Book Review

The Annotated Flatland: A Romance of Many Dimensions

Reviewed by A. K. Dewdney

The Annotated Flatland: A Romance of Many Dimensions
Edwin A. Abbott, with Introduction and Notes by Ian Stewart
Perseus Publishing, 2001
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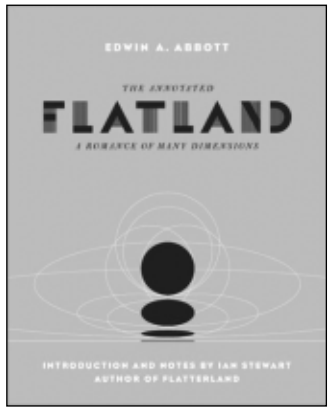
I must confess at the outset to never having been a great fan of *Flatland*. Yet this new edition, decorated with insightful and helpful notes by Ian Stewart, has added a new depth (or is it thickness?) to the novel. Stewart is a fan of *Flatland* and has recently published a fantasy sendup called *Flatterland*.

Imagine then a lesson from Euclid brought to life by a nineteenth-century Englishman, teacher, and ordained minister. The characters are line segments, triangles of various persuasions, and polygons, some of them multisided enough to masquerade as circles. How do they move about in their plane world? No one, not Abbott, not Stewart, nor I, knows. Yet move they do, with passions and pleasures no less real, for all their lack of anatomy, than our own.

Writing a work of science fiction (Stewart identifies the work as pre-science fiction) that is based on certain rigid premises, such as dwelling entirely in the plane and being a polygon of one kind or another, entails certain consequences which it becomes the delight of the author to explore. That the premises lead sometimes to an awkward physics



or society becomes a challenge to be surmounted, sometimes by a clever twist of plot, sometimes by the creaking of heavy backstage machinery that barely saves the situation. Most works of science fiction get over their premises in the first few pages, thereafter launching into the plot. But for the average Victorian reader, Flatland had to be explained very carefully, and Abbott was therefore led to write a natural history, rather than a science (or pre-science) fiction novel. Natural history was a popular topic in nineteenth-century England, including accounts of travels in exotic lands, many of them imperial territories and possessions.

But Abbott's was a natural history with a difference. Here was a land that everyone had visited in school, being subjected to plane figures, theorems, and constructions from Euclid. But it was also, under Abbott's hand, a land of passion and violence, not to mention extreme political incorrectness. Shortly after *Flatland* appeared, according to one of Stewart's notes, some readers objected to Abbott's characterization of Flatland females. Males could be many-sided, social rank increasing with the number of sides, but females were digons, their

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sides coincident, mere segments. According to Stewart, Abbott was merely satirizing Victorian views of women, not expressing his own. Although the class structure reminds me more strongly of Imperial India, satire or lack of it may be a red herring. Abbott had made his choices and pressed on, tongue in cheek, to see what would come out of it all. Thus women, who were lower in caste than the narrowest "irregular" triangle and (apparently) completely without brains, nevertheless possessed a deadly penetrating power in their infinitesimal heads or tails, being able to pierce the side of any triangle or other polygon with instantly fatal effect. To warn others of their dangerous presence, women must utter "peace cries" and constantly wiggle their behinds. Hmmm.

The central metaphor of *Flatland* is of course the ability to appreciate higher, as well as lower, dimensions. As a teacher, Abbott had discovered a marvelous door to this world. He first explains how things in Flatland appear to Flatlanders and then demonstrates their utter helplessness in imagining something higher. The real target is you, the reader. Can you put yourself in a Flatlander's shoes? (Excuse me, they have no feet.) If so, you will discover that you have no more knowledge of the fourth dimension than a Flatlander does of the third. Yet you may embrace the analogy far enough to know in which direction the Flatlander must point, even if his arm (if he had one) must leave the plane entirely. Aha! If you were to point in the direction of a fourth dimension, would your arm not also disappear?

