Learning from Liu Hui? A Different Way to Do Mathematics

Christopher Cullen

Could we have done mathematics differently? At a logical level this question is trivial: research mathematicians spend their time exploring all the ways one *can* "do it differently" and then doing them. There are no signs that the mathematical enterprise has any artificial barriers round it that stand in the way of this task. But my question refers to something a little less foundational. What if we make the "counterfactual move" of trying to imagine the history of mathematics with one of its great monuments no longer there—say Euclid, for example. Could we imagine a possible history of mathematics without Euclid? By "mathematics without Euclid" I do not of course mean non-Euclidean geometry, but rather a mathematics stripped of the whole axiomatic-deductive scheme for which Euclid's writing served as the great exemplar and entry point for generations of western mathematicians. At first glance the likely course of development of such a mathematics seems so different from our own that it might deserve a place in one of the more intellectually inclined episodes of *Star Trek* (Spock: "It's mathematics, Jim, but not as we know it").

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But in fact it is not even necessary to imagine such an alternative history of mathematics, since one already exists. Ancient China developed its own mathematical culture based on a radically different approach to the structuring of mathematical thought. Unfortunately, not only western but also Chinese historians of mathematics have often failed to see this point. As a result, early Chinese mathematicians have been portrayed as if they were doing the same job as Euclid, only—to be blunt—a whole lot worse. In history, as in other disciplines, using the wrong tools for the job often breaks the thing you were trying to fix. But what *were* ancient Chinese mathematicians up to if they were not playing the Euclidean game? I hope that the answer to this question may be of more than historical interest, since it bears directly on the pressing question of how mathematicians are to be made and made more effectively.

The Right Triangle Relation in China

The problem of what Chinese mathematicians were up to emerges very clearly if we start to ask about the history of Pythagoras' theorem in China.

Before we face the main issue, there are a few smaller problems in the way of our shift from west to east. For a start, we can hardly name the relation as used in China after a Greek thinker of around 520 B.C. whose name was not even mentioned in China until many centuries later.[1] More significantly, ancient Chinese mathematicians did not talk about right-angled triangles, because they did not talk about triangles in any general sense: there is no ancient Chinese term corresponding to Greek *trigonon*. They did, however, talk quite specifically about the ensemble of a horizontal 'hook' *gou* 句 extended out at right angles from the foot of a vertical line, or 'leg' *gu* 股. Joining the ends of the hook and leg was what westerners call the hypotenuse, but the Chinese called the 'bowstring' *xian* 弦. One does not, however, usually speak of *gouguxian*, but just of *gougu* 'the hook and the leg'.

When was the *gougu* relation (or whatever we choose to call it) first known and used in China? The first evidence that *gou*, *gu,* and *xian* were known to have a simple and useful relationship is found in the two earliest works of the classical Chinese mathematical canon, the *Zhou bi* 周髀 (*Gnomon of Zhou*), and the *Jiu zhang suan shu* 九章算術 (*Mathematical Methods in a Nine-fold Categorisation*).[2] Both these works were close to their present form by the first century A.D. In the *Zhou bi* the *gougu* relation is put to practical use in four instances,[3] while in the *Jiu zhang suan shu* it is the main theme in all the problems of the

[1]*It is in any case well known that the "Pythagorean" relation between the sides of a right-angled triangle was used in Mesopotamia long before the study of mathematics began in the Hellenic world. See, for instance, Otto Neugebauer,* The Exact Sciences in Antiquity *(New York, 1969), pp. 34–6, and plate 6a showing a clay tablet of the Old Babylonian period (c. 1800–1600 B.C.) with the length of the diagonal of a unit square marked with a number equivalent to* 1.414213 … *which is in error by one in only the seventh significant figure. As Neugebauer notes, Ptolemy used the same value in computing his table of chords two thousand years later.*

