How the Other Half Thinks: Adventures in Mathematical Reasoning
and
Mathematics Galore! Masterclasses, Workshops, and Team Projects in Mathematics and Its Applications

Reviewed by Edward J. Barbeau

How the Other Half Thinks: Adventures in Mathematical Reasoning
Sherman Stein
McGraw-Hill, 2001
xi + 177 pages
ISBN 0-07-137339-X

Mathematics Galore! Masterclasses, Workshops, and Team Projects in Mathematics and Its Applications
C. J. Budd and *C. J. Sangwin*
Oxford University Press, 2001
viii + 254 pages
ISBN 0-19-850769-0 (hardcover), 0-19-850770-4 (softcover)

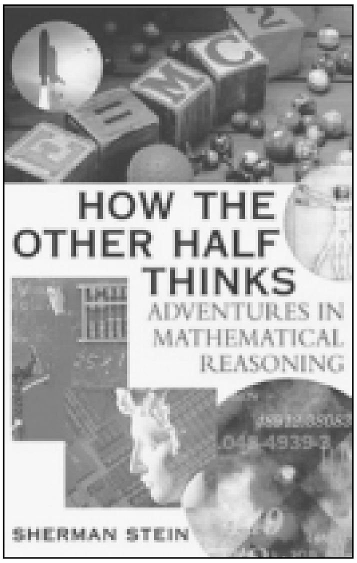
How does it happen that there are people who do not understand mathematics? If mathematics invokes only the rules of logic, such as are accepted by all normal minds; if its evidence is based on principles common to all men, and that none could deny it without being mad, how does it come about that so many persons are here refractory?…[T]hat not everyone can understand mathematical reasoning when explained appears very surprising when we think of it.

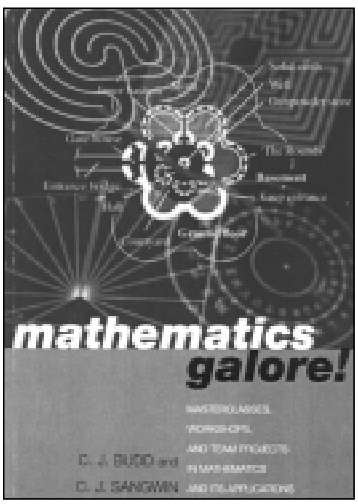
—Henri Poincaré, *Mathematical Creation* [49]

I am often aware of a deep cultural divide between me as a mathematician and members of the general public, even those who are well educated in other respects. It is not so much a difference of knowledge and technical background as of distinct patterns of thought and analysis. Over the past two centuries many mathematical writers have sought to bridge this gap.

Early authors like Lewis Carroll [8], Courant and Robbins [14], and Hilbert and Cohn-Vossen [22] presumed upon some mathematical background and experience. Much of the literature consists of collections of problems for amateurs (such as [8], [12], [13], [27]). The prolific Martin Gardner [9], [10] has probably done more than anyone to keep the amateur abreast of the most interesting and elegant mathematical problems and results. Other books have been written for a wider audience. The classics of W. W. Rouse Ball and H. S. M. Coxeter [7], Hugo Steinhaus [40], George Gamow [19], and Hans Rademacher and Otto Toeplitz [33], for example, showed that the scope of mathematics goes far beyond what is learned in school. Recent writers such as Ian Stewart [41], Ivars Peterson [30], [31], and A. K. Dewdney [18] follow in this tradition. One of the first attempts to reach a very broad audience was the publication of the 1956 anthology *The World of Mathematics* [28], which drew together a variety of essays touching different aspects of mathematics by many authors, including some nonmathematicians. History and biography are other ways of conveying the significance of mathematics and what it feels like to be a mathematician [1], [2], [3], [4], [5], [6], [24], [45]. A growing and receptive public has accounted for the popularity of such recent books as *Gödel, Escher,*

Edward J. Barbeau is professor of mathematics at the University of Toronto. His e-mail address is barbeau@math.utoronto.ca.





