Book Review

Does God Play Dice: The New Mathematics of Chaos and *What Shape Is a Snowflake? Magical Numbers in Nature*

Reviewed by Philip Holmes

Does God Play Dice: The New Mathematics of Chaos
Ian Stewart
Second Edition
Blackwell Publishers Inc., Malden, MA, 2002
$21.95, xii+ 401 pages, softcover
ISBN 0-631-23251-6

What Shape Is a Snowflake? Magical Numbers in Nature
Ian Stewart
W. H. Freeman and Co., New York, 2002
$29.95, 224 pages, hardcover
ISBN 0-7167-4794-4

Ian Stewart is certainly the most prolific, and probably the most successful, among the current popularizers of mathematics. ("Further Reading" in the second of the books reviewed here lists no fewer than seven of his other books on topics ranging from bifurcation and catastrophe to philosophy, including the first reviewed here.) He is also a serious research mathematician, working in dynamical systems with an emphasis on systems with symmetries; indeed, he was recently elected a Fellow of the Royal Society, and in 1995 he won the Society's Faraday Medal for his contributions to the public understanding of science. He is thus unusually and ideally placed to reveal to the vast world that has never heard of the AMS or its *Notices* what readers of this journal spend their working hours doing. How well does he succeed in the pair of books under review? I have to say, "Pretty well." I qualify this at length below, but of course I, along with most of you, do not belong to his primary target audience, so my opinions may be irrelevant. Nonetheless, the ways in which mathematics does or does not get beyond the pages of technical and learned journals should be of some interest to all of us.

First, some remarks on style. In his popular writings Stewart favors a punchy (perhaps sometimes punch-drunk) mode. Reading the second edition of his 1989 (mathematical) bestseller *Does God Play Dice*, which appeared in the U.K. in 1997 but was published in the U.S. only this year, I noted closing paragraphs as short as: "And was horrified by it" (Poincaré, on discovering chaos), "So, 'nonlinear' it is", and simply "We don't" (on human ignorance of quantum states). Sentences may begin with conjunctions or may be missing a subject or verb; an early pair of paragraphs reads:

"Simple. Elegant. Elusive.

Order from chaos."

Epigraphs drawn from literature suggest the author's wide reading but often have little to do with the chapter's contents beyond echoing a key word. From time to time I chuckled or winced at puns such as "the goat in the machine" (*Dice*, p. 370), and I wonder if we need the "bug-eyed monster…Worzel of Velantia" to flap its "scaly reptilian wings" in a neighboring galaxy to illustrate infinitesimal

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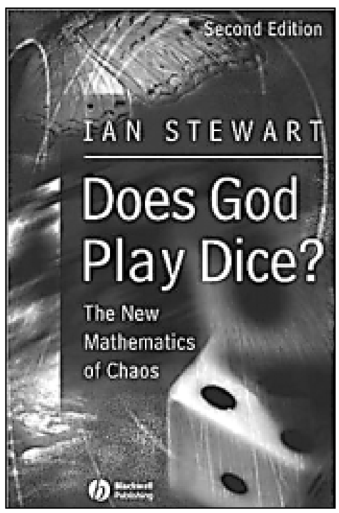
perturbations (p. 52), or "Mummy Bear" to complain that "someone's been iterating [her] Poincaré map" (p. 106). On the other hand, some of the one-liners really do hit the mark: "On any exponential curve, yesterday is invisible and tomorrow is explosive" (p. 131).

So, I don't always warm to the style.
But I had to get this out of the way.
For these *are* pretty good books.
At least, the first is.

Does God Know the Initial Conditions?

The second edition of *Does God Play Dice* adds three new chapters (14–16), on prediction and control of chaos, and quantum mechanics and chaos. There are a few updates elsewhere, but the rest of the book remains, according to the preface, "virtually untouched." (Alas, the poor quality of some figures is also untouched, a constraint evidently imposed by the publishers.) For those who missed the first edition of this account of the mathematics and applications of dynamical systems theory, I shall outline the contents as I comment on them.



