Reflections on the Future of Mathematics

Felix Browder

Felix Browder served as president of the AMS during 1999–2000. On January 6, 2002, at the Joint Mathematics Meetings in San Diego, he delivered his Retiring Presidential Address. What follows is an abridged version of his address.

One of the mandated duties of the president of the AMS is to deliver a Retiring Presidential Address. The fact that this duty has not been met by a few presidents does not mean that one should not take it seriously. The content of this address is not mandated by the bylaws of the Society; one can presumably deliver a mathematical lecture or talk about policy issues.

The title of this lecture is intended to be as general as possible. One can reflect on the future; one cannot predict the future with precision. For example, I cannot predict with any measure of authority whether the Riemann Hypothesis will be proved or disproved. But I can predict that people will work on the Riemann Hypothesis until the question is settled. That is a prediction with a high degree of probability. The same applies for the Poincaré Conjecture and other celebrated problems.

This is one kind of reflection that is possible: to consider where some of our topics of concern are probably going to come from. Another way to reflect is to consider the internal workings of mathematics on the one hand and, on the other, the way in which mathematics fits into institutional structures, such as universities, government, private foundations, etc. This is the area where policy comes in.

I shall take seriously some recent articles and speeches on these and related themes. One article is based on the address Michael Atiyah gave at a meeting at the Fields Institute in Toronto. The article has just been published [1]. There is an interesting programmatic article by Mikhael Gromov, which appeared originally in a report of the National Science Foundation—a very influential report, as a matter of fact, on the problems of American mathematics in the future [2]. There was also a lecture in the same direction given by Robert MacPherson of the Institute for Advanced Study in Princeton. This was the one and only official lecture I heard about mathematical policy given at the Institute; it was entitled "Can One Predict the Future of Mathematics?" MacPherson remarked on a cyclical process in the twentieth century in which, after a mid-century emphasis on general abstract theories, we have returned to an emphasis on concrete problems and scientific applications reminiscent of 1900.

Of Hilbert and Poincaré

One of the most famous attempts to set the course for the future of mathematics was made in the early twentieth century, in a paper by David Hilbert. Hilbert delivered the paper, or rather part of it, at the International Congress of Mathematicians in Paris in 1900. He was not specifically invited to give such a paper, but he did it anyway. The speech was instigated by his friend Hermann Minkowski, to whom he was very close personally and mathematically at Göttingen. The speech was an attempt to answer an earlier paper presented by Henri Poincaré at the first International Congress of Mathematicians, held in 1897 in Zurich. Poincaré's paper did not specifically describe the future of

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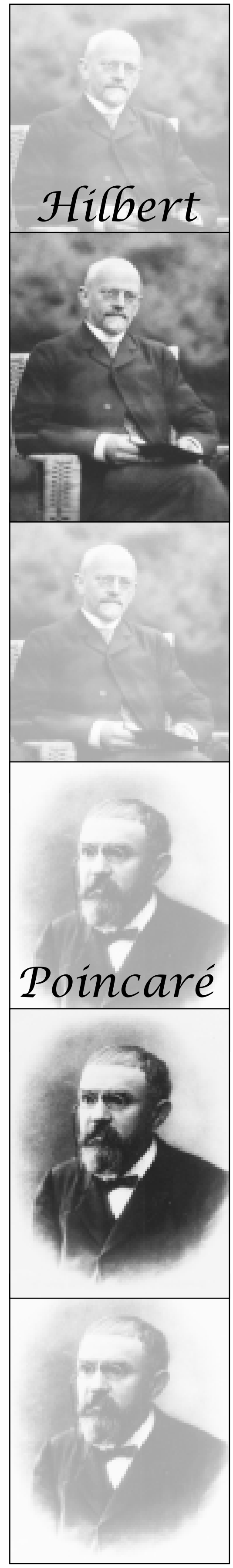

mathematics, but it was centered on the theme of how mathematical problems arising in physics would probably be very influential in mathematics. The thesis that Hilbert wanted to develop is that mathematical problems arising in other directions would also be influential, in particular in number theory. The Hilbert Problems that have been influential fall into three sections: the problems on logic and foundations, the problems on number theory, and the problems on partial differential equations. (For those who are interested in the genesis and influence of the Hilbert Problems, I strongly recommend two publications: the recent book by Jeremy Gray called *The Hilbert Challenge* [3] and an article by the historian David Rowe [4].)