Bach (Hofstadter) [23]; *The Mathematical Experience* and *Descartes' Dream* (Davis and Hersh) [15], [16]; *The Emperor's New Mind* (Penrose) [29]; and *A Brief History of Time* (Hawking) [21], which offer pretty challenging reading as they delve into social, philosophical, and epistemological matters. The book by Lakoff and Núñez [47] is a recent foray in this direction.

Some authors, alarmed by failures of the education system and the innumeracy of much of the public, have turned to polemic [46], [48] and propaganda [17]. There is a vast literature intended to inspire young people to find in mathematics more substance and excitement than is seen in the regular curriculum. An early example is the work of W. W. Sawyer (for example, [34]), which explored the context and conceptual foundation of the school syllabus, but we also have recent books of David Wells [43] and of Anthony D. Gardiner (such as [20]) that engage students in problem solving and investigation, and of Ravi Vakil [42], who while a student himself sought to inspire other students with mathematical lore, elegant problems and solutions, and achievements of other young mathematicians.

Both books under review belong to a relatively small part of the literature that presents readers with case studies of mathematical investigation. Their purpose is to share with the novice a feeling for how mathematicians select objects of study, how they work, and how they validate their results. The audience for *Mathematics Galore!* consists of students, their teachers, mentors and parents; and for *How the Other Half Thinks*, intelligent lay persons with intellectual curiosity and the patience and resolve to satisfy it.

Sherman Stein is probably best known for his book *Mathematics: The Man-Made Universe* [38], which has become a staple of general mathematics courses in colleges. In the last decade he has turned his attention to a broader audience than students and become a missionary to the world at large. In the introduction to a recent collection of essays, *Strength in Numbers* [39], he says that his "purpose is to spread the gospel of mathematics, to carry the word to unbelievers and believers alike." In careful and direct prose he invites his readers to take another, more discriminating, look at the discipline. *How the Other Half Thinks* is even more ambitious. Stein wants to do more than just talk about mathematics. He guides the reader through the experience of solving some significant problems. As the title indicates, Stein recognizes that some readers will be entering alien territory and will need to acclimatize themselves to a mathematics quite different from what they are used to.

What should the lay person appreciate about mathematics? Mathematics has invaded almost every area of human activity, from art to the making of decisions, insofar as there is structure to be analyzed and manipulated. However, while argumentation and analysis occur in many walks of life, in mathematics they have their own peculiar characteristics. People seem to be perfectly capable of reasoning in everyday situations with a context to buttress their thinking, but often have trouble with logically similar tasks framed abstractly when familiar cues are removed. Furthermore, many arguments in ordinary affairs depend on premises weighted according to the context, experience, and values of the proponents; it may be possible for disputants to agree on the facts but not on the interpretation and consequences of them.

Mathematical arguments, on the other hand, are self-contained and ineluctable. They are not contextualized by anything outside themselves so that, in particular, once the premises are clarified and accepted, the conclusions become inevitable. As Leonhard Euler put it, "Wherefore, even if analysis is not without occasions for dispute, nevertheless they are distinguished from all other occasions in that when eventually all the evidence has been thoroughly weighed the matter can be completely settled."[1] There is economy in a good mathematical argument, and different ways of approaching an investigation may cast more light but can never contradict. Mathematicians try to identify and strip away any hypotheses not essential to a conclusion; this leads to results of astonishing generality that are hard to grasp by a neophyte who may become distracted by irrelevant details.

Mathematical discourse occurs at different levels of intensity, from discursive and suggestive to focussed and formal. Experienced mathematicians know the strengths and limitations of each and can switch easily among them. Insight and validation are based on analogy and metaphor, as well as on rigorous argumentation. The ease of understanding mathematical situations depends on finding appropriate representations, and it is hard for

[1] *From the opening paragraph of* "De seriebus divergentibus", *by Leonhard Euler,* Novi Comm. Sci. Petrop. ***5*** *(1754–55), 1760, 205–237* = Opera Omnia ***I, 14****, 585–617.*

the novice to see commonality between two apparently different formulations and to move from one to another.