[2]*For a full translation and study of the first of these two works, see C. Cullen,* Astronomy and Mathematics in Ancient China *(Cambridge, 1996). The title is often found in the form* Zhou bi suan jing 周髀算經 (Mathematical canon of the Zhou gnomon), *but the words* suan jing *were not added before approximately A.D. 600. The second work has been translated in Anthony W. C. Lun, J. N. Crossley, and Kangshen Shen,* The Nine Chapters on the Mathematical Art: Companion and Commentary *(Oxford, 1999). The title is translated by Lun et al. in what is now the traditional manner; the version given by me is intended to suggest more clearly the significance of the original Chinese.*

[3]*See sections B11, B29 (twice), and B33 in the version of Cullen (1996). Most of the* Zhou bi *is concerned with astronomy rather than mathematics per se.*

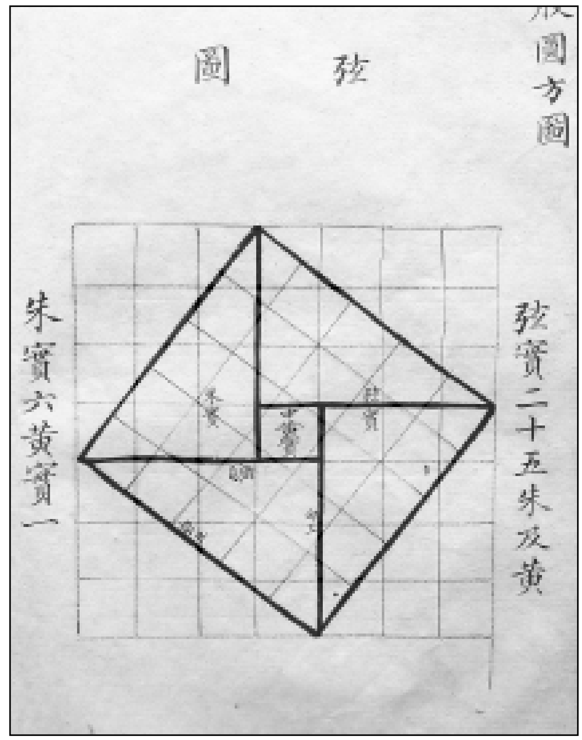

Figure 1.

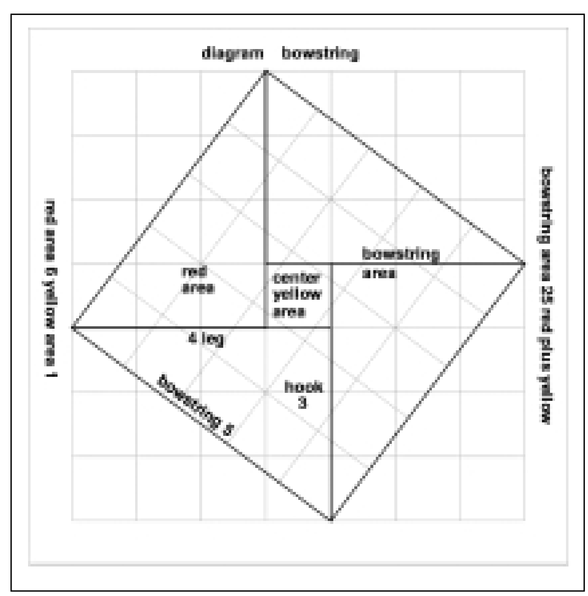

Figure 2.

ninth and final chapter. A manuscript of a pre-canonical mathematical text recently found in a tomb of the second century B.C. does not make use of the *gougu* relation, even where one might have expected to find it, in problems relating to sawing a square beam out of a round log.[4] So it seems that we have at least a rough fix on when *gougu* thinking began in China.