Chapter 1 opens with the Newtonian universe and Laplace's remark from his *Philosophical Essays on Probabilities* that for an intellect knowing the equations of motion and initial conditions of "Nature…nothing would be uncertain; and the future…be present before its eyes" (I paraphrase). Stewart then exhorts the reader to iterate the quadratic function $x \mapsto kx^2 - 1$ on a calculator (I am among the laptopless few who still carry them) and thereby convince himself that, for certain values of k, the future is not quite so clear. This quickly gets one involved.

Two historical chapters exploring deterministic and probabilistic equations and laws follow, leading to "The Last Universalist". Here there is a fairly good account of Henri Poincaré's work on the planar, restricted, circular, three-body problem (*not*, as Stewart states, Hill's problem, which further simplifies the system by moving one of the primaries to infinity) and of Poincaré's entry [8] to the prize competition sponsored by King Oscar II of Sweden and Norway. This work essentially founded the modern theory of dynamical systems and led in turn to the rich and disorganized riot of nonlinear dynamics and chaos theory that is Stewart's subject. The historical study by June Barrow-Green [2], which he mentions but does not cite, gives considerably greater detail for those interested.

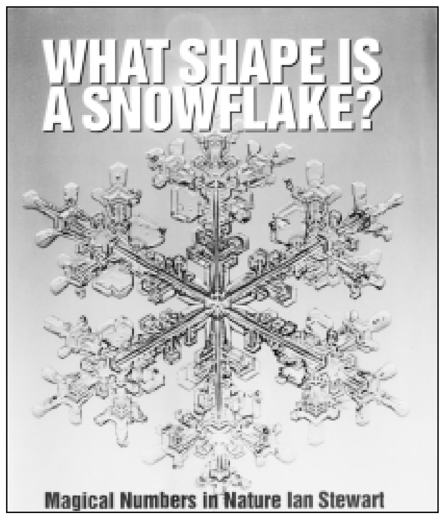
In Chapters 5 and 6 the book begins to engage its subject more substantially, with a rather good discussion of the dynamics of a simple pendulum from a phase-space and energy-conservation perspective, moving on to general phase spaces, invariant sets, and strange attractors, using a mod 10 version of the solenoid mapping to illustrate chaos and sensitive dependence on initial conditions. This avoids introducing mod 2 arithmetic, as in the usual discussions of symbolic dynamics and coding for quadratic-like maps. The mathematical foundations thus sketched, Stewart discusses weather forecasting and Edward Lorenz's now-famous 1963 paper [4], although he does not cite Lorenz's own delightful book [5] (developed from lectures given at the University of Washington in the early 1990s), which, while it does not have the range of the one under review, in some respects does a better job and in any case retells the story of Lorenz's discoveries from his own viewpoint.

Chapter 8 returns to sketching more of the basic ingredients of chaotic dynamics: stretching and folding in phase space, Stephen Smale's horseshoe map and historical developments that motivated it, the work of physicists such as Hénon and Chirikov, and the logistic map (reprise). Chapter 9 considers turbulent fluids and notes the Ruelle-Takens paper [9], another landmark, and the key experiments of Swinney and Gollub revealing that these abstract ideas really are relevant in practice. Feigenbaum's discovery of universal scaling in cascades of period-doubling bifurcations and his proposal that renormalization methods account for it appear in Chapter 10. At this point "Feigenvalues" are inevitable, as will be "Mandelbrots" in the discussion of complex-analytic dynamics and fractals in Chapter 11, appropriately titled "The Texture of Reality". Here as throughout the book science (mostly physics) and mathematics are nicely interwoven, although the lack of precision makes the discussions of fractal structures in phase space (attractors) and in the physical space of mixing fluids somewhat confusing.

Stewart returns to astronomy (the chaotic rotation of Hyperion, a Saturnian satellite) and Hamiltonian mechanics in Chapter 12 and moves on to biology in Chapter 13, with sketches of dynamical population and disease models and of the entrainment of oscillators with reference to cardiac dynamics. Apart from the final chapter (portentously titled "Farewell, Deep Thought"), this is where the first edition ended.