These are some of the themes that come up in Sherman Stein's beautiful book. His writing is spare and clear, and he has selected eight problems, loosely connected by their formulation in terms of strings of the two letters a and b, to illuminate some of these points. The reader needs no specialized knowledge, only some elementary arithmetic and reasoning ability to negotiate the examples. The solutions to the problems are developed in great detail, with the evolution of the ideas indicated in some cases. However, he tends to let the mathematics speak for itself and does not intrude with editorial comment on the process that the reader is being guided through. He hopes that the book will help bridge "that notorious gap" that separates the humanities and sciences by demonstrating that mathematics is holistic and intuitive as well as analytic and numerical. He "kept in mind two types of readers: those who enjoyed mathematics until they were turned off by an unpleasant episode, usually around fifth grade, and mathematics aficionados who will find much that is new throughout the book."

While I can attest confidently that he has succeeded with the latter audience, I am less certain about my ability to judge how well he might reach the former. So I enlisted some volunteers from a class of third-year undergraduates to read and comment on individual chapters. These students, planning to become elementary teachers, had a variety of backgrounds in mathematics, and some had not ventured beyond compulsory secondary courses. With one exception, none of them caught fire. They appreciated how the material was organized, particularly the use of charts and tables to summarize information. While some found the explanations reasonably clear, others had considerable difficulty. Stein's analogies were appreciated by some, but others had difficulty negotiating changes in perspective. There were differing levels of satisfaction with the motivation for the examples. One student, who found that her chapter was "easy to follow and understand," did, however, have to "slow down and reread some proofs of the chapter to fully follow the author's argument." Another student, self-described as more of a nonmathematical person, "was delightfully surprised" to discover that she had much less difficulty than expected and praised the "simple, clear steps" of the explanations. While they were not in agreement on recommending the book, their comments underscored for me what a difficult challenge Stein undertook.

The sixth chapter, "Counting Ballots", will give an idea of the flavour of the book. Barbara has won an election against Ann. As the votes are counted, one by one, what is the chance that Barbara always stays ahead of Ann throughout the count? Equivalently, given all possible strings of a's and b's, with more b's than a's, what fraction of them are such that every prefix has more b's than a's? Stein guides the reader through an empirical investigation leading to the conjecture that when Ann has $N(a)$ votes and Barbara has $N(b)$ votes, the likelihood that Barbara is continually ahead is

$$\frac{N(b) - N(a)}{N(b) + N(a)}.$$

Various checks are made. It should be less than the likelihood that the first vote counted is for Barbara. It should be 1 when Ann gets no votes. It should be small when Ann has close to the number of votes for Barbara. It should work when we test it against other specific cases where the number of ballots is small. If we try a simulation with poker chips of two colours, then we should come close to this value. Such an elegant formula must have an elegant proof, so we cast around for a productive way to look at the situation.

Eventually we arrive at a reformulation where we associate each counting of the vote to a zigzag graph starting at the origin of the cartesian plane, where a graph remains in the positive quadrant if and only if Barbara stays ahead throughout the count. This provides a geometrical representation of the problem and opens the door for a reinterpretation and a solution invoking the reflection principle. Stein closes the chapter with a brief account of how the principle can be applied to a minimum path problem to make the point that it is not an isolated trick but a well-used tool.

The remaining chapters highlight other characteristics of mathematical research, including the use of a theoretical model and simulation (runs of consecutive wins in a match between two competitors), existence proofs that are or are not constructive (Sperner's lemma), reduction to a canonical situation (a generalization of the Buffon needle problem), and visualization. The most conventional chapter treats cardinality and Cantor's diagonal problem; this seemed somewhat out of place both in depth of treatment and subject.

However, the final chapter, while not long, packs a particularly effective punch. The question under discussion is:

> *How long can a string in the letters a, b, c be if one cannot find two adjacent shorter strings of any length right next to each other that are completely identical?*

The string *cbabcabcac* with two adjacent copies of *abc* does not satisfy the condition, nor does *baabc* with two adjacent a's. But *acbcabcbacb* works. As Stein tells us,

> This question and related ones were raised and answered by the Norwegian mathematician Axel Thue (1863–1922) in 1912. His motivation was simply the desire to know the answer. As he explained at the beginning of his paper, "For the development of the logical sciences it is important to find large areas for speculation about difficult problems without any consideration of possible applications."

