The Hilbert Challenge *and* The Honors Class

Reviewed by Brian E. Blank

The Hilbert Challenge
Jeremy Gray
Oxford University Press, 2000
ISBN 0-198-50651-1, 240 pages, $34.95

The Honors Class
Benjamin Yandell
A K Peters, 2002
ISBN 1-56881-141-1, 500 pages, $39.00

The year 1897 was a momentous one for the development of mathematics. A lasting tradition was initiated that summer when Zürich hosted the first International Congress of Mathematicians. It followed by a few months another event that would have great significance for mathematics: the publication of David Hilbert's *Zahlbericht*. Charged with the task of summarizing the rapid progress that had been made in algebraic number theory, Hilbert delivered a masterpiece that simplified, clarified, and unified existing theories. Above all, it pointed the way to a vast program of further research that would be carried out by the next generation of number theorists. As one of his followers remarked, "Hilbert was not only very thorough but also very fertile for other mathematicians." The *Zahlbericht* alone sufficed to confirm Hilbert's position among the leaders of mathematics. Three years later, when the new century dawned, Hilbert was extended an invitation to give a plenary address at the second International Congress of Mathematicians that was to be held in Paris.

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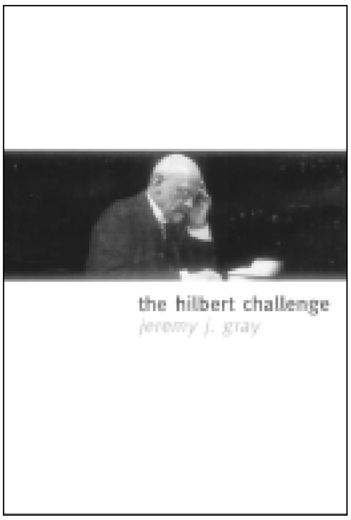
Although Hilbert had not attended the Zürich Congress, he was familiar with the published version of Henri Poincaré's talk on the relationships between analysis and mathematical physics. Poincaré had used his address as a vehicle for reflecting on the role that mathematics would have in the scientific investigation of nature. The result was a philosophical discourse of meager impact. Hermann Minkowski, who had been present for Poincaré's lecture, dismissed it as a "mere chat." Hilbert had a more favorable opinion despite a mathematical philosophy that diverged sharply from that of Poincaré. Indeed, he considered using his own talk as a counterpoint. His other idea was to examine the direction that the study of certain important problems would impart to mathematics in the coming century.

For a dangerously long time Hilbert vacillated. At the end of March 1900 Hilbert confessed to Adolf Hurwitz, "I must start preparing for a major talk at Paris, and I am hesitating about a subject." The month of June arrived, and Hilbert still had not produced his lecture. As a result, the program for the second Congress was mailed without listing Hilbert's talk, the organizers having scheduled another talk for the opening session in its stead. Just two weeks before the start of the Congress, the proof sheets for *Mathematical Problems*, the title of the talk Hilbert eventually settled on, materialized. A slot was found for Hilbert's talk on August 8, 1900, and an important new chapter for twentieth-century mathematics was assured.

"Who of us would not be glad to lift the veil behind which the future lies hidden: to cast a glance at the next advances of our science and at the secrets of its development during future centuries?" So began Hilbert's lecture. It seems safe to say that no mathematical talk has ever had so great an effect. Although Hilbert had only enough time to consider a selection of ten problems during his oral presentation, the full text of his manuscript issued twenty-three challenges for the gifted masters of the new century to solve. It was published in the *Göttingen Nachrichten* and summarized in French shortly thereafter. An English translation of the entire paper appeared in the 1902 *Bulletin of the American Mathematical Society*.