The first of the three additional chapters concerns prediction in chaotic systems. After again making the important distinction between the real world and (mathematical) *models* of (parts of) it, Stewart returns to sensitive dependence and Liapunov exponents and shows that short-term prediction is possible. He mounts a spirited defense of academic life and pure research and ends with a lengthy description of an industrial project on which he and his colleagues at the University of Warwick consulted for a U.K. manufacturer of coil springs, prompting the building of a prototype "coilability" testing machine that uses phase-space reconstruction methods to improve predictability.

In the second he describes the control of chaos work due to Ott, Grebogi, Yorke, et al. This has already received a fair bit of notice in the scientific press, although, contrary to some impressions, nonlinear control theory was a flourishing subject before their work appeared (albeit largely confined to electrical engineering departments). Again, applications—to steering space missions and to cardiac pace-makers—are included.



Stewart admits that his third new chapter, on quantum mechanics, is "much more speculative," and indeed it is. Here he enlarges on five pages from the final chapter of the first edition, sketching familiar puzzles associated with interpreting the physical meaning of QM (Bell's inequality and the Einstein-Podolsky-Rosen paradox) and arguing that sensitive dependence of underlying *deterministic* processes, as exemplified in throwing a die, might lie behind quantum indeterminacy. The discussion remains pretty vague, and Stewart does not convince me that dynamical systems theory will elucidate quantum mechanics much more than Roger Penrose manages to illuminate the role of quantum mechanics in consciousness [7].

The closing chapter has been expanded to include sections on scientific theories and why the reductionist viewpoint beloved of fundamental-laws-style physicists is not necessarily appropriate to understanding complex systems. In an annoying but provocative figure, between the long-sought "theory of everything" and "nature", Stewart interposes Chris Langton's cellular automaton "ant country" with its emergent behavior and uses this to plead for a series of "levels of description." Unfortunately he does not cite what is still one of the finest comments on the failings of reductionism: Philip Anderson's wonderful 1972 article in *Science* [1].

The editing is occasionally careless. For example, on page 350 the conjectured existence of "technically 'nice' invariant measures" is mentioned in passing, as if the reader were expected to take this in stride, but four pages later we are cautioned "for theoretical understanding [to] appeal to things known as 'invariant measures', which are very similar to probabilities." I noted numerous minor and a few more serious misleading or incorrect statements, including persistent errors from the first edition. On page 186 the spacing of successive period-doubling bifurcations is said to *increase* by a factor of about four, when *decrease* is intended. On page 291 an italicized *not* inverts the intended sense (that one is trying to shoot down the suggestion that chaos *does* fit the data). In Chapter 5, discussing a nice example of a mechanical system with a higher-dimensional phase space, the bicycle, Stewart notes that the five main moving parts require five position and five velocity variables to characterize the dynamical state. An engineer calls this five degrees of freedom, not ten. He gets the terminology wrong in the text on page 81 but right in the picture on page 82. A trivial point perhaps, but then he appeals to the bicycle again in discussing Liouville's $2n$-volume conservation theorem for (canonical) Hamiltonian systems. The problem here is that this lovely result does *not* apply to steerable wheeled vehicles such as bicycles, which are *non-holonomically* constrained, are not canonical, and can exhibit asymptotic stability with respect to some variables, corresponding to phase-space contraction, even while energy is conserved [6]. In a popular book it is fine to omit obsessive technical detail, but in this case imprecision misleads.

Such complaints aside, this is a lively and broad introduction to a fascinating field of research in which mathematicians and scientists of all kinds are once more interacting in a manner almost worthy of the eighteenth and nineteenth centuries. Stewart ties his story together well with examples and leitmotivs spanning the chapters. And, as I've noted, he interweaves mathematics and science, theory and experiment, ideas and human actors in a manner appropriate to the subject and provides much of the solid science missing from James Gleick's entertaining account [3]. He may sometimes frustrate, but he almost always informs and provokes.

Does God Play Dice is aimed at nonmathematicians and nonscientists, and, as I've argued, it does a good job of conveying the main ideas and excitement of a relatively new field to a general audience. It can also be read with profit by mathematicians and even by specialists in dynamical systems. *What Shape Is a Snowflake* is much more of a coffee table book, with attractive photographs of things natural and manmade on almost every page. It is basically a rumination on patterns and symmetries in nature and as such covers some of the same ground as Marty Golubitsky's books with Stewart and Mike Field [11].