It is particularly helpful to be accompanied on this voyage by a presence on the sidelines, someone who can explain the twists and turns of dimensionality as cleverly and completely as Stewart. And besides four-dimensional spacetime and space-filling curves, there is the cellular automaton (which demonstrates a substrate for thought, thanks to von Neumann), Hilbert space (which has infinitely many dimensions), geometrical construction, and much else. Indeed, for those with a thirst for more dimensions and more about them, there is an excellent essay at the end of the book entitled "The Fourth Dimension in Mathematics".

Our three-dimensional companion also provides ample historical material from his two-dimensional space beside the text. There are crucial notes on Abbott's personal and professional life, many notes about historical figures, from British Prime Ministers to noted scientists of the day, whimsical observations on calendars, verse in scientific papers, sex in Flatland, and enough other entertaining material to divert the reader from drier stretches of the work.

The hero of the story is one A. Square who happens to be a square and therefore of higher caste, somewhere between a merchant and a physician. Thanks to a visit from a mysterious spherical being from three dimensions, Mr. Square has the mathematical equivalent of a religious experience and ultimately gets locked up for proclaiming his gospel. He witnesses a sphere manifesting in Flatland as a circle that gradually widens until it reaches its fullest extent, thereafter diminishing. A perfect being able to change its size at will, what else could it be but semidivine? As Stewart explains, however, Abbott himself took no spiritual message from his own book. As a minister he was at pains to point out (elsewhere) that seemingly miraculous powers, by themselves, do not warrant worship.

Abbott lays his chapters out with methodical precision. In the first he sketches the two-dimensional nature of Flatland, and in most of this he goes to great lengths to explain how Flatlanders see only lines. In Chapter 2 he discusses the climate and houses of Flatland. The houses are pentagonal because square (or worse, triangular) houses would have sharp corners that might do injury to passers-by. Instead of gravity created by large, nearby bodies, there is a "constant attraction to the South" (felt keenly by many English people even today). Thus rain must fall from the North. By this point in my original reading of the book, I can recall feeling distinctly disoriented, not to say puzzled, by such a peculiar way of arranging things. Yet, with no central planet, what else can Abbott do if he wants to have rain? There seems no reason why rain could not fall into Flatland from the third dimension, for this is the origin of all light in Flatland!

Chapter 3 sketches the inhabitants of Flatland as line segments (females), isosceles triangles (low-caste workers and soldiers), equilateral triangles ("middle class"), squares and pentagons ("professional class"), hexagons and higher ("nobility"). At the highest level of society are the near-circles ("priestly order"). A curious progression is made by the sons of regular polygons, who enjoy one more side than their fathers had. But the poor isosceles triangles are doomed to remain isosceles unless, by dint of perseverance and high achievement, an isosceles father will have an equilateral son (shades of Lamarck). "The birth of a True Equilateral triangle from Isosceles parents [sic] is the subject for rejoicing in our country for many furlongs round." There is social unrest among the lower classes, of course, but revolutions are nipped in the bud by a curious law of compensation: "Thus, in the most brutal and formidable of the soldier class—creatures almost on a level with women in their lack of intelligence—it is found that, as they wax in the mental ability necessary to employ their tremendous penetrating power to advantage, so do they wane in the power of penetration itself."

Chapter 4 is about women. To what I have already said, I might only add that, although brainless, they

suffer from being confined too much and are wont to rebel. "Hence, in their fits of fury, they remember no claims and recognize no distinctions. I have actually known a case where a woman has exterminated her whole household and half an hour afterwards, when her rage was over and the fragments swept away, has asked what became of her husband and children." Abbott has the bit in his teeth by now, perhaps relishing the public reaction to this shocking state of affairs.