The situation has unfortunately been made much more confusing by the fact that the book traditionally placed first in the canonical mathematical series, the *Zhou bi*, was for centuries commonly thought to date from the beginning of the Zhou dynasty, around 1000 B.C., since it begins with a short dialogue between the Duke of Zhou (who ruled as regent near the start of the Zhou dynasty) and Shang Gao, a sage of the preceding dynasty, in which the *gougu* relation is mentioned. No scholar now believes in this early dating. I have argued that this dialogue is in fact probably one of the last sections of this somewhat heterogeneous work to be written and might have been added to the book to make it more consonant with the cosmological numerology popular in the imperial court in the early first century A.D.[5]

However, putting the dating problem to one side, a further issue arises from the fact that the opening dialogue includes a passage which (though rather obscurely phrased) amounts to no more than the statement of the *gougu* relation for the case where the 'hook' is 3 units, the 'leg' is 4 units, and the 'bowstring' is 5 units. No premodern Chinese commentator has ever claimed to see anything more substantial here, including some eighteenth- and nineteenth-century scholars well versed in western as well as Chinese mathematics. Nevertheless, a number of twentieth-century historians of mathematics in the East and West have felt obliged to extract a "proof of Pythagoras" from this material by hook or by crook. I think the unfortunate results of this will be clear if I exhibit first my own fairly literal translation of the relevant part of the dialogue, followed by one of the more creative versions that have been suggested:[6]

> #A1 [13b] Long ago, the Duke of Zhou 周公 asked Shang Gao 商高, "I have heard, sir, that you excel in numbers. May I ask how Bao Xi 包羲[7] laid out the successive degrees of the circumference of heaven in ancient times? Heaven cannot be scaled like a staircase, and earth cannot be measured out with a footrule. Where do the numbers come from?"
>
> #A2 [13f] Shang Gao replied, "The patterns for these numbers come from the circle and the square. The circle comes from the square, the square comes from the trysquare, and the trysquare comes

[4] *This text, bearing the title* Suan shu shu 筭數術 (A Book of Reckoning and Numbers) *was recovered in 1983 from a tomb at Zhangjiashan* 張家山 *in Hubei* 湖北 *province, China. Evidence from other material found there suggests the tomb was closed in 186 B.C. Like most other Chinese books of its period, the* Suan shu shu *was written in ink on a series of bamboo strips bound together with strings, rather like a roller blind. The strings have perished over the centuries, leaving archaeologists with the tricky task of recovering the original order of 190 jumbled strips on which many characters were partly or wholly obliterated. These and other difficulties no doubt explain why a preliminary transcription of this material was published for the first time in 2000. The standard text is now that given in Peng Hao* 彭 浩 Zhangjiashan Han jian «Suan shu shu» zhu shi 張家山 漢簡《筭數術》註釋 (The Han Dynasty Book on Wooden Strips *Suan shu shu* found at Zhangjiashan, with a Commentary and Explanation) (*Beijing, 2001). Unlike the highly structured* Jiu zhang suan shu, *the* Suan shu shu *lacks any kind of chapter divisions and consists of a series of about 68 problems falling into a dozen or so groupings with broadly common themes. The problems differ in style and terminology to a degree that suggests they were taken from a variety of sources. The material involving the log is found on pp. 110–3 of Peng's book.*

[5] *Cullen (1996), pp. 153–6.*

[6] *Translations from the* Zhou bi *are labelled according to the system used in Cullen,* Astronomy and Mathematics in Ancient China *(Cambridge, 1996).*

[7] *Bao Xi, also known as Fu Xi* 伏羲, *is a mythical figure credited with having conceived the basic structure of the* Yi Jing 易經 (Book of Change).

> from [the fact that] nine nines are eighty-one."[8]
>
> #A3 [14b] "Therefore fold a trysquare[9] so that the base is three in breadth, the altitude is four in extension, and the diameter is five aslant. Having squared its outside, halve it [to obtain] one trysquare.[10] Placing them round together in a ring, one can form three, four, and five. The two trysquares have a combined length of twenty-five. This is called the accumulation of trysquares. Thus we see that what made it possible for Yu 禹[11] to set the realm in order was what numbers engender."

