in applied PDEs. He was very careful not to influence his students too much, saying that he saw himself as a counselor, trying only to help his students develop the best of their possibilities.

Lions also had regular scientific contacts with many high-level scientists worldwide whom he visited regularly or who visited him in Paris. Visitors usually spoke at one of the seminars Lions was directing, thus providing up-to-date information to his students and collaborators. For thirty-six years he directed one or two weekly high-level seminars, one applied and one theoretical from 1962 to 1984. When he left INRIA, the two seminars merged at the Collège de France, and, until 1998, there would regularly be one or two (or possibly three) lectures on Friday afternoons. For several years at the beginning, Haim Brezis assisted him in the organization of the Collège de France seminar, and they published the proceedings of the seminar in twelve volumes of the *Pitman Research Notes* series. The long list of regular visitors included Felix Browder, Louis Caffarelli, Peter Lax, Andrew Majda, Louis Nirenberg, and Paul Rabinowitz from the U.S.; Ennio De Giorgi, Enrico Magenes, and Guido Stampacchia from Italy; Shmuel Agmon from Israel; John Ball from the United Kingdom; Li Ta-tsien from China; and Sergei Sobolev and Mark Vishik from the former USSR.

As Enrico Magenes recalls below, among the countless mathematical initiatives of Lions is that at the end of World War II he was the first French mathematician (along with Laurent Schwartz) to reestablish contact with the Italian mathematical community and to visit Italy. Thence followed the lasting and very active interaction and collaboration with Ennio De Giorgi, Enrico Magenes, Giovanni Prodi, and Guido Stampacchia. Lions started to guide a long series of Italian postdoctoral researchers who eventually became themselves well-established mathematicians, the first one in Nancy being Emilio Gagliardo. Lions also helped the development of applied mathematics in Spain and in India (Bangalore), always very generous of his time with young people for correspondence, advising, and visits.

Scientific Responsibilities and Other Activities

The scientific research of Jacques-Louis Lions was only part of his professional activity; the other part was his role as manager and consultant, his responsibilities in governmental organizations, and later his role in high-level industrial companies. He seems to be one of very few mathematicians in modern history to have had at the same time important research activities and important positions in governmental and industrial organizations.

In 1980 IRIA became INRIA (N for National) and Lions became its first president, a position he held until 1984. Lions was both the manager and the scientific head of this new institute, which he literally molded. As much as possible he got involved in all the scientific and organizational aspects. INRIA played and still plays an important role in the development of computer science in France.

Lions initiated the expansion of INRIA to Sophia-Antipolis, near Nice, and to Rennes. He designed

[2] http://mathgenealogy.mnsu.edu/.

the organization of the institute through projects (a very relevant and efficient approach to motivate teams) with precise objectives, budget assignments, and managerial responsibilities, as well as frequent reports and evaluations. He created Simulog, the first subsidiary of INRIA, which was followed by a long series of successes with spin-off companies. He always advocated that a project should have three pillars: scientific excellence, application relevance, and international cooperation. During his four years as president of INRIA, he gave the institute the basic principles on which a significant and long-standing success has been built.

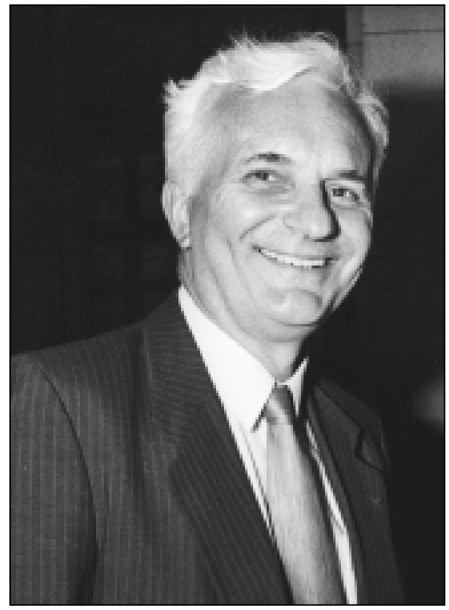
In 1984 Lions became president of the Centre National d'Études Spatiales (CNES), the French space agency. The physicist Hubert Curien, previous president of CNES and subsequently Minister of Research, foresaw the important role that mathematics would play in space research, and he asked Lions to accept this responsibility. In his new position Lions was confronted with new challenges. Beside the scientific ones (to supervise work on mathematics, physics, chemistry, and engineering), he went from directing INRIA, a new institute that he fully shaped, to presiding over a large, active, and well-established institution. Furthermore, he was the first mathematician to hold this position. He certainly did well, since he was reappointed for a second four-year term and since the current president, Alain Bensoussan, first appointed in 1996, is also a mathematician.

Lions was president of CNES in a period of economic growth which allowed the launching of big programs such as Ariane 4, Ariane 5, and the successor of the SPOT series. He was very effective in promoting CNES programs with the ministries in charge and, more generally, with politicians, industrialists, and decision makers. A man of conviction, he put his credibility at stake on these programs and created confidence.