Hilbert prefaced his list of problems with some general thoughts on the nature of mathematical inquiry. Whereas Poincaré had offered lofty platitudes about the purpose of mathematics, Hilbert cut directly to the chase. He echoed Johann Bernoulli in stating that mathematical science is frequently advanced by nothing more than the challenge of a difficult problem. Time after time the medium *is* the message in mathematics: how often do we judge results not by what they say but by the ingenuity and depth of thought that they require? As Hilbert phrased it, "A mathematical problem should be difficult in order to entice us, yet not completely inaccessible lest it mock our efforts." Several other passages from Hilbert's foreword have been quoted again and again. Surely there has never been a more inspirational call-to-arms than Hilbert's "We hear within us the perpetual call: There is the problem. Seek its solution! You can find it by pure reason, for in mathematics there is no *ignorabimus*." We know a bit more about that now than Hilbert did then, but, even so, who of us on reading his words does not hear the siren call of a problem that has been mocking our efforts?

The future is not always easy to divine, even for a Hilbert. Consider, for example, his third problem, which asks for two tetrahedra of equal volume, neither of which can be cut up into finitely many tetrahedra that can be reassembled to take the form of the other. This is a neat task in solid geometry, but Hilbert probably would not have placed it alongside the Riemann Hypothesis as a challenge for the twentieth century had he an inkling that his own student, Max Dehn, would solve it even before the list of problems appeared in print. Many years later Carl Ludwig Siegel was present at a lecture during which Hilbert told his audience that he was hopeful that he would live to see the Riemann Hypothesis proved (Problem 8), that the youngest members of that 1919 audience might live to see Fermat's Last Theorem proved, but that he did not expect anybody in the hall to see the transcendence of $2^{\sqrt{2}}$ (Problem 7). Hilbert's optimism about the eighth problem did not pan out, and his pessimism about the seventh was equally misplaced. Within a decade of these prognostications Alexander Osipovich Gelfond established the required transcendence and "passed on to the honors class of the mathematical community," the accolade conferred by Hermann Weyl on solvers of the Hilbert problems.

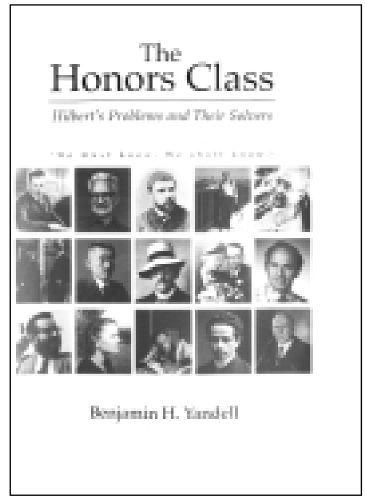
No less than anyone else we mathematicians are creatures of base 10. Now, as in Hilbert's time, there is nothing like the turn of a century to prompt us to take stock of past progress and speculate about what lies ahead. When the new century coincides with the centennial of so influential a wellspring as Hilbert's problems, it is a sure thing that at least a few historians of mathematics will have been busy in anticipation. Indeed, a brief synopsis of the Hilbert problems appeared in the *Notices* [2] during the centenary, but, as its author demurred, nothing less than a "formidable but worthwhile monograph" could do the subject justice. Actually, a retrospective that had been undertaken twenty-five years earlier required *two* such volumes [1]. As it happens, two books that survey the research inspired by the Hilbert problems, Jeremy Gray's *The Hilbert Challenge* and Benjamin Yandell's *The Honors Class*, have recently appeared. Neither attempts to be formidable, but they are both worthwhile in their different ways.

The first thing that must be acknowledged is the boldness that is needed for such a project. When Weyl wrote that Hilbert's Paris address "straddles all fields of our science," he was availing himself of the sort of discreet exaggeration that is overlooked in a eulogy. Famously missing in Hilbert's list—especially to those who ponder the selection Poincaré might have made—is the important role that topology was poised to assume. That there were omissions is not unexpected: by the end of the nineteenth century the mathematical enterprise was already so diverse that there would have been lacunae in any short list of problems. Nevertheless, Hilbert's problems *do* straddle enough logic, algebra, number theory, geometry, and analysis that one does not envy Gray or Yandell for his chosen task. The intensity of effort inspired by Hilbert's lecture is truly astonishing. As Kronecker liked to say about mathematical activity, "When kings build, carters work." The honors class may be extremely exclusive, but over the course of one hundred years the cast of important contributors has become substantial. The bibliographies of the two volumes under review speak to this: 256 items in Gray's book, 320 in Yandell's. The size of the intersection is even more suggestive: there are only about thirty references common to both!