Then follows a brief account of the works of others, some ignorant of what their predecessors achieved and some who, in fact, investigated the question in order to solve other problems.

If we allow only two letters, then we cannot have a string with more than three entries. So it is astonishing to learn that indefinitely long strings can be created with three letters. The strategy is to begin with a suitable string and replace each a, b, c by strings with several letters so that there continue to be no adjacent pairs of identical substrings. The solution presented is due to P. A. B. Pleasants, who in 1970 found his replacement strings by "just elementary messing about." An appendix comments on Thue's approach and on a computer attempt to construct an arbitrarily long string one element at a time.

Is this a book for the general reader? The answer is a qualified but enthusiastic yes. Judging from the responses of my students, the reader has to be prepared to read slowly and methodically, to sometimes backtrack to pick up a missed point, and to try things out. This is a type of reading that mathematicians are familiar with, but lay persons need to learn to do. The mathematician will enjoy this book, both for the new material that is likely to be found and for some topics that might be presented to students. Perhaps for a class one might best handle this book by assigning readings to the students, to be followed by presentations and class discussions. In a similar way, other adults may wish to use this as a book club selection. Many individuals would find it hard to negotiate the book alone, but with the chance for discussion, they might be encouraged to study it and be enriched thereby.

The second book is of quite a different character. In 1981 Sir Christopher Zeeman induced the Royal Institute of Great Britain to establish mathematics master classes for young teens. Such classes, designed to inform and excite the youngsters about the breadth and creativity of mathematics, have sprung up all over the United Kingdom. *Mathematics Galore!* provides a description of eight of these in the region of Bristol and Bath, which has had these workshops for local schoolchildren since 1990. It can be used as a handbook for others organizing such events.

A master class is a tightly organized event that occupies about 2 1/2 hours of a Saturday morning. Two half-hour talks alternate with two workshops, with a break for refreshments; the goal is to actively involve students with material accessible to the weakest but sufficiently challenging to the more able. The pupils work in small groups with parents and "experts" on hand. Each chapter opens with a description of the mathematical setting, includes the exercises given at the workshop, suggests field trips and extensions, and lists some resources; answers are provided. Insofar as some of the topics are closely aligned to the British situation, readers abroad will need to make some adaptations.

The authors get off to a flying start with a chapter that touches many bases. The classical tale of Theseus and the Minotaur provides the occasion to discuss the construction of the Cretan labyrinth, and we learn that a labyrinth is a tortuous path that leads directly from entrance to centre with no alternatives that might be taken. On the other hand, in a puzzle maze there are different routes that could be taken and one could easily get lost. We come to our first theorem: *If the walls are connected, then we can reach the centre or escape by keeping our hand on the wall at our right.* Mazes that cannot be solved in this way can be recast as a network of nodes and edges, where each node represents a place where alternative routes might be taken. This allows for a more convenient analysis and provides the occasion for a little history (the Königsberg bridges) and a reference to current applications of graph theory.

The chapter entitled "Dancing with Mathematics" uses folk dancing, change ringing of bells, and knitting to illustrate the ideas of group theory. Fortifications provide an incentive to discuss the isoperimetric problem, along with ways of offering attackers the maximum amount of frustration. A description of the work of the cryptographers at Bletchley Park during the Second World War leads to a discussion of ciphers, in particular those that apply the ideas of modular arithmetic.

The magical mathematics discussed in the fourth chapter probably appealed to the young participants. Four easy tricks are presented and analyzed, with only some elementary algebra and arithmetic. Arithmetic intervenes in two other chapters as well. The students learn about different systems of numeration and how to operate in base -2. The final chapter reminds pupils that there was life before calculators and computers when multiplication of large numbers was a significant task. They learn about Napier's logarithms (as well as about inverse functions), are given a little bit of practice with a logarithm table, and learn about that marvellous analogue tool, the slide rule.