Here now is a version of the third of these sections, intended to show that there is a proof of Pythagoras somewhere in there:

> "Thus let us cut a rectangle (diagonally) and make the width 3 (units) wide and the length 4 (units) long. The diagonal between the (two) corners will be 5 (units) long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a (square) plate. Thus the four (outer) half-rectangles of width 3, length 4, and diagonal 5 together make two rectangles (of area 24); then (when this is subtracted from the square plate of area 49) the remainder is of area 25. This process is called 'piling up the rectangles'."[12]

No comment seems necessary, apart from noting that the insertions and very free interpretations of the original text evidenced in this version have no basis in the *Zhou bi*, its commentaries, or indeed any other premodern Chinese text. To make matters worse, one frequently finds it assumed that the text of the *Zhou bi* is referring to the diagram reproduced here as Figure 2, which was in fact only added—as he tells us himself—by the third-century A.D. commentator Zhao Shuang 趙爽, who uses it to illustrate his own essay on the *gougu* relation, a piece of writing which is more or less independent of the main text.[13] Zhao Shuang, it may be said, gives a pedestrian explanation of this passage that in the first place shows that it was no clearer

[8] *This statement exploits a number of data. The circumference of a unit circle was taken to be three, while the perimeter of a unit square is four; "the nine nines" is the name for the traditional multiplication table, framed by two rows of numbers at right angles in the form of the trysquare mentioned below.*

[9] *The trysquare* ju 矩 *is the familiar L-shaped carpenter's tool. In an ancient Chinese context it also refers to an L-shaped area made of two rectangular strips at right angles. In the* Zhou bi *and its commentaries it never means a square or rectangle. Although this shape is sometimes called a "gnomon" in western usage, I avoid this term, since I need to reserve it for the vertical shadow-casting pole after which the* Zhou bi *is named.*

[10] *This is one possible rendering of a sentence which is clearly corrupt. None of the variant versions that are found in textual sources makes good sense in Chinese. See Cullen (1996), p. 94, note 91.*

[11] *A mythical figure whose work as a surveyor and hydraulic engineer rescued the world from a terrible flood.*

[12] *This version, due to Arnold Koslow, is quoted in Joseph Needham,* Science and Civilisation in China *(Cambridge, 1959), vol. 3, pp. 22–3. I offer this translation as a representative of the lengths to which one may be tempted to go if one is determined to translate this text as if it was intended to convey a proof. I deliberately do not cite other examples (of which there are plenty, some by scholars whose other work I respect a great deal), and obviously each of them must be judged on its own merits. But I am regretfully convinced that all such efforts amount to making hamburgers without any ground beef to put inside the bun. One may admire the ingenuity of the attempt while declining to eat the result.*

[13] *See Cullen (1996), p. 171. What is more, it is clear from Zhao's commentary that the diagram he used was not in the form seen in most versions of the* Zhou bi *nowadays, in which a 7 by 7 square has four 3-4-5 triangles inscribed in its corners, so as to enclose an inclined 5 by 5 square in which four further 3-4-5 triangles are inscribed so as to enclose a unit square. Such a diagram might be used to give a graphical dissection proof of the* gougu *relation,*

to him than it is to us and also shows no sign of his seeing a proof of any kind in it.[14]

Up to this point then the record seems to be "Greek mathematicians 10, Chinese mathematicians nil", despite the attempts to smuggle a ball over the line for the East Asian team. But that is where you get if you mix baseball and football. We need to look more closely at the way Chinese mathematical writing actually functions to see what the Chinese score actually was.

How Chinese Mathematics Worked

The earliest explicit statement of the relations between the lengths of the *gou*, *gu*, and *xian* occurs in the *Jiu zhang suan shu*, an anonymous text which, as already mentioned, had probably reached something close to its present form by the first century A.D.:

> 術日。句股各自乘。并，而開方除之。即弦。
>
> Method: Let *gou* and *gu* each multiply themselves. Add, and find the side of the square,[15] which is the *xian*. (*Jiu zhang suan shu*, p. 419 in the edition of Guo Shuchun 郭書春, Shenyang, 1990)[16]