The symmetric difference of bibliographic materials notwithstanding, the two books have much in common. Each includes a brief biography of Hilbert, drawn primarily from Constance Reid's superb full-length treatment. Each author sketches Hilbert's mathematical development prior to the Paris Congress, explaining, for example, invariant theory and quadratic reciprocity from scratch. Though the levels of exposition here, as well as for the twenty-three Hilbert problems, are elementary, they suffice to provide the novice reader with a reasonable understanding of what is at stake. Both books conclude with a translation of Hilbert's lecture, a scorecard describing the status of the problems at the time of writing, and an inadequate index.

There are also many differences between the two treatments. Yandell considers the twenty-three Hilbert problems in Hilbert's own order, subject to a little rearrangement according to mathematical branch. Such a sequence makes for some temporal choppiness, but given Yandell's perspective, which is directed more toward the members of the honors class than to the problems that they solved, the shuffling of time has little consequence. By contrast, Gray discusses the problems in chronological order of progress: it suits his historian's point of view, allowing him to chart the development of mathematics as the twentieth century unfolds. He relates his story with a writing style that is contemplative and sober but not starchy. Yandell is informal, with prose that is often amusingly breezy. Where Gray opts for "tortuous paths" in his translation of Hilbert, Yandell prefers "mazy paths." To Yandell, numbers and shapes are "oddball," L-functions "proliferate like rabbits," Jean van Heijenoort, historian of logic, is "the epitome of cool," and "Gödel liked Ike."

Not everything about Yandell's relaxed style, however, will draw the reader in. For one thing, his writing can be confused. Every once in a while the reader will be interrupted by a sentence that might fairly be described as a non sequitur. In the story of Problem 10, the question of algorithmic solvability of Diophantine equations, Yandell devotes a paragraph to a priority claim made by Gregory Chudnovsky. The reader is left in wonder when Yandell concludes the paragraph with a sentence that fast-forwards twenty years to tell us about the Chudnovsky brothers calculating mountains of pi in a hot New York apartment. In other instances just one errant word is enough to throw the reader. A paragraph devoted to the geographic origins of Andrew Gleason begins the trail in Fresno and then mentions in short order the San Joaquin Valley, British Guiana, Switzerland, Los Angeles, and Malaga. The paragraph concludes with a puzzling phrase that tells us that the Gleason family "moved *back* to New York, to Bronxville." (As in the upcoming quotes, the emphasis here has been added by the reviewer.) Consider as well Ernst Hellinger, who "reached mandatory retirement age *quickly*," a trick anyone of us might envy after an especially trying class. To be fair, these are minor blemishes. Nevertheless, it is troubling that so many recent mathematics books suffer from inattentive editing. If the "eagle-eyed editor" has become extinct, then the loss of a cliché is poor compensation for passages such as the following one (in which the "she" refers to Adele Gödel): "At the end of their lives, when she fell ill and could no longer take care of him, Gödel died."



Between them the two books display the dangers of concision and prolixity. None too rarely Gray's succinct style will suggest a question that is left unanswered. The reader, for instance, may wonder what Gray has in mind when he asserts, "Koebe published prolifically, pursuing a *ruthless* strategy to gain the attention of the leading mathematicians in Germany." A case can be made for Gray's choice of adjective, but you will not find it in his book. For his part, Yandell usually tells us all we want to know and then some, packing his book with entertaining anecdotes as well as bizarre detail of less certain interest. Name your topic and chances are you will find something. Food? Picture Emil Artin newly arrived in Bloomington consulting the *Encyclopædia Britannica* for instructions on carving a Thanksgiving Day turkey! Or Carl Ludwig Siegel, who, having invited Harold Davenport to dinner, served a large trout as the first course and a large trout as the second course! Religion? Deane Montgomery did not enroll in the University of Minnesota, because his mother did not think it safe for Methodists. Circumcision? Quilted toilet paper? Those too are there. Yandell's exertions in tracking down minutiae at the periphery of his story are praiseworthy, but his indiscriminate presentation suggests a brain dump that was not curbed by a disciplined editor.