In Chapter 5, Abbott must get around the inability of Flatlanders to recognize each other, especially to discern social class in encounters. They "feel" each other (without hands), and Abbott goes on at great length about the social conventions that arise from this practice: "Let me ask you to feel Mr. So-and-so." And on it goes. Flatlanders have an exquisite sense of touch that permits them to distinguish between the rather small differences in the angles of a pentagon and a hexagon. Not content with his earlier treatment of vision, Abbott returns to the subject by invoking yet another Deus ex Machina, a pervasive fog that causes farther surfaces to appear fainter.

The next three chapters concern history and an earlier use of colour to identify individuals. But then—you guessed it—things got out of hand, with lower classes painting themselves as upper, resulting in the "Chromatic Sedition," a move by the lower classes to erase all social distinctions. The revolt was put down with great violence and loss of life, something rather common in these pages.

But loss of life is an aspect of mortality and I cannot continue this review without a digression into the population biology of Flatland, a subject neglected by Abbott. There is a danger to the stability of Flatland society in the tendency for n-sided polygons (n-gons) to have $(n+1)$-sided sons.

Neglecting the scalene rabble and the isosceles workers for the moment, let there be m regular polygons with three or more sides, the average being x. In the next generation there will be $2m$ such polygons, with an average of $(2x+1)/2$ sides, unless, of course, not all polygons reproduce in every generation. The tendency to exponential growth in this class, along with a gradual but inexorable increase in average sidedness, will nevertheless be inevitable. The isosceles and scalene classes will grow apace, but the number of sides will remain the same at three. Losses due to violence and graduation to the polygon class will reduce this number slightly, but Abbott gives no discussion of mortality in general. Perhaps he has no need to, since the plane of Flatland is infinite and populations have no natural limits.

The danger to Flatland society is now clear. If the increase in x goes unchecked, the sidedness gap $x-3$ must grow with every generation, and stability of the society is in doubt. Is there yet another "Natural Law" that Abbott forgot to mention? In order to prevent the number x from increasing forever, the relative distribution of individuals within sidedness classes must remain the same over time. For this to happen, the distribution should be monotone decreasing in the direction of higher sidedness, there always being the same ratio of priests to those in lower classes for example. The number $f(n)$ of n-gons may increase with every generation, but the ratio $f(n+1)/f(n)$ must remain approximately constant at c, with $c<1$.

Now the number of n-gons will increase to $r(n)f(n)$ at the next generation, where r is the fecundity of that class. It follows that the fecundity r, as a function of n, must decline with increasing n, because the ratio of two populations at the next generation will be the same:

$$\frac{r(n+1)f(n+1)}{r(n)f(n)} = c$$

so that

$$\frac{r(n+1)}{r(n)} < c\frac{f(n)}{f(n+1)} = c(1/c) = 1,$$

The Greatest Circle (for so I call the highest priest) reproduces once in a blue moon. Indeed, Abbott implicitly recognizes this situation, stating that "as the race climbs higher in the scale of development, ... the race shall become less fertile." For some reason, he then spoils everything by declaring that development accelerates, with the number of sides increasing by leaps and bounds with every generation.

I had not intended to analyze the situation at such length, but such is the seductive charm of the Flatland setting that readers may well think of puzzles of their own. Should any prove insoluble, the validity of Abbott's Flatland will fall under the shadow of a grave inconsistency.

By Chapter 11, Abbott arrives at the priestly class and admits that the preceding natural history is but a preface to A. Square's "initiation into the mysteries of space". Before launching into the book's ultimate plot, he takes a last swipe at the woefully incomplete explanations of how Flatlanders get along or accomplish anything without hands or feet. He, through the person of his hero, A. Square, must neglect things like "the nature of our hills and mines, our trees and vegetables ..." in favour of the main event. This is just as well, since I cannot imagine what he could say about them. Unfortunately Abbott is not quite done with his natural history and societal exposition.