There are some ironies that people have remarked on in the Hilbert Problems and the attempts to predict what would be done in mathematics after the year 1900. First of all, the Hilbert Problems did not predict what Hilbert would do. For example, there is not a word in the Hilbert Problems about functional analysis or integral equations, which are the subjects that Hilbert worked on in the succeeding two decades. Second, the problems ignored many of the major trends that were going on in mathematics at the time that he spoke. There is very little on topology, which Poincaré had founded and which a few years after Hilbert's speech was revolutionized by L. E. J. Brouwer. Third, the section on logic is very interesting from a psychological point of view and is probably the most influential part of the Hilbert Problems. It gave rise to Kurt Gödel's incompleteness theorem and the work of Gödel and Paul Cohen on the continuum hypothesis. These results constituted a negation of Hilbert's most powerful presupposition: that every problem could be solved. But the whole modern thrust of foundations was to prove the contrary.

Nevertheless, I think anybody who really dares to try such a thing as Hilbert tried should be commended, because after all it stimulates research. Whatever the ultimate influence of the Hilbert Problems has been, they certainly were a major stimulus in research and in diversifying mathematics.

At the 1908 Rome International Congress, Poincaré gave a response to Hilbert's speech in a speech entitled "The Future of Mathematics". Poincaré's approach is much broader and much more tolerant than Hilbert's, emphasizing, among other things, the connection of mathematics with theoretical physics, of which Poincaré was the foremost practitioner. One of Hilbert's problems, for example, was to axiomatize physics. This was a misguided endeavor at that time, which was just before relativity theory and quantum theory began. Even if one could have axiomatized physics, one could just as well ignore the axioms, because physics changed so drastically shortly thereafter. Poincaré was much closer to the physicists than was Hilbert (although Hilbert had a good deal of interaction with his local physicists). I have been told that physicists considered Poincaré to be one of the great physicists of his time, particularly because of his work on thermodynamics and relativity theory, where in effect he competed with Einstein for the origins of that subject. This is not to speak of areas like celestial mechanics, where Poincaré was the chief contributor for many years. His work earned him more nominations for the Nobel Prize in Physics than probably everybody else put together. In fact, in 1908, Gösta Mittag-Leffler went to every major physicist in Western Europe and got all but a very few to sign a petition saying that Poincaré deserved the Nobel Prize in Physics. Hundreds signed, but he still did not get it. Much of this is described in Jeremy Gray's book.

A number of years ago, at a symposium at the National Academy of Sciences, the mathematical physicist Elliott Montroll said, "The first half of the twentieth century in physics was the era of Hilbert, and the second half was the era of Poincaré." What did he mean? He meant that Hilbert's influence in setting up the research program that other people practiced in spectral theory had a very crucial influence on the development of quantum mechanics in the 1920s and 1930s. From Montroll's point of view, Poincaré's creative work in founding what we now call chaos theory is probably the decisive influence on physics in the latter half of the twentieth century.

In the 1940s, that habitually courageous mathematician, André Weil, wrote an essay entitled "The Future of Mathematics" [5]. I personally find this essay very sympathetic. It is also entirely free of the ideology that we call today Bourbaki-ism—Weil does not refer to Bourbaki; he does not praise formalization and the abstract movement in mathematics. He ignores it completely, even though he was one of the founders and principal figures of Bourbaki. Weil begins the essay by quoting Poincaré—even though Bourbaki, until the late 1970s, rarely said a good word about Poincaré. Weil also discusses, not in a very sanguine spirit, the institutional influences on mathematics. He had just finished a stint teaching at American universities, and he was not very optimistic about their intellectual future, because they did not have much intellectual structure at the time. This essay actually does to a large extent predict Weil's own intellectual tendencies in the years that followed. It is a very penetrating essay, and the spirit is so vigorous and free of prejudice—surprisingly so—that it is well worth reading.