The most challenging chapter of the book concerns sundials, the construction and use of which are greatly complicated by the tilt of the earth on its axis and the eccentricity of its orbit around the Sun. A lot of detail is provided to those who would construct their own sundials, although the mathematical underpinnings are left to the cited literature. The authors use just enough trigonometry to provide an adequate description.

Since the audience for each presentation is "50–100 dynamic teenagers," one has to be impressed by the level of mathematics. There is much for even the brightest of them to follow up on. Undoubtedly, the young people embraced it with differing levels of understanding, but it is highly likely that most were captured by the activity and impressed that mathematics had something to say about a wide range of human activity.

Like the Stein book, *Mathematics Galore!* works better for readers who can meet together rather than for lone readers. It has practical value for those of us trying to organize activities for the young. In particular, one might hope that it will be read by educators who will be inspired to replace some of the more banal and sterile activities that are inflicted upon modern school children by activities with real mathematical substance that are related to everyday interests of people.

Both books deserve a wide circulation. While difficult for many readers, they provide an authentic perspective of our discipline. Those prepared to see in the refined rigour of mathematical arguments "principles common to all men" can rejoice in the potential of mathematics to reach across differences of time and culture to reveal a shared core of human experience.

A complete listing of books written for students and the general public would go on for hundreds of pages. The following items are intended to suggest the scope of the literature and to indicate some possible starting places for the general reader.

History/Biography

[1] Donald J. Albers and G. L. Alexanderson (eds.), *Mathematical People, Profiles and Interviews*, Birkhäuser, Boston, 1985.

[2] Paul R. Halmos, *I Want to Be a Mathematician: An Automathography*, Springer, New York, 1985.

[3] Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford Univ. Press, New York, 1972.

[4] Richard Mankiewicz, *The Story of Mathematics*, Princeton Univ. Press, 2000. [Reviewed by Karen Hunger Parshall, *Notices Amer. Math. Soc.* **49**:4 (April 2002), 458–461.]

[5] Stanislaw M. Ulam, *Adventures of a Mathematician*, Scribner's, New York, 1976.

[6] Norbert Wiener, *I Am a Mathematician: The Later Life of a Prodigy*, Doubleday, Garden City, NY, 1956.

Recreational Mathematics/Puzzles/Problem Solving

[7] W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, twelfth ed., University of Toronto, 1974.

[8] Lewis Carroll, *Pillow Problems and a Tangled Tale*, Dover, New York (reprint of fourth ed. of 1895).

[9] Martin Gardner, *The Colossal Book of Mathematics*, W. W. Norton, New York, 2001.

[10] ____ , *A Gardner's Workout: Training the Mind and Entertaining the Spirit*, A K Peters, Natick, MA, 2001.

[11] L. A. Graham, *The Surprise Attack in Mathematical Problems*, Dover, New York, 1968.

[12] Maurice Kraitchik, *Mathematical Recreations*, Dover, New York, 1942.

[13] T. H. O'Beirne, *Puzzles and Paradoxes*, Oxford University Press, London, 1965.

Popular Exposition

[14] Richard Courant and Herbert Robbins, *What Is Mathematics?* second ed. (revised by Ian Stewart), Oxford Univ. Press, New York, 1996 (1941). [Reviewed by Brian Blank, *Notices Amer. Math. Soc.* **48**:10 (November 2001), 1182–1188.]

[15] Philip J. Davis and Reuben Hersh, *The Mathematical Experience*, Birkhäuser, Boston, 1981.

[16] ____ , *Descartes' Dream: The World according to Mathematics*, Harcourt, Brace, Jovanovich, San Diego, CA, 1986.

[17] Keith Devlin, *Life by the Numbers*, Wiley, New York, 1998.

[18] A. K. Dewdney, *A Mathematical Mystery Tour: Discovering the Truth and Beauty of the Cosmos*, Wiley, New York, 1999. [Reviewed by Lynn Arthur Steen, *Notices Amer. Math. Soc.* **47**:2 (February 2002), 221–224.]

[19] George Gamow, *One, Two, Three, …, Infinity: Facts and Speculations of Science*, Dover, New York, 1988 (1947).