The priestly class is hardly religious, but administrative. In this and the subsequent chapter, we discover that the priestly class has succeeded in convincing the rest of Flatland society that configuration is everything, that individual effort leads

to no improvement in one's status. The doctrine has drawbacks, of course, and when Abbott's hero states that some scalene fellows plead in court that their crimes were the result of their poor configuration, Stewart leaps in with a discussion of the "nature versus nurture" question in our own three-dimensional society, comparing the scalene's plea to that of the drunk driver: "I couldn't help it, Your Honour, I was drunk."

Finally, in Chapter 13, Abbott settles to the main event, beginning with a visit to Lineland in a dream of A. Square's. Lineland is inhabited by line segments. Has he found the Planet of the Women? Not so. Yet every man has two wives in a world that permits no proximity whatever between male and female. Courtship (and, presumably, fertilization) is carried out entirely by sound, all inhabitants of Lineland being given to song. In this way Abbott gets around the difficulty that proximate courtship would create. Things get sillier than this in Lineland, but Abbott hastens on to the next chapter, in which his hero must explain to the King of Lineland the nature of Flatland. A. Square pleads with the King to imagine an extra dimension, all to no avail.

I was wrong. Finally, in Chapter 15, Abbott settles to the main event. It is the turn of the millennium (1999 in the Flatland era). In the midst of a geometry lesson for his hexagonal grandson, A. Square demonstrates how an array of squares three on a side has nine (the square of three) squares in it. But then the bright little fellow recalls that Grandpa has been teaching him to cube numbers, as well. "I suppose three cubed must mean something in Geometry. What does it mean?" The child tries to construct a higher dimension and Grandpa must send him off to bed with, "If you would talk less nonsense, you would remember more sense." Presently A. Square and his wife have a visitor from the third dimension, a sphere who asks to be left alone with the hero.

The sphere appears as a circle and therefore exercises great authority over our hero, who is nevertheless hard pressed to imagine what the circle means by being able to see into Square's house from a third dimension. The sphere describes the sleeping servants and other household contents of Square's house without having visited any of the rooms. Failing to convince Square that he comes from a higher dimension, the alien visitor proceeds in Chapter 17 to a demonstration of the reality of the third dimension. He filches an accounting book from a closed cupboard without opening the door. Square is even more horrified when the sphere pokes him from the inside.

The hero throws himself against the figure, judging him to be an evil magician. In the next chapter the sphere reacts by plucking Square right out of Flatland. Giddily, Square witnesses the spherical surface, as well as the contents of his own house. Technically, Square should be capable only of views that are one-dimensional cross-sections of these views, but by now the reader is swept up in the drama. If there is any place in the book where technical objections fall by the wayside, it is here. The two float through the third dimension until they can see the General Assembly Hall of the States of Flatland. As it happens, the administrators are in the midst of passing a law that dictates death or imprisonment for anyone preaching the "gospel of the third dimension." Apparently the Sphere has been visiting other households. However, even a visit by the sphere to the council chamber itself changes nothing. The presiding circle instructs those present to take no notice. This happens every millennium, it turns out.

There are further adventures in Spaceland, including a visit to various three-dimensional solids and an extended conversation with the Sphere in which Square asks whether spaces higher than three might exist. The Sphere pooh-poohs this idea, thereby granting Square a kind of moral and intellectual superiority by default. In the end, only Square knows the truth, but he is imprisoned for it. He writes from prison.

Abbott also writes from a prison of sorts. He has followed his premises, adding more or less arbitrary ideas (further premises) that allow a humanlike existence to his subjects. If the narrative creaks and groans under these burdens, Abbott nevertheless carries it off to the point of convincing the naive reader that perhaps higher dimensions do exist, which is all he set out to do. In this edition, Stewart is the tour operator, clearly enamored of this place and keen to give us the widest possible view. He includes every conceivable scrap of history and mathematics that bears on the strange adventures of A. Square.

With me, Stewart succeeds as well. How else did I become motivated to explore the population biology of Flatland? How else to explain my renewed interest in the subject of two-dimensional existence?