Snowflake contains sixteen short chapters organized into three parts: Principles and Patterns, The Mathematical World, and Simplicity and Complexity. The investigation starts and ends with snowflakes but visits a vast range of patterns and environments in between, including "Order in Chaos", a twenty-page précis of *Does God Play Dice*. Each chapter is in turn divided into 1–3 page segments, with text and extensively captioned color and line illustrations intermingled in a busy fashion. The result is that the story proceeds on two levels: in the main text and in the pictures and captions (like a *Scientific American* article). One could therefore skim by just looking at the pictures and reading their captions and pick up the main ideas without ever dipping into the text. The text, in fact, adds less than one might hope, for the treatment is rather superficial.

Mathematical ideas and applications are again interwoven, but the short sections mean that each little nugget is rapidly followed by another. One-, two-, and three-dimensional patterns; crystal lattices; spots and stripes on animal coats; waves in sea, sand, and cloud; scales in animal size and music; seashell patterns; spirals of sunflower seeds, chemical reactions, and hurricanes; space-time footfall patterns of animal gaits (a topic that Stewart, Golubitsky, and Jim Collins have studied in detail); fractals, image compression, and seashells (again); chaos and cosmology follow in dizzying succession. A rich and wonderful series of snapshots is presented in an engaging manner, and we are told over and over that the patterns "are a consequence of simple mathematical rules" (caption, p. 125). Names are dropped (Turing's reaction-diffusion systems, p. 164), but things move so fast that specific "rules" are rarely vouchsafed. The discussions of symmetry breaking on pages 152–5, of fractals on pages 158–63, of intrinsic geometry and the universe in Chapter 15, and the closing explanation of snowflake forms of Chapter 16 are notable exceptions, but the blur of images tends to obscure the main message: that fundamental mathematical principles (the symmetries of Euclidean space, for example) determine a catalog of what we *expect* to see, while physical laws and the mathematical models encoding them determine what we actually *do* see. Stewart stresses that mathematics helps us idealize and hence better understand the world, but this could have been a deeper and stronger book had he applied his considerable talents to explaining some elements of his own professional interests—normal form theory and bifurcation with symmetry—thereby revealing more of the "underlying mathematics."

Not only does the mathematics remain largely implicit, but many of the images that appear in the illustrations are incompletely identified, although some are acknowledged. For example, Constable's painting *The White Horse* (now in the Frick Collection, New York), which is reproduced unnamed on page 101 in connection with a discussion on mathematics and beauty, is mysteriously credited to "Geoffrey Clements". Specificity is thereby lost, and it begins to seem as if the book was partly composed by graphic artists in the publishing house.[1] In the discussion of Penrose tilings and quasicrystals in Chapter 7, for instance, the former are nicely illustrated, but inclusion of electron micrographs (or their Fourier transforms) of quasicrystals would have demonstrated the link more strongly. In summary, the book *looks* very attractive, but with properly integrated illustrations and a more modest range of phenomena treated at greater length, the story could have been made tighter and clearer.

So, while it would make a nice gift for a nonscientist friend or relative who will enjoy the pictures and wonder at the range of phenomena touched on, this book is at best an appetizer. Others, including *Does God Play Dice* and [11], will be needed to appreciate the work of mathematicians and scientists in understanding those phenomena.

In closing, I must confess that I'm not an entirely unbiased reviewer, since Florin Diacu and I have also taken aim at that elusive quarry, the intelligent layperson. Moreover, Ian Stewart reviewed our book [10], praising the story but faulting our "attempt to explain the mathematics at an advanced technical level." (I tend to agree that we probably *did* tilt too far in that direction.) In these books Stewart avoids the details and moves from subtopic to subtopic in an attractive way. One can safely put down either of them without losing the thread. We took a more pedestrian approach. Boooring? Maybe.

Ian isn't.

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[1]*Ian Stewart confirms that this is the case: the images and layout were handled by Ivy Press, a "packager" that set the book for the publishers. Copyright laws and licensing agreements evidently determined how pictures were acknowledged. Realities of trade publishing!*

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