[20] Anthony D. Gardiner, *Discovering Mathematics: The Art of Investigation*, Oxford Univ. Press, New York, 1997.

[21] Stephen Hawking, *A Brief History of Time: A Reader's Companion*, Bantam, New York, 1992.

[22] David Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie*, translated as *Geometry and the Imagination*, Chelsea, New York, 1952.

[23] Douglas R. Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid*, Basic Books, New York, 1979.

[24] Leopold Infeld, *Whom the Gods Love*, National Council of Teachers of Mathematics, 1978 (1948).

[25] Morris Kline (ed.), *Mathematics in the Modern World: Readings from* Scientific American, W. H. Freeman, San Francisco, CA, 1968.

[26] ____ (ed.), *Mathematics: An Introduction to Its Spirit and Use*, W. H. Freeman, San Francisco, CA, 1979.

[27] Joseph S. Madachy, *Mathematics on Vacation*, Scribner's, New York, 1966.

[28] James R. Newman (ed.), *The World of Mathematics*, Simon & Schuster, New York, 1956.

[29] Roger Penrose, *The Emperor's New Mind*, Vintage, London, 1990.

[30] Ivars Peterson, *The Mathematical Tourist*, W. H. Freeman, New York, 1988, 1998.

[31] ____ , *Fragments of Infinity, a Kaleidoscope of Mathematics and Art*, Wiley, New York, 2001.

[32] George Pólya, *How to Solve It*, Doubleday, Garden City, NY, 1957 (1945).
[33] Hans Rademacher and Otto Toeplitz, *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur*, Princeton Univ. Press, 1957.
[34] W. W. Sawyer, *Prelude to Mathematics*, Dover, New York, 1982 (1955).
[35] Simon Singh, *Fermat's Enigma*, Walker, New York, 1997. [Reviewed by Allyn Jackson, *Notices Amer. Math. Soc.* **44**:10 (November 1997), 1304–1306.]
[36] Sherman K. Stein, *Algebra and Tiling: Homomorphisms in the Service of Geometry*, Mathematical Association of America, Washington, DC, 1994.
[37] ____ , *Archimedes; What Did He Do besides Cry Eureka?* Mathematical Association of America, Washington, DC, 1999.
[38] ____ , *Mathematics: The Man-Made Universe*, W. H. Freeman, San Francisco, CA, 1963.
[39] ____ , *Strength in Numbers: Discovering the Joy and Power of Mathematics in Everyday Life*, Wiley, New York, 1996. [Reviewed by Giuliana Davidoff, *Notices Amer. Math. Soc.* **46**:5 (May 1991), 552–554.]
[40] Hugo Steinhaus, *Mathematical Snapshots* (3rd U.S. ed.), Oxford University Press, New York, 1969 (1950).
[41] Ian Stewart, *The Magical Maze: Seeing the World through Mathematical Eyes*, Wiley, New York, 1997.
[42] Ravi Vakil, *A Mathematical Mosaic, Patterns and Problem Solving*, Brendan Kelley Publishing Co., 1996.
[43] David Wells, *You Are a Mathematician*, Wiley, 1995.

Cultural/Psychological Aspects

[44] Jacques Hadamard, *The Psychology of Invention in the Mathematical Field*, Princeton University Press, 1945.
[45] Morris Kline, *Mathematics in Western Culture*, Oxford University Press, New York, 1953.
[46] ____ , *Why Johnny Can't Add: The Failure of the New Math*, St. Martin's Press, New York, 1973.
[47] George Lakoff and Rafael E. Núñez, *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*, Basic Books, New York, 2000. [Reviewed by James J. Madden, *Notices Amer. Math. Soc.* **48**:10 (November 2001), 1182–1188.]
[48] John Allen Paulos, *Innumeracy: Mathematical Illiteracy and Its Consequences*, Vintage, Random House, New York, 1990.
[49] Henri Poincaré, L'intuition et la logique en mathématiques, translated by G. B. Halsted in *The Foundations of Science*, The Science Press, New York, 1913. Also see the translation Mathematical Creation in *Mathematics in the Modern World*, pp. 14–17, and in *Scientific American*, August 1948.