The Mathematics of Bisociation

During my AMS presidency I organized a symposium called "Mathematical Challenges of the 21st Century", which took place at the University of California, Los Angeles, in August 2000. There are many axes on which one could organize such a meeting—or on which one can organize one's thinking about the future of mathematics. I chose an unorthodox axis, the precursor for which was an essay by Dieudonné published almost twenty-five years ago [6]. This essay provides a framework for discussing two aspects of developments in mathematics: its internal structure and its interdisciplinary nature. Many of these developments can be seen as examples of "bisociation", a word coined by the political and scientific writer Arthur Koestler. Bisociation occurs when two seemingly unrelated things are shown to have unanticipated connections. Arthur Koestler argues that all creativity is bisociation. In his book on humor, he argues that all humor is bisociation. One can accept these theses or not. But I believe that "bisociation" is a good description of many significant developments in mathematics in recent decades. This is put elegantly by Poincaré as follows:

> In proportion as science develops, its total comprehension becomes more difficult; then we seek to cut it in pieces and to be satisfied with one of these pieces: in a word, to specialize. If we went on in this way, it would be a grievous obstacle to the progress of science. As we have said, it is by unexpected union between its diverse parts that it progresses.

Mathematics is now extremely diverse. First of all, the size of the mathematical community since I started fifty-odd years ago has multiplied by a factor of more than ten. The first meetings of the AMS I attended were minuscule compared to today's meetings. At that time the AMS had about 4,000 members. And now we have 30,000 members all over the world. The number of mathematical specialties has increased greatly. There are many people working on a lot of subjects, and some people know nothing but their own narrow subject. This is a very unusual situation, in historical terms. To some extent it influences what the mathematical community thinks is of high significance. For example, the solution to historically famous problems is considered significant. Nobody in the mathematical world could doubt that Fermat's Last Theorem and the Riemann Hypothesis are of high significance. The same can be said for the "Millennium Prize Problems" identified by the Clay Mathematics Institute [7], although some of those problems may be less famous. Nevertheless, those problems at least are plausible candidates for being considered significant.

What else is significant, besides such problems? That is where the term "bisociation" comes in. Developments that have been regarded by the tastemakers of the mathematical community as being highly significant are almost always developments in which ideas and techniques from one set of mathematical sources impinge fruitfully on the same thing from another set of mathematical sources. In modern history, a major example of that in my experience was the Atiyah-Singer Index theorem, where K-theory and differential geometry on the one hand and elliptic partial differential equations on the other hand were identified in a very crucial way. Another example, which is perhaps of even greater importance in terms of its influence on contemporary mathematical research, was the thesis of Simon Donaldson, where he attained revolutionary new results on the topology of four-dimensional differentiable manifolds by using techniques of quantum field theory, particularly the study of Yang-Mills fields. An even more significant example is the interaction between quantum field theory and topology in the work of Edward Witten in the last two decades.

Many examples arose in lectures at the UCLA symposium. One was the Langlands Program. Andrew Wiles' solution of the Fermat problem consists of verifying one important special case of the Langlands Program. And what is the Langlands Program? It is essentially to establish systematically an interrelation between number theory and certain problems in group representations and automorphic forms. The program continues to develop, and every new case that is verified will be regarded as a very significant development. In his essay, Atiyah emphasizes quantum algebra, arguing that the mathematics of the future will consist of algebraic developments related to the fact that quantum field theory is both nonlinear and infinite-dimensional and therefore falls outside of the frame of reference of most of the classical mathematics that people try to apply to it.

Another interesting example is stochastic processes, which is a fairly classical field by now. The so-called Malliavin Program consists of putting stochastic processes in the framework of Sobolev spaces. Sobolev spaces were designed explicitly for studying partial differential equations, and Paul Malliavin has revolutionized this field by putting stochastic processes into this framework. There are other interesting variants of bisociation that can be mentioned. For example, there are developments in the topology of Sobolev manifolds. This is another case of the intrusion of Sobolev spaces where ordinarily one would not see them. I asked one of the practitioners of this field whether the topologists take his work seriously, and he said they do and he has great interactions with them.

I believe that a great deal of the internal development of mathematical ideas will consist of examples of this kind, where ideas from one area suddenly impinge on another area. Sometimes this causes displeasure, but I think that displeasure has to be quelled. It appears that, at least in the major mathematical centers, the students have learned to adapt to such changes. In fact, sometimes they flock from one fad to another. It is sometimes comical to see a group of people decide to change their minds about what they are interested in.