Accuracy is usually a thorny issue in popular books pertaining to mathematical history. In this regard Gray strikes no false notes, whether mathematical or historical. That is not the case with Yandell. We read, for example, "Weierstrass introduced the epsilons and deltas that are used to define the basic concepts of calculus." In reality, Weierstrass was of kindergarten age when Cauchy introduced the symbols ϵ ("erreur") and δ ("différence"), using them exactly as we do now. Ironically, Yandell forgoes epsilons and deltas in his misleading definition of the convergence of an infinite *sequence*: "An infinite series [sic] of numbers converges if the numbers, as one proceeds through the list, move *consistently* closer to a specific number that is said to be the limit. Any wandering that the numbers do from the limit value gets *progressively* smaller." Granted, the harm here is not too great, but the language worsens when Yandell takes another crack at the concept a few pages later: "When we say an infinite series converges to a number, we are saying that the completed infinite of the series equals the number." I suppose that if you cannot fathom the meaning of a sentence, then you cannot get all riled up. On the other hand, a statement such as "logic is about classes and 'class' is a close synonym of 'set'" can induce apoplexy. So too can Yandell's assessment of Antoni Zygmund, who Yandell says "could best be described as a classical analyst persevering amid modernism." The image is reinforced when Yandell refers to "the modern machinery that, except for Zygmund, was the style at Chicago" in the 1950s. Opinions may legitimately differ: a dissenting point of view is that during the decade of which Yandell speaks, Zygmund, together with Alberto Calderón, introduced a significant share of the modern machinery of harmonic analysis.

When it comes to the treatment of individual problems, each author has his successes and failures. Of course, the reader cannot expect exposition that is either deep or complete. At best these books can only hint at what has been done, supplying references that update [1]. On balance it is Gray who most often proves to be the more enlightening guide. Examination of a few particular examples—Problems 6, 8, 17, and 18—will illustrate.

Problem 6 asks for the axiomatization of physics, a program about which the two authors come to near opposite conclusions. On his scorecard Yandell puts Problem 6 down as unsolved, whereas Gray lists the axiomatization of classical mechanics by Hamel, thermodynamics by Carathéodory, special relativity by Robb and Carathéodory, and quantum field theory by Wightman. The situation is virtually reversed for Problem 8, the Riemann Hypothesis, about which Gray has almost nothing to say. That decision is puzzling in view of the many remarkable things that have been learned. Yandell also does not give Problem 8 its due, according it a section that cannot be ranked among his best. At least he is more forthcoming than Gray: Bohr, Landau, and Littlewood are mentioned only in passing, but Hardy's theorem *is* stated. Not content with that, Yandell uses the hint of primes as the segue into a two-page diversion on Ramanujan (which in turn offers the opportunity for a brief digression about novelist F. R. Keating's fictional Inspector Ghote).

In the 1890s Hilbert deduced that a positive polynomial of two variables cannot necessarily be expressed as a sum of squares of polynomials. This led him to pose Problem 17, which asks whether a positive polynomial in many variables can be expressed as a sum of squares of rational functions. In 1927 Emil Artin answered this question in the affirmative for fields in which -1 is not a sum of squares. Like the majority of Hilbert problems, No. 17 has not been a dead end that offers no path of research beyond its solution. In 1940 Habicht used an interesting theorem of Pólya to give a constructive proof of Artin's result. Habicht's paper is not mentioned by either Gray or Yandell, but the latter does refer to a second constructive proof, which was published by Kreisel in 1957. It was Kreisel's work that prompted Artin to confirm his preference for a clear existence proof over a construction with $2^{2^{100}}$ steps. Yandell ignores the flurry of activity subsequent to Kreisel, whereas Gray informs us of Motzkin's explicit counterexample in the polynomial ring, the negative answer to Problem 17 that Dubois established for arbitrary ground fields, and the bounds Pfister found for the number of summands that are needed.