The rest of the chapter goes on to apply this relation to the solution of problems of increasing complexity, of which the following is a sample:

> 今有戶高多於廣六尺八寸，兩隅相去適一丈。
> 問戶高，廣各 幾何？
>
> Now there is a door whose height is greater than its breadth by 6.8 feet. Two [diagonally opposite] corners are 10 feet apart. It is asked: what are the height and breadth of the door? (*Jiu zhang suan shu,* p. 423)

The text then gives the solution: the breadth is 2.8 feet, and the height is 9.6 feet.[17] It continues:

> 術日 。 令一丈自乘為實 。 半相多 。 令自乘 。
> 倍之。減實。半其餘 。以開方除之 。 所得，
> 減相多之半 。 即戶廣 。 加相多之 半 。
> 即戶高。
>
> Method: Let the 10 feet multiply itself to make the product. Halve the difference, and let it multiply itself. Double it, subtract from the product. Find the side of the square. With what you obtain, subtract from the halved difference, and that is the breadth of the door. Add to the halved difference, and that is the height of the door.

The reader may easily verify algebraically that this method works. No justification of the method is given in the original text, nor is any attempt

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although Zhao does not do this, and neither does any other premodern commentator. In fact, it is clear from his description that his xian tu 弦圖 *'hypotenuse diagram' consisted only of the 5 by 5 square with its inscribed triangles and central unit square. The outer part of the usual diagram is never referred to by Zhao and probably originates in the draftsman's construction lines used to construct the inner square and its triangles.*

[14] *Zhao's commentary is fully translated and explained in Cullen (1996), pp. 82–8.*

[15] *"Find the side of the square" is my attempt at a reasonably faithful but still comprehensible rendering of the conventional Chinese expression* Kai fang chu zhi 開方除之*, literally "opening the square, eliminate it", which would have been about as puzzling to a nonmathematical ancient Chinese reader as our own "find the square root" still is to a nonmathematical modern English reader. The expression* chu zhi *(eliminate it) is probably a reference to the parallel between the algorithms for square root extraction and division, which both make use of the same procedure for eliminating that which corresponds to a given digit in the number to be divided or whose root is being sought (Professor Karine Chemla, REHSEIS Paris, private communication, May 2002).*

[16] *Readers may compare my interpretations with those in the version of Anthony W. C. Lun, J. N. Crossley, and Kangshen Shen,* The Nine Chapters on the Mathematical Art: Companion and Commentary, *(Oxford, 1999).*

[17] *For simplicity I have reduced all the units used in the original text to the* chi 尺*, roughly one foot.*

made to justify the original statement of the general *gougu* relation. So far the reader may have the feeling that ancient Chinese mathematics consists of no more than a body of rules of thumb adopted blindly and without interest in whether or why the rules worked. Actually the truth is rather stranger than that.

Like many ancient Chinese texts, the *Jiu zhang suan shu* has a commentary, in this instance by the great mathematician Liu Hui 劉徽, who was active around 260 A.D. What his commentary does is to follow through all the problems of the book, in effect taking the "Method" statements of the original text to pieces and reconstructing them in a way that enables the reader to see clearly how they work. In the case of the chapter on *gougu* problems, he does this mainly with reference to diagrams which are now lost but which can in most cases be easily reconstructed. But one case in which the reconstruction is by no means easy is that of the original statement of the *gougu* relation. Liu Hui's explanation reads:

> 句自乘為朱方。 股自乘為青方 。 令出入相補，各從其類 。因就其餘不移動也 ， 合成弦方之幕。開方除之，即弦也。
>
> The *gou* multiplied by itself makes the red square, and the *gu* multiplied by itself makes the blue square. Let there be taking away, and putting in, and being made complete, each following its kind. Thus one reaches [a state where] the differences no longer are to be adjusted. Together they form the area of the *xian* square. Find the side of this square, and this is the *xian*. (*Jiu zhang suan shu,* p. 419)