His action was particularly effective in three areas. First, when the V15 Ariane launch failed in September 1985, he insisted that the scientists and engineers invest heavily in modelling and numerical treatment in order to sort out the causes of the failure. Later on, he insisted on having a CNES basic Research and Technology Program carried out in close cooperation with other institutes and research bodies. He successfully negotiated with NASA the France-U.S. space oceanography program Topex/Poséidon at a time when the flight of NASA payloads on Ariane launch vehicles was banned by the U.S. Congress.

Lions was very active in French-Soviet, and then French-Russian, negotiations to get flight opportunities for French astronauts. Numerous manned space missions such as Jean-Loup Chrétien's second flight and Michel Tognini's mission have to be credited to his actions.

In 1996, after Lions left the presidency of CNES, the new rocket Ariane 501 failed on its maiden flight. He was appointed chairman of the European Space Agency committee investigating the reasons for the failure. The conclusions of the investigations conducted under his direction appeared in *SIAM News* in October 1996. The failure was due to a calculation overflow that was not properly anticipated. This is now a textbook and classroom example of overflow in computing.



Photograph courtesy of INRIA Archives.

Jacques-Louis Lions

Although a new appointment was proposed to him, Lions retired from CNES in 1992, deciding once more to confront a new challenge: the industrial world. For many years he had been working on mathematical problems originating from industry, and as president of INRIA and CNES he had many contacts with industry. He then decided to enter the industrial establishment itself, and soon we were to see him as a member of the scientific council or of the board of directors of large industrial groups. He was president of the Scientific Committees of Pechiney, Gaz de France, Électricité de France, and France Telecom; high-level scientific consultant at Dassault-Aviation and Elf; and member of the board of directors of Dassault-Systems, Pechiney, Compagnie de Saint-Gobain, and Thomson Multimedia.

Lions was president of the French Academy of Sciences from 1997 to 1999. He was secretary (1978–1990) and then president (1991–1994) of the International Mathematical Union, and he initiated the World Mathematical Year 2000. He was also a member, the secretary, or the chairman of countless committees in research institutions. He never spared his efforts to help scientifically isolated individuals or young groups in Eastern Europe, in developing countries, and in other places.

Lions received many awards honoring his various activities. He was a member or foreign member of about twenty academies, including the French Academy of Sciences, the (U.S.) National Academy of Sciences, the American Academy of Arts and Sciences, the USSR and then the Russian Academy of Sciences, and the Third World Academy of Sciences. He received about twenty honorary degrees. He was awarded the von Neumann Prize in 1986, the Japan Prize and the Harvey Prize in 1991, the Lagrange Prize in 1999, among others. In France he was Commandeur de la Légion d'Honneur and Grand Officier dans l'Ordre National du Mérite.

Jacques-Louis Lions was an exceptional person in many respects. He was a charismatic man, generous, very open and accessible, avoiding conflicts and contentious situations. One of the most striking aspects of his personality was his long-term vision; he was able to see and get involved in things that came to maturity five, ten, or twenty years later. He had many good ideas, and he had the mathematical talent, the physical strength, and the understanding of people needed to implement them.

Jean Céa wrote during the celebration of Lions's sixtieth birthday that he was at the same time a very simple and a complex person. He was indeed. Always kind, he could nevertheless make difficult decisions when needed and stick to them. He had a sense of humor and even knew how to put humor into serious matters or difficult situations. He would set high standards but remain kind to those who did not reach those standards. His long-term vision put him ahead of others and gave him time to elaborate subtle strategies, but he would withdraw and avoid conflicts when his propositions did not go through.

We said that Jacques-Louis Lions received many prizes, awards, and distinctions. Nevertheless, we believe that all in all he has given much more than he received. He will be very much missed by his friends and colleagues worldwide.

Peter D. Lax

Jacques-Louis Lions had many admirers and friends in the United States. We were only vaguely aware that he had been ill, so the news of his death came with the suddenness of a shock wave. We were—and are—stunned by the sense of loss.

Lions first came to our notice when he published his elegant generalization of the Titchmarsh convolution theorem to n dimensions. Subsequently he chose to work on the theory of linear partial differential equations; he was among the first to use systematically the theory of distributions. Under the influence of Jean Leray he turned to the theory of nonlinear partial differential equations, in particular the equations of fluid dynamics. He combined his interest in questions of existence with the equally intriguing questions of how to calculate these solutions numerically. This philosophy went against the then current intellectual fashion in France, formalized by Bourbaki, which gloried in looking for sources of mathematics in mathematics itself. That Lions was able to turn around almost single-handedly this intellectual climate is one of his great achievements.

Lions had an open, friendly, generous personality, with a light touch and a subtle sense of humor. When an American whose desire to lecture in French vastly exceeded his ability to do so visited the University of Nancy, Lions tactfully told him that the graduate students and postdocs were very eager to improve their English and were very much looking forward to hearing lectures in that language. Much later, Lions was a member of a committee of the National Academy of Sciences whose task was to evaluate the standing of mathematics in America on an international scale. In the course of the discussion of applied mathematics, the British participant criticized the French school of applied mathematics for being overly devoted to finding the right function spaces rather than concentrating on the physical phenomenon. Lions disarmed the potentially painful confrontation by saying, in mock horror, "We knew we couldn't trust the English ever since they burnt Joan of Arc."