The most important area of bisociation lies in the interaction between mathematics and the sciences.

One major external influence on mathematics is, of course, the computer. Everything that relates to this sphere will be of central importance to mathematics in the coming century or as long as our civilization continues on its present trajectory—which means as long as it lasts! The influence of quantum computation is hard to predict, because nobody actually knows how to carry it out. It is still in the realm of mathematical modeling. But many other things—such as complexity of computation or number theory applied to codes—provide very striking examples of bisociation that certainly nobody would have predicted thirty years ago.

In molecular biology, mathematics has a much greater role to play than people realize, even though mathematics has had, for example, a significant effect on the course of the genome project. There will be an even larger effect when it comes to analyzing how the genome actually creates living cells. This development has caused a great upsurge of interest in mathematics on the part of biologists.

In his brief paper, Gromov mentions the essential developments in data analysis, which have an impact on such areas as genomics. David Donoho, an eminent practitioner of data analysis, gave a talk about this at the UCLA symposium. He showed how sophisticated mathematical tools such as wavelets are being used in data analysis. The rituals of classical statistics no longer suffice to deal with many problems that people face, especially when they have large masses of data—and large masses of data are the basic ingredient of the modern world.

Relations of Mathematics to Societal Structures

The interactions between mathematics and the sciences are important for mathematics not just for its internal development, but also for the future of mathematics within societal structures such as the federal government. There is now an initiative of the National Science Foundation (NSF) to try to boost the NSF mathematics budget—an initiative coming directly from the director of the NSF, Rita Colwell, a distinguished biologist, who is firmly convinced that a good deal of the future of biology rests on its interactions with mathematics. She has taken a proactive attitude of favor towards mathematics and plans to increase funding for it. Some may dislike the initiative's emphasis on interdisciplinary research. The emphasis reflects the fact that mathematicians, especially young mathematicians, ought to be interested in how mathematics is applied to other things.

The importance of the relations of mathematics to the institutions of society can be seen in the following example. American mathematicians often think of France as a country where mathematics is much encouraged. However, the relations of mathematics and mathematicians to the French government have been extremely problematic. In fact, there was a major crisis in 2000, when Claude Allègre was the French minister of science, research, and education. He is a reasonably distinguished geophysicist and has written several books, one called *The Defeat of Plato*—Plato being a representative of mathematics. In this book he clearly reveals that he does not know about the mathematics pervading fundamental physics after the year 1900. And he is firmly convinced that with the computer one does not need to know much mathematics of any kind to do science. While he was minister, Allègre decided that mathematics was overemphasized in French education. He advocated the elimination of mathematics classes in the schools. Because he was a very close friend of the prime minister, this looked like a very dangerous situation. Mathematicians were saved from the consequences of Allègre's dogmas, but not through their own influence. Allègre got very much under the skin of the major teachers' unions in France, who were very angry about his proposed reforms. In France, which has a socialist regime, you do not irritate the teachers' unions! So no matter how close Allègre was to the prime minister, he had to go. And he went. I do not think his successors have taken up his ideas. This episode illustrates that mathematicians have to be aware and alert in responding when these kinds of problems arise in the institutions that determine our destiny.

Fortunately, the AMS has taken a very proactive attitude toward dealing with many of the problems to which I just referred, and as long as it continues in this role, the Society will be a vital influence on the future of American mathematics. The AMS is playing an extremely important role, which most of its members do not realize, in organizing and focusing attention on policy matters. I urge the membership to find out what is going on in this sphere. One can go to meetings of the committees that deal with these matters. The meetings are held in Washington, D.C., and are open to the mathematical public.

The report in which Gromov's essay appeared noted that mathematics in the United States is flourishing, but it is not guaranteed that it will continue to flourish, for two reasons. One is the scarcity of resources. But even more important is the scarcity of recruitment, especially of talented American students, to mathematical vocations. Both of these are very crucial questions, and questions that we must continue to address.

References

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[5] The future of mathematics, by André Weil, *Amer. Math. Monthly* **57** (1950), 295–306.

[6] Present trends in pure mathematics, by Jean Dieudonné, *Adv. in Math.* **27** (1978), no. 3, 235–55.

[7] The Millennium Prize Problems are described on the Clay Institute's website, `http://www.claymath.org/`.