It is clear that Liu Hui assumes his readers can see a diagram in which the two smaller squares on the *gou* and the *gu* are in some way dissected and reassembled to form the larger square. For him no further explanation is necessary. The reader may enjoy pausing to try to make such a dissection before turning to one possible and ingenious solution suggested by Don Wagner. It was originally published as "A proof of the Pythagorean Theorem by Liu Hui (third century AD)" (*Historia Math.* **12** (1985), pp. 71–3), but may be more conveniently accessed through Wagner's website at:

`http://www.staff.hum.ku.dk/dbwagner/Pythagoras/Pythagoras.html`

For our present purposes, the point is that Liu Hui simply gets this explanation out of the way as a preliminary to the main business of the chapter: this is not a major issue on which he feels the need to dwell.[18] His explanation does not use any word that could be mapped onto modern English "proof" or "theorem" or onto their Greek equivalents. As explicit issues, these were not his concern. What then was Liu Hui concerned with, as an ancient Chinese mathematician? Fortunately he tells us. In his preface to the commented edition of the *Jiu zhang suan shu*, he says:

> When I was young I learned the *Jiu Zhang* and when I grew up I went over it again carefully. I looked into the breaking apart of Yin and Yang, took a comprehensive view of the basis of mathematical methods, and of the suppositions involved in seeking the unknown, and thus attained to realisation of [the work's] meaning. Therefore I have ventured to exert my meagre capacities to the utmost, and to select from what I have seen [in other books?] in order to make a commentary. The categories under which the matters [treated herein fall] extend each other [when compared], so that each benefits [from the comparison]. So even though the branches are separate they come from the same root, and one may know that they each show a separate tip [of the same tree] (事類相推，各有攸歸，故枝條雖分，而同本幹 者，知發其一端而已). (*Jiu zhang suan shu*, preface, p. 177)

And indeed his words seem to echo those found in the second section of the *Zhou bi* itself, in which Chen Zi 陳子 (a figure unknown to history) is represented as explaining to his confused student

[18]*He certainly shows no signs of being inclined to sacrifice an ox in celebration, as the legendary western account of the theorem's discovery claims Pythagoras did: see Ivor Thomas,* Greek Mathematical Works: I Thales to Euclid *(Harvard, 1980), p. 185, citing Proclus.*

Rong Fang 榮方 the essentials of mathematical thinking:

> #B6 [24k] Chen Zi replied, "You thought about it, but not to [the point of] maturity. This means you have not been able to grasp the method of surveying distances and rising to the heights, and so in mathematics you are unable to extend categories (*tong lei* 通類).… If one asks about one category, and applies [this knowledge] to a myriad affairs, one is said to know the Way. …Therefore one studies similar methods in comparison with each other, and one examines similar affairs in comparison with each other. This is what makes the difference between stupid and intelligent scholars, between the worthy and the unworthy. Therefore, it is the ability to distinguish categories in order to unite categories (*neng lei yi he lei* 能類以合類) which is the substance of how the worthy one's scholarly patrimony is pure, and of how he applies himself to the practice of understanding."

What we have here is a concise statement of a twofold heuristic strategy, summed up in the words "distinguish categories in order to unite categories" (*lei yi he lei* 類以合類). On the one hand, the mathematician performs the analytic task of distinguishing different problem types, each with their own methods *shu* 術, from each other. On the other hand, the very act of analysis brings together groups of similar problems which may be treated synthetically. Further, one can then attempt to "unite categories" at a higher level by finding common structures underlying different problem categories.