Lions the scientist left his mark on the scope of mathematics; Lions the person left his mark in our hearts. We mourn his loss.

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Enrico Magenes

The first meeting of J.-L. Lions with the Italian mathematicians took place, I believe, in May 1954 on the occasion of a conference in Brussels on partial differential equations. In this conference Mauro Picone proposed a problem in the theory of elasticity for which the existence of a solution was not yet known. Lions promptly gave the answer to Picone's problem and was happy to publish his paper in the *Annali di Matematica Pura ed Applicata* [vol. 41 (1956), 201–219].

But the decisive meeting occurred in Nice in the summer of 1957 at the Réunion des Mathématiciens d'Expression Latine, when Guido Stampacchia and I had the opportunity to meet Lions and to become friends with him because of common scientific interests and life experiences.[3]

Stampacchia and I wanted to know and make known in Italy the results of the school of Laurent Schwartz on distributions and on partial

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[3]*Editor's Note: The allusion here is to fighting against Nazism during World War II.*

differential equations. At our invitation Lions came to Genoa in April 1958 to give a series of lectures on mixed problems in the sense of Hadamard, which were published by Luigi Amerio in *Rendiconti del Seminario Matematico e Fisico di Milano* [vol. 28 (1959)].

After that year Lions's meetings with Italian mathematicians became very frequent. He gave lectures and courses at many university mathematics institutes (Genoa, Pavia, Milan, Rome, Naples,…), at the Scuola Normale Superiore di Pisa, at S.I.S.S.A. (Scuola Internazionale Superiore di Studi Avanzati, Trieste), at C.I.M.E. (Centro Internazionale Matematico Estivo), at I.A.C. (Istituto per le Applicazioni del Calcolo, Rome) of the Italian CNR (Consiglio Nazionale delle Ricerche), at the National Academy of Sciences in Rome, at the Istituto Lombardo of the Academy of Sciences and Letters of Milan, and finally at I.A.N. (the Institute of Numerical Analysis) of the CNR at Pavia. Concerning the I.A.N., one should recall that he conducted his activities and gave advice as a member of the scientific council, starting from the establishment of this institute in 1970 and continuing until 1992.

Even last year, when he was already ill, he gave a course on mathematical modeling at the University of Pavia. From start to finish his lectures were very clear and up to date, and he always proposed open problems.

He welcomed in France, to work with him or with his collaborators, many young Italian mathematicians, all of whom remember him with admiration and thanks. Mentioning here only researchers coming from Pavia, I recall C. Baiocchi, F. Brezzi, P. L. Colli, V. Comincioli, G. Geymonat, L. D. Marini, A. Quarteroni, A. Visintin,…. But the list of all the Italian researchers welcomed by Lions would be very long, including scientists of a very high level.

He worked personally with G. Stampacchia on variational inequalities and with G. Prodi on Navier-Stokes equations. With E. De Giorgi he was linked not only because of scientific interests, notably in the calculus of variations and in homogenizations in PDE, but also because of a great friendship, revealed especially when De Giorgi fell ill.

Finally, it is with deep emotion that I recall here our collaboration, which for me was an inestimable source of ideas, for which I will always be grateful to him. It began in 1958 and was carried on especially during the years 1958–1972, when we studied nonhomogeneous boundary-value problems for linear PDE, and later in the scientific council of the I.A.N. But even in the last months we thought about a regularity problem for solutions of a linear integro-differential equation that he posed.

I would like to recall one episode concerning our collaboration. After finishing our third paper on elliptic problems, we noticed a mistake. The error was of this kind: we had thought that the extension by zero outside a regular open set Ω in $\mathbb{R}^n$ was continuous from $H_0^{1/2}(\Omega)$ to $H^{1/2}(\mathbb{R}^n)$, which is false. Discovering a mistake in one of one's own papers is not a pleasant thing, but we recalled what Renato Caccioppoli used to say, with his Neapolitan humor: the only sure way to avoid making mistakes is to do nothing at all! In any case, the reaction of Lions and the speed at which he overcame the difficulty, proposing to me to introduce the space $H_{00}^{1/2}(\Omega)$ (that is, the interpolated space of order $1/2$ between $H_0^1(\Omega)$ and $L^2(\Omega)$), were astonishing to me.

Obviously we met each other many times during all those years (in Nancy, Paris, Grasse, Genoa, Pavia,…). This deepened our friendship, which extended to our families.

I wished to stress our collaboration to show that I had the opportunity to fully appreciate the intellectual and human qualities of Lions: his unaffected manners; his commitment and energy in work; his rapidity of intuition and decision; his openness to new ideas and new problems in a body of knowledge that increased more and more over time, even outside mathematics; his love of freedom; and his respect for the opinions of others. And I can also attest to his willingness to collaborate with all mathematicians and scientists, independently of their cultural tradition, as well as with the scientific institutions of all countries, be they countries with an old heritage or countries on the path of development.

We have lost not only a great mathematician but also a great man.