Hilbert's eighteenth problem is one of his questions with multiple parts. One component of Problem 18, namely the sphere packing problem, has proved especially difficult and contentious. Gray skirts the controversy, electing not to explicitly mention either Wu-Yi Hsiang's disputed paper or the manuscript of Thomas Hales that was announced in [3] but which was already circulating in 1998 (and which, at the time of this writing, remains unpublished). Instead, he refers to Hsiang and Hales obliquely, stating, "Everyone expects that the cannonball arrangement will prove to be best possible in 3-dimensional space but even in 1999 this still has not been proved (although a final proof is thought to be close)." Yandell confronts the discord head-on, interviewing Hsiang (on Hales) and Conway (on Hsiang and Hales). Neither Gray nor Yandell says much about packings in higher dimensions, although Gray alerts his readers to the Leech lattice and thereby to unexpected connections with other areas of mathematics.

The evidence presented up to this point may seem to suggest that *The Hilbert Challenge* has it all over *The Honors Class*. I should make it clear that the final evaluation is not so clear-cut. Although I came to Yandell's book after I had finished Gray's, I did not have the sense of reading a twice-told tale. Yandell has new things to tell us, and there is no doubt how he manages to do so. A century may seem like a very long time, but *The Honors Class* demonstrates that the history of the Hilbert problems has largely occurred within living memory. Just as a generation ago Reid put us in her debt by endeavoring to interview, before it was too late, those who had known Hilbert, so has Yandell done with the honors class. Whether by telephone, correspondence, or email exchange, he has accumulated a wealth of fascinating information that otherwise would likely have gone unrecorded, eventually to pass out of existence. His sources include not only mathematicians but also their family members. He contacted, for example, all of Dehn's children and three of Artin's. The result is a sequence of original character sketches that will not fail to interest any mathematician who reads them. I found the biographies of Artin, Dehn, Kolmogorov, Siegel, and Takagi to be of particular value, but no member of the honors class has been slighted. The entire narrative is enhanced by the inclusion of photographs that, like the reminiscences Yandell sought out, would have remained private but for his efforts. (With regret I note that the publisher did not reproduce the photographs in decent size or on appropriate paper. One image in particular, a photograph taken by Natascha Artin in 1927, cried out to be turned sideways and given its own page: it features Artin, Herglotz, Rademacher, Schreier, Blaschke, and van der Waerden seated at a dinner table in the Hamburg City Hall!)

Every mathematics library should have a browsing shelf, and every browsing shelf should hold both of these books. Anybody who has read this review would enjoy either volume, but I suppose that few will need or want to read both. If you require a concise summary of the work generated by the Hilbert problems, a book that gives a clear overview of each problem with plenty of references to more technical treatments, then *The Hilbert Challenge* is the one to go for. If you enjoy biography or if you have a fondness for mathematical anecdote, then you cannot miss *The Honors Class*.

References

[1] F. E. Browder (ed.), *Mathematical Developments Arising from Hilbert Problems*, 2 vols., Proceedings of Symposia in Pure Mathematics, vol. 28, Amer. Math. Soc., Providence, RI, 1976.

[2] Ivor Grattan-Guinness, A Sideways Look at Hilbert's Twenty-three Problems of 1900, *Notices Amer. Math. Soc.* **47** (August 2000), 752–757.

[3] Thomas C. Hales, Cannonballs and Honeycombs, *Notices Amer. Math. Soc.* **47** (April 2000), 440–449.