Chen Zi's analytic/synthetic approach is in fact not particularly well exemplified in the *Zhou bi* itself. It is, however, clearly (or so it seems to me) the main rationale of the *Jiu zhang* mentioned earlier. Whereas Euclid was concerned to show how a great number of true propositions could be deduced from a small number of axioms, the anonymous author of the *Jiu zhang* followed a different but no less rational route in the reverse direction. He started from the almost infinite variety of possible problems and aimed to show that those known to him could all be reduced to nine basic categories solvable by nine basic methods. To a great extent he succeeded, although the contents of some sections still show a degree of diversity. It did not strike him as worthwhile to try to argue explicitly that his methods would always work for the appropriate problem type. In the first place, he already knew they *did* work—the examples are before us to this day. Secondly, if it ever turned out that the method failed on a new problem, that would not have been taken as a sign that the method was wrong, but rather that it was necessary to distinguish a new problem category with a new common method for all problems of the new type—*lei yi he lei* 類以合類 distinguish categories in order to unite categories, in fact, as Chen Zi says.

As a person whose initial mathematical training beyond the level of arithmetic was based firmly on an initiation into a Euclidean structure of axiomatic deduction and stacking one theorem on another,[19] with problems serving only to show that one had understood the theorems, it is clear to me that the problem-centred approach of the ancient Chinese mathematician deals directly with the difficulty pointed out by one very perceptive student of the history of science in the West. Students, Thomas Kuhn tells us:

> …regularly report that they have read through a chapter of their text, understood it perfectly, but nonetheless had difficulty solving a number of the problems at the chapter's end. Ordinarily, also, those difficulties dissolve in the same way. The student discovers, with or without the assistance of his instructor, a way to see his problem as *like* a problem he has already encountered. Having seen the resemblance, grasped the analogy between two or more distinct problems, he can interrelate symbols and attach them to nature in the ways that have proved effective before. … The resultant ability to see a variety of situations as like each other … is, I

[19] *I enjoyed it, by the way; this was Britain in the late 1950s. Thank you, Mr. B. G. Worsdall (Bedford School), for introducing me to the delights of Euclidean proof!*

> think, the main thing a student acquires by doing exemplary problems, whether with a pencil and paper or in a well-designed laboratory. After he has completed a certain number, which may vary widely from one individual to the next, he views the situations that confront him as a scientist in the same gestalt as other members of his specialists' group. For him they are no longer the same situations he had encountered when his training began. He has meanwhile assimilated a time-tested and group-licensed way of seeing. (Thomas Kuhn, *The Structure of Scientific Revolutions* (second edition) (Chicago, 1970), p. 189)

Things may be very different in mathematics education today from the situation I recall, in which, as Kuhn hints, the art of problem solving was something one all too often had to discover without the help of one's teacher. Could it be that students of mathematics in ancient China had a more effective introduction to this aspect of their craft than at least some in the West have received? Since it seems that well-trained and highly motivated students of mathematics (let alone teachers of mathematics) are not overly common nowadays, it may still be worthwhile to ask what there is in the ancient Chinese approach that might still be useful to us today.

About the Cover

This month's cover accompanies the article by Christopher Cullen and shows the so-called "hypotenuse diagram" from the *Zhou bi*, presumably drawn in its original form by the third-century commentator Zhao Shuang. It is the source of the logo of the 2002 International Congress of Mathematicians (ICM), to be held in Beijing in August 2002. The particular image is a photograph of a page from a manuscript of the late eighteenth century, now located in the Asian Studies Library of the University of British Columbia.

This handsome copy was made by hand, presumably from a printed book. The *Zhou bi* was, as far as we know, the first mathematics book ever to be printed. The first edition was produced in the eleventh century, but the earliest extant copy is from the next century, preserved in a library in Shanghai. Printing in China at this time was from inked wood blocks, on which the content of a pair of successive pages had been cut in relief. This technique, which was invented in China, played only a small role in European printing, but it was so suitable for the Chinese writing system, which has a huge character set, that it persisted long beyond when movable type had been first introduced.

The coloring scheme is not original, but is suggested by the text in the diagram, as shown in Figure 2 of Cullen's article. The color "red" is actually vermilion, an orange-red that since ancient times has been the color of seal stamps in much of eastern Asia.

Good references are the short book *Printing and Publishing in Medieval China* by Denis Twitchett and the more comprehensive *Paper and Printing* by Tsien Tsuen-Hsuin, volume V, part 1, of *Science and Civilization